Junior Problems

J505. Solve the equation

$$2x^3 + x\{x\} + 2\{x\}^3 = \frac{1}{108},$$

where $\{x\}$ denotes the fractional part of x.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J506. Prove that any integer $n > 6$ can be written as $n = p + m$, where p is a prime less than $n/2$ and p does not divide m.

Proposed by Li Zhou, Polk State College, USA

J507. Consider a real number a,

$$b = (a^2 + 2a + 2)\left(a^2 - (1 - \sqrt{3})a + 2\right)\left(a^2 + (1 + \sqrt{3})a + 2\right)$$

and

$$c = (a^2 - 2a + 2)\left(a^2 + (1 - \sqrt{3})a + 2\right)\left(a^2 - (1 + \sqrt{3})a + 2\right).$$

Find a knowing that $b + c = 16$.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J508. Let a, b, c be positive numbers such that $a + b + c + 2 = abc$. Prove that

$$(1 + ab)(1 + bc)(1 + ca) \geq 125.$$

Proposed by An Zhenping, Xianyang Normal University, China

J509. Find the least 4-digit prime of the form $6k - 1$ that divides $8^{1010}11^{2020} + 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J510. Let a, b, c be positive real numbers. Prove that

$$(1 + a)(1 + b)(1 + c) \geq \left(1 + \frac{2ab}{a + b}\right)\left(1 + \frac{2bc}{b + c}\right)\left(1 + \frac{2ca}{c + a}\right)$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Senior Problems

S505. Find k such that a triangle with sides a, b, c is right if and only if

$$\sqrt[6]{a^6+b^6+c^6+3a^2b^2c^2} = k\max\{a,b,c\}$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S506. Let x, y, z, t be real numbers, $0 \le x, y, z, t \le 1$, such that

$$(1-x)(1-y)(1-z)(1-t) = xyzt.$$

Prove that

$$x^2+y^2+z^2+t^2 \ge 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S507. If a, b, c are real numbers such that $ax^2+bx+c \ge 0$ for all real numbers x, prove that $4a^3-b^3+4c^3 \ge 0$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S508. Prove that in any triangle ABC,

$$\left(\frac{h_a}{\ell_a}\right)^2+\left(\frac{h_b}{\ell_b}\right)^2+\left(\frac{h_c}{\ell_c}\right)^2-2\frac{h_a}{\ell_a}\frac{h_b}{\ell_b}\frac{h_c}{\ell_c}=1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S509. Solve in integers the equation

$$2(xy+2)^2-6(x+y)^2=(x+y-1)^3-6.$$

Proposed by Alessandro Ventullo, Milan, Italy

S510. Consider an array of 49 consecutive integers whose median is a perfect square. Prove that the sum of the cubes of the 49 integers can be written as a sum of four perfect squares two of which are equal.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U505. Let K be a field. Prove that the polynomial

$$X^n + X^2Y + XY + XY^2 + Y^n$$

is irreducible in the ring $K[X, Y]$, for all $n \geq 2$.

Proposed by Mircea Becheanu, Montreal, Canada

U506. Find all functions $f : (0, \infty) \longrightarrow (0, \infty)$ such that

$$f(1 + x) = 1 + f(x) \text{ and } f\left(\frac{1}{x}\right) = \frac{1}{f(x)}$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U507. Evaluate

$$\int_{-1/3}^{1} \frac{1}{2x + \sqrt{x^2 + x + 2}} dx.$$

Proposed by Titu Andreescu, University of Texas a Dallas, USA

U508. For positive integer n, let S_1, S_2, $\ldots$, S_{2^n-1} be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is $m_{ij} = |S_i \cup S_j|$. Find the determinant of $M = (m_{ij})$.

Proposed by Li Zhou, Polk State College, USA

U509. Prove that for any $x > 1$, the following inequalities hold.

$$\log\left(\frac{1 + x^2}{x^2 - 2x + 2}\right)^{\frac{1}{2x-1}} < \arctan(x) - \arctan(x - 1) < \log\left(\frac{1 + x^2}{x^2 - 2x + 2}\right)^{\frac{1}{2(x-1)}}$$

Proposed by Besfort Shala, University of Primorska, Slovenia

U510. Evaluate

$$\int_0^{\pi} \frac{x \sin x}{2021 + 4 \sin^2 x} dx$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Olympiad Problems

O505. Let a, b, c, d be positive real numbers such that

$$a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$\frac{3(a^2+b^2+c^2+d^2)}{a+b+c+d}+1\geq a+b+c+d.$$

Proposed by Marius Stănean, Zalău, Romani

O506. Let a be a nonnegative integer. Find all pairs (x, y) of nonnegative integers such that

$$(a^2+1)(x^3-2axy+y^3)=a^2-xy.$$

Proposed by Mircea Becheanu, Montreal, Canada

O507. Let a, b, c, d be positive numbers such that $a+b+c+d=2$. Prove that

$$\frac{a^2b}{a^4+b^3+c^2+d}+\frac{b^2c}{b^4+c^3+d^2+a}+\frac{c^2d}{c^4+d^3+a^2+b}+\frac{d^2a}{d^4+a^3+b^2+c}\leq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

O508. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{b(a+5c)^2}+\frac{b}{c(b+5a)^2}+\frac{c}{a(c+5b)^2}\geq\frac{1}{4(\sqrt{a}+\sqrt{b}+\sqrt{c})}.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

O509. Prove that for any positive real numbers a, b, c

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq\frac{27(a^3+b^3+c^3)}{(a+b+c)^3}+\frac{21}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O510. Let $ABCDE$ be a convex pentagon with

$$\angle BCD=\angle ADE \quad \text{and} \quad \angle BDC=\angle AED\,.$$

The circumcircle of triangle CDE meets lines DA and DB for the second time at points P and Q, respectively. Lines CP and QE intersect at X. Prove that $ADBX$ is a parallelogram.

Proposed by Waldemar Pompe, Warsaw, Poland