Junior problems

J499. Let a, b, c, d be positive real numbers such that

$$a(a-1)^2+b(b-1)^2+c(c-1)^2+d(d-1)^2=a+b+c+d.$$

Prove that

$$(a-1)^2+(b-1)^2+(c-1)^2+(d-1)^2\le 4.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

First solution by Polyahedra, Polk State College, USA
The given condition is equivalent to $a^3+b^3+c^3+d^3 = 2(a^2+b^2+c^2+d^2)$ and the claimed inequality is equivalent to $a^2+b^2+c^2+d^2\le 2(a+b+c+d)$. By the Cauchy-Schwarz inequality,

$$(a^2+b^2+c^2+d^2)^2\le (a^3+b^3+c^3+d^3)(a+b+c+d)=2(a^2+b^2+c^2+d^2)(a+b+c+d),$$

and the proof is complete.

Second solution by Lilit Ghazanchyan, Vanadzor, Armenia
According to Sedrakyan-Engel-Titu inequality

$$a+b+c+d=a(a-1)^2+b(b-1)^2+c(c-1)^2+d(d-1)^2=$$

$$=\frac{(a(a-1))^2}{a}+\frac{(b(b-1))^2}{b}+\frac{(c(c-1))^2}{c}+\frac{(d(d-1))^2}{d}\ge$$

$$\ge\frac{(a(a-1)+b(b-1)+c(c-1)+d(d-1))^2}{a+b+c+d}$$

from which we immediately infer the inequality

$$(a+b+c+d)^2\ge(a(a-1)+b(b-1)+c(c-1)+d(d-1))^2$$

The latter gives

$$a+b+c+d\ge a(a-1)+b(b-1)+c(c-1)+d(d-1),$$

which can be easily transformed into the desired inequality

$$(a-1)^2+(b-1)^2+(c-1)^2+(d-1)^2\le 4$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA.

J500. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{5a^2-2a+1}+\frac{1}{5b^2-2b+1}+\frac{1}{5c^2-2c+1}+\frac{1}{5d^2-2d+1}\geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Arkady Alt, San Jose, CA, USA

By replacing (a, b, c, d) in the problem with $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right)$ we obtain equivalent setting of the originl problem:

Prove that $\sum_{cyc}\frac{a^2}{a^2-2a+5}\geq 1 \quad$ if $abcd = 1$.

By Cauchy Inequality

$$\sum_{cyc}\frac{a^2}{a^2-2a+5}\geq\frac{(a+b+c+d)^2}{a^2+b^2+c^2+d^2-2(a+b+c+d)+20}\geq$$

$$(a+b+c+d)^2\geq a^2+b^2+c^2+d^2-2(a+b+c+d)+20 \iff$$
$$ab+ac+ad+bc+bd+cd+a+b+c+d\geq 10$$

where latter inequality holds because by AM-GM Inequality $ab+ac+ad+bc+bd+cd\geq 6\sqrt[6]{a^3b^3c^3d^3}=6$ and $a+b+c+d\geq 4\sqrt[4]{abcd}=4$.

Also solved by Polyahedra, Polk State College, USA; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Mohamed Ali, Houari Boumedien School, Algeria; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Suhas Sheikh, Indian Institute of Science, India; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania.

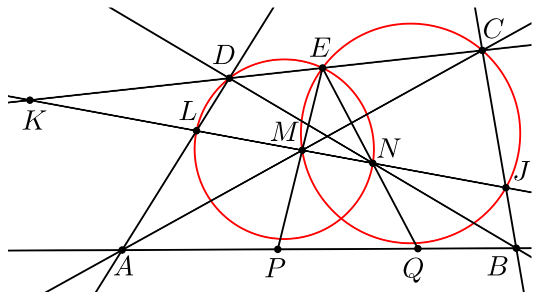
J501. In a convex quadrilateral $ABCD$, M and N are the midpoints of diagonals AC and BD, respectively. The intersection of the diagonals lies on segments CM and DN, while points P and Q lie on segment AB and satisfy

$$\angle PMN = \angle BCD \quad \text{and} \quad \angle QNM = \angle ADC.$$

Prove that lines PM and QN meet at a point lying in line CD.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Polyahedra, Polk State College, USA



Suppose that MN intersects BC, CD, and DA at J, K, and L, respectively. Applying Menelaus' theorem to triangles ACD and BCD with transversal MN, we get

$$\frac{AL}{LD} \cdot \frac{KD}{KC} \cdot \frac{CM}{MA} = 1 \quad \text{and} \quad \frac{BJ}{JC} \cdot \frac{KC}{KD} \cdot \frac{DN}{NB} = 1.$$

Hence, $AL/LD = KC/KD = JC/BJ$, thus $AD/LD = BC/BJ$ as well. Applying Menelaus' theorem to triangles ALM and BJN with transversal CD, we get

$$\frac{AD}{LD} \cdot \frac{KL}{KM} \cdot \frac{MC}{AC} = 1 \quad \text{and} \quad \frac{BC}{JC} \cdot \frac{KJ}{KN} \cdot \frac{ND}{BD} = 1.$$

Therefore, $KL/KM = 2LD/AD$ and $KJ/KN = 2JC/BC$, so

$$\frac{KD}{KC} \cdot \frac{KM}{KL} \cdot \frac{KJ}{KN} = \frac{BJ}{JC} \cdot \frac{AD}{2LD} \cdot \frac{2JC}{BC} = 1.$$

Now suppose that PM intersects CD at E. Then C, E, M, and J are concyclic. By the power of a point, $KC \cdot KE = KM \cdot KJ$. Consequently, $KD \cdot KE = KD \cdot KM \cdot KJ/KC = KL \cdot KN$, that is, D, E, N, and L are concyclic as well. Hence, E, N, and Q are collinear.

Also solved by Taes Padhihary, Disha Delphi Public School, India.

J502. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{c\left(a^2+bc\right)}+\frac{b^3}{a\left(b^2+ca\right)}+\frac{c^3}{b\left(c^2+ab\right)}\geq\frac{3}{2}.$$

Proposed by Konstantinos Metaxas, Athens, Greece

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain
By changing the variables to $x=\frac{a}{c}$, $y=\frac{b}{a}$, $z=\frac{c}{b}$, the inequality reads as

$$\frac{x^2}{(x+y)}+\frac{y^2}{(y+z)}+\frac{z^2}{(z+x)}\geq\frac{3}{2},$$

where $xyz=1$.

This inequality follows by the Cauchy-Schwarz inequality in Engel form and the AM-GM inequality:

$$\begin{aligned}\frac{x^2}{(x+y)}+\frac{y^2}{(y+z)}+\frac{z^2}{(z+x)} &\geq \frac{(x+y+z)^2}{2(x+y+z)}\\ &= \frac{x+y+z}{2}\\ &\geq \frac{3\sqrt[3]{xyz}}{2}=\frac{3}{2}.\end{aligned}$$

Also solved by Polyahedra, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Dumitru Barac, Sibiu, Romania; Adarsh Kumar, IIT Bombay, India; Taes Padhihary, Disha Delphi Public School, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece.

J503. Solve in positive integers the equation

$$\min\left(x^4+8y, 8x+y^4\right) = (x+y)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Since we may exchange x, y without altering the problem, we may assume wlog that $8x + y^4 = (x+y)^2$, or $(x+y-4)^2 = y^4 - 8y + 16$. Therefore, $y^4 - 8y + 16$ must be a perfect square. If $y \geq 3$, note that $2y^2 - 8y + 15 = 2(y-1)(y-3) + 9 > 0$, or $y^4 - 8y + 15 > \left(y^2-1\right)^2$, wherereas $8y - 16 > 0$ for $y^4 - 8y + 16 < \left(y^2\right)^2$. Therefore, $y^4 - 8y + 16$ is a perfect square for positive integer y only when $y \in \{1, 2\}$.

For $y = 1$, we have $(x-3)^2 = 9 = 3^2$, yielding either $x = 0$ which is not a positive integer, or $x = 6$, in which case $x^4 + 8y > y^4 + 8x = 49 = (6+1)^2$ is indeed a solution.

For $y = 2$, we have $(x-2)^2 = 16 = 4^2$, yielding either $x = -2$ which is not a positive integer, or $x = 6$, in which case $x^4 + 8y > y^4 + 8x = 64 = (6+2)^2$ is indeed a solution.

Restoring generality, all solutions are

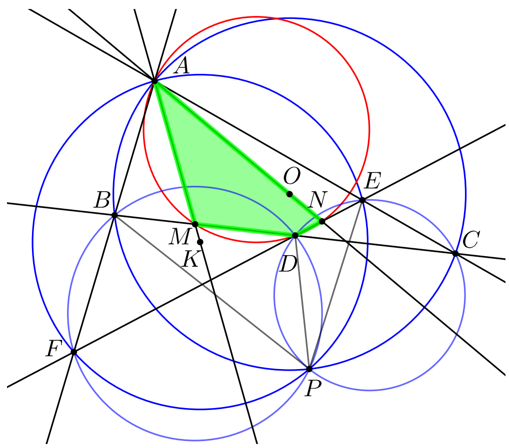
$$(x,y) = (6,2), \qquad (x,y) = (6,1), \qquad (x,y) = (1,6), \qquad (x,y) = (2,6).$$

Also solved by George Theodoropoulos, National Technical University of Athens, Athens, Greece; Polyahedra, Polk State College, USA; Dumitru Barac, Sibiu, Romania; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Ioannis D. Sfikas, Athens, Greece.

J504. Let ABC be a triangle with circumcenter O, E be an arbitrary point on AC and F a point on AB such that B lies between A and F. Let K be the circumcenter of the triangle AEF. Denote by D the intersection of lines BC and EF, M the intersection of lines AK and BC, and N the intersection of lines AO and EF. Prove that points A, D, M, N are concyclic.

Proposed by Mihai Miculiţa, Oradea, România

First solution by Polyahedra, Polk State College, USA



Let P be the second intersection point of the circumcircles of ABC and AEF. By Miquel's theorem, P is on the circumcircles of FBD and ECD as well. Using directed angles and taking all equalities modulo π, we have

$$\begin{aligned}&\angle NDM = \angle FDC = \angle FDP + \angle PDC = \angle FBP + \angle PEC\\&= -\angle PBC - \angle CBA - \angle FEP - \angle AEF = -\angle PAC - \frac{\angle COA}{2} - \angle FAP - \frac{\angle AKF}{2}\\&= -\angle FAC - \left(\frac{\pi}{2} - \angle OAC\right) - \left(\frac{\pi}{2} - \angle FAK\right) = -\angle FAC + \angle OAC + \angle FAK = -\angle MAN,\end{aligned}$$

thus A, D, M, and N are concyclic.

Second solution by Daniel Lasaosa, Pamplona, Spain
Note that

$$\angle AND = 180^\circ - \angle ANE = \angle NAE + \angle AEN = \angle OAC + \angle AEF = 90^\circ + \angle AEF - \angle B.$$

Similarly,

$$\angle AMD = 180^\circ - \angle AMB = \angle MBA + \angle MAB = \angle B + \angle KAF = \angle B + 90^\circ - \angle AEF.$$

Since $\angle AND + \angle AMD = 180^\circ$, we conclude that $AMDN$ is cyclic.

Note: The proof is conducted in the case where M, N are on opposite sides of line AD. Certain choices of E, F may result on M, N being on the same side of AD, in which case we may analogously prove that $\angle AND = \angle AMD$, using that in this case angles $\angle AND$ and $\angle ANE$ are equal instead of adding up to 180°. Similarly, the proof relies on ABC and AEF being acute-angled triangles. If ABC is obtuse at B, then O is on the opposite side of line AC with respect to B, and E is inside segment DN, or we have again $\angle AND = \angle ANE$, but $\angle NAE = \angle OAC = \angle B - 90^\circ$, and $\angle AEN = 180^\circ - \angle AEF$, resulting once more in $\angle AND = 90^\circ + \angle AEF - \angle B$. We may similarly treat the case where AEF is obtuse at E.

Also solved by Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.

Senior problems

S499. Let a and b be distinct real numbers. Prove that $27ab\left(\sqrt[3]{a}+\sqrt[3]{b}\right)^3 = 1$ if and only if $27ab(a+b+1) = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Let $\sqrt[3]{a} = x, \sqrt[3]{b} = y$. The problem may be restate as follows: Prove that $3xy(x+y) = 1$ if and only if $27x^3y^3(x^3+y^3+1) = 1$. Now we have $27x^3y^3(x^3+y^3+1) = 1$ is equivalent to one of the following

$$x^3+y^3+1 = \frac{1}{(3xy)^3},$$

$$x^3+y^3+\left(\frac{-1}{3xy}\right)^3 - 3xy\left(\frac{-1}{3xy}\right) = 0,$$

$$x^3+y^3+z^3-3xyz = 0 \qquad \left(\text{where } z = \frac{-1}{3xy}\right),$$

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx) = 0,$$

$$(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2] = 0.$$

Since $x \neq y$ so this equation is equivalent to

$$x+y+z = 0.$$

That is

$$x+y-\frac{1}{3xy} = 0.$$

This is exactly $3xy(x+y) = 1$. Therefore the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Albert Stadler, Herrliberg, Switzerland; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Suhas Sheikh, Indian Institute of Science, India; Arkady Alt, San Jose, CA, USA.

S500. Let a, b, c be pairwise distinct real numbers. Prove that

$$\left(\frac{a-b}{b-c}-2\right)^2+\left(\frac{b-c}{c-a}-2\right)^2+\left(\frac{c-a}{a-b}-2\right)^2 \geq 17.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Albert Stadler, Herrliberg, Switzerland

Note that

$$\left(\frac{a-b}{b-c}-2\right)^2+\left(\frac{b-c}{c-a}-2\right)^2+\left(\frac{c-a}{a-b}-2\right)^2-17=$$

$$\left(\frac{a^3+b^3+c^3-4a^2b-4b^2c-4c^2a+ab^2+bc^2+ca^2+6abc}{(b-c)(c-a)(a-b)}\right)^2 \geq 0.$$

Second solution by Daniel Lasaosa, Pamplona, Spain

Since $a-c \neq 0$, the inequality is homogeneous in a, b, c, and invariant under simultaneous exchange of the signs of a, b, c, we may substitute a, b, c respectively by $\frac{a}{a-c}$, $\frac{b}{a-c}$ and $\frac{c}{a-c}$, so that $a-b=1$, and we may define $x=a-b$, and consequently $b-c=1-x$, where clearly $x \notin \{0,1\}$. The inequality then rewrites as

$$17 \leq \left(\frac{3x-2}{1-x}\right)^2+(x-3)^2+\left(\frac{2x+1}{x}\right)^2=\frac{x^6-8x^5+35x^4-40x^3+10x^2+2x+1}{(1-x)^2x^2},$$

or equivalently after multiplying by the (clearly nonzero) denominator and rearranging terms,

$$0 \leq x^6-8x^5+18x^4-6x^3-7x^2+2x+1=\left(x^3-4x^2+x+1\right)^2.$$

The inequality therefore always holds. Note that equality occurs when x is one of the roots of x^3-4x^2+x+1, or equivalently, when $y=1-x$ is one of the roots of $y^3+y^2-4y+1=0$. Denote therefore by r each one of the three roots of $y^3+y^2-4x+1=0$, and denoting by d the original value of $a-c$, we find that $a=c+d$, $b=c+rd$, or equality follows iff

$$(a,b,c)=(c+d,c+rd,c),$$

where c takes any real value, d takes any nonzero real value, and r is any one of the roots of y^3+y^2-4x+1.

Also solved by Ioannis D. Sfikas, Athens, Greece; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

S501. Solve the equation $\lfloor x \rfloor \{8x\} = 2x^2$, where $\lfloor a \rfloor$ and $\{a\}$ are the greatest integers less than or equal to a and the fractional part of a, respectively.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Alessandro Ventullo, Milan, Italy and Sarah B. Seales, Northern Arizona University, USA
Clearly, $\lfloor x \rfloor \geq 0$. Let $\lfloor x \rfloor = n$ and $\{x\} = t$. The equation becomes

$$2(n+t)^2 = \lfloor x \rfloor \{8x\} \leq 8\lfloor x \rfloor \{x\} = 8nt,$$

which gives $(n-t)^2 \leq 0$. It follows that $n = t$. Since $0 \leq t < 1$ and n is an integer, then $n = 0$ and so $t = 0$, i.e $x = 0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

S502. Find all positive integers n such that

$$a+b+c|a^n+b^n+c^n-nabc$$

for all positive integers a,b,c.

Proposed by Oleg Muskarov, Sofia, Bulgaria

Solution by Dumitru Barac, Sibiu, Romania

We particularize $b=c=1$, hence $a+2|a^n-na+2$, for all positive integers a. From $a^n-na+2=(a^n-(-2)^n)-n(a+2)+2+2n+(-2)^n$, we deduce that $a+2|2+2n+(-2)^n$, namely the number $2+2n+(-2)^n=0$. We conclde that n is an odd number, hence $2^n-2n-2=0, 2^{n-1}=n+1$. Clearly, $n=3$ is a solution and $n<3$ is not. Indeed, for $n=3$, we have

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\vdots(a+b+c).$$

We claim that $2^{n-1}>n+1$, for all $n\geq 4$. The assertion is clearly true for $n=4$. We suppose $2^{n-1}>n+1$, and we deduce:

$$2^n=2\cdot 2^{n-1}>2(n+1)=2n+2>n+2.$$

Finally, only $n=3$ is the solution of the problem.

Also solved by Albert Stadler, Herrliberg, Switzerland; Chakib Belgani and Mahmoud Ezzaki, Morocco; Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Taes Padhihary, Disha Delphi Public School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.

S503. Solve in positive integers the equation

$$101x^3 - 2019xy + 101y^3 = 100.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that $101\left(x^3 + y^3 - 20xy - 1\right) + xy + 1 = 0$, or 101 divides $xy+1$, and since $xy > 0$, we have $xy+1 \geq 101$, for $xy \geq 100$. Moreover,

$$1 \leq \frac{xy+1}{101} = 1 + xy(20 - x - y) - (x+y)(x-y)^2 \leq 1 + xy(20 - x - y),$$

or $x + y \leq 20$. But then the AM of x, y is at most 10, and their GM is at least 10, or by the AM-GM inequality we must have $x = y = 10$, which since it satisfies the proposed equation, it is clearly its only solution in positive integers.

Also solved by Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Dumitru Barac, Sibiu, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Titu Zvonaru, Comănești, Romania.

S504. Let $a \geq b \geq c \geq 0$ be real numbers such that $a+b+c=3$ Prove that

$$ab^2+bc^2+ca^2+\frac{3}{8}abc \leq \frac{27}{8}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Li Zhou, Polk State College, USA

By assumption, $a^2b+b^2c+c^2a-ab^2-bc^2-ca^2=(a-b)(b-c)(a-c) \geq 0$. Therefore,

$$\begin{aligned}
&27-8(ab^2+bc^2+ca^2)-3abc \geq (a+b+c)^3-4(ab^2+bc^2+ca^2+a^2b+b^2c+c^2a)-3abc\\
&=a^3+b^3+c^3-(ab^2+bc^2+ca^2+a^2b+b^2c+c^2a)+3abc\\
&=a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)
\end{aligned}$$

which is non-negative by Schur's inequality.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Kevin Soto Palacios Huarmey, Perú; Arkady Alt, San Jose, CA, USA.

U499. Let a, b, c be positive real numbers not greater than 2. The sequence $(x_n)_{n\geq 0}$ is defined by $x_0 = a$, $x_1 = b$, $x_2 = c$ and

$$x_{n+1} = \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}$$

for all $n \geq 2$. Prove that $(x_n)_{n\geq 0}$ is convergent and find its limit.

Proposed by Mircea Becheanu, Canada and Nicolae Secelean, România

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that if $0 < x_{n-2}, x_{n-1}, x_n \leq 2$, then $0 < x_{n+1} \leq \sqrt{2 + \sqrt{2+2}} = 2$, or after trivial induction we have $0 < x_n \leq 2$ for all $n \geq 0$. Note also that if $x_n > 1$, then $x_{n+1} > \sqrt{x_n} > 1$. Note next that if $x_n > 2^{-2^m}$ for some positive integer m, then $x_{n+1} > \sqrt{x_n} > 2^{-2^{m-1}}$, or after trivial induction, $x_{n+m-1} > \frac{1}{4}$ and $x_{n+m} > \frac{1}{2}$, for $x_{n+m+1} > \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4}}} = 1$. Therefore, we eventually have $x_n > 1$ for all $n \geq N$ for some integer N.

Denote now $\delta_n = 2 - x_n$, where clearly $0 \leq \delta_n < 2$ for all positive integer n, or for all $n \geq N + 2$ with N defined as above, we have

$$\delta_{n+1} = 2 - x_{n+1} = \frac{4 - x_n - \sqrt{x_{n-1} + x_{n-2}}}{2 + \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}} =$$

$$= \frac{\delta_n}{2 + \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}} + \frac{\delta_{n-1} + \delta_{n-2}}{\left(2 + \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}\right)\left(2 + \sqrt{x_{n-1} + x_{n-2}}\right)} \leq$$

$$\leq \frac{\delta_n}{2 + \sqrt{1 + \sqrt{2}}} + \frac{\delta_{n-1} + \delta_{n-2}}{\left(2 + \sqrt{1 + \sqrt{2}}\right)\left(2 + \sqrt{2}\right)} \leq \frac{6\delta_n + 2\delta_{n-1} + 2\delta_{n-2}}{21},$$

where for proving the last inequality, it suffices to prove that $\sqrt{1 + \sqrt{2}} > \frac{3}{2}$ and $2 + \sqrt{2} > 3$. The latter is obvious from $\sqrt{2} > 1$, whereas the former is equivalent to $\sqrt{2} > \frac{5}{4}$, also clearly true since $4\sqrt{2} = \sqrt{32} > \sqrt{25} = 5$. It follows that $21\delta_{n+1} < 6\delta_n + 2\delta_{n-1} + 2\delta_{n-2}$, or

$$21\delta_{n+1} + 8\delta_n + \frac{10\delta_{n-1}}{3} \leq \frac{2}{3}\left(21\delta_n + 8\delta_{n-1} + \frac{10\delta_{n-2}}{3}\right) - \frac{2\delta_{n-2}}{9} \leq$$

$$\leq \frac{2}{3}\left(21\delta_n + 8\delta_{n-1} + \frac{10\delta_{n-2}}{3}\right).$$

Or, denoting $\Delta_n = 21\delta_{n+1} + 8\delta_n + \frac{10\delta_{n-1}}{3}$, we have $0 \leq \Delta_n < \frac{2\Delta_{n-1}}{3}$ for all $n \geq N + 2$, hence $\lim_{n\to\infty} \Delta_n = 0$, or since $0 \leq \delta_n \leq \frac{\Delta_{n-1}}{21}$, we finally have $\lim_{n\to\infty} \delta_n = 0$, and $\lim_{n\to\infty} x_n = 2$.

Also solved by Albert Stadler, Herrliberg, Switzerland.

U500. Evaluate

$$\lim_{n\to\infty} \tan \pi\sqrt{4n^2+n}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Joel Schlosberg, Bayside, NY, USA

For $r \in \mathbb{R}$ and $m \in \mathbb{N}$, arbitrarily large values

$$n = \frac{1}{2}\sqrt{\left(\frac{\pi m + \arctan r}{\pi}\right)^2 + \frac{1}{16}} - \frac{1}{8}$$

satisfy $\tan \pi\sqrt{4n^2+n} = r$ if noninteger values of n are allowed. To find a well-defined limit, assume that $n \in \mathbb{N}$.

For $x \in \left((2n-\frac{1}{2})\pi, (2n+\frac{1}{2})\pi\right)$, $\tan x$ is strictly increasing. Since

$$4n^2 + n < 4n^2 + n + \frac{1}{16} = (2n+1/4)^2 < (2n+1/2)^2,$$

$$\tan \pi\sqrt{4n^2+n} < \tan \pi(2n+1/4) = \tan(\pi/4) = 1.$$

For any $s \in (0,1)$, if $n > \frac{s^2}{16(1-s)}$, then

$$4n^2 + n > 4n^2 + sn + \frac{s^2}{16} = (2n+s/4)^2 > (2n-1/2)^2,$$

so

$$\tan \pi\sqrt{4n^2+n} > \tan \pi(2n+s/4) = \tan(\pi s/4).$$

Therefore,

$$\lim_{n\to\infty} \tan \pi\sqrt{4n^2+n} \geq \tan(\pi s/4)$$

and so

$$\lim_{n\to\infty} \tan \pi\sqrt{4n^2+n} \geq \lim_{s\to 1^-} \tan(\pi s/4) = \tan(\pi/4) = 1.$$

By the squeeze theorem, $\lim_{n\to\infty} \tan \pi\sqrt{4n^2+n} = 1$.

Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Ioannis D. Sfikas, Athens, Greece; Suhas Sheikh, Indian Institute of Science, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; S.Chandrasekhar, Chennai, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA.

U501. Let $a_1, a_2, \dots, a_n \geq 1$ be real numbers such that $a_1 a_2 \cdots a_n = 2^n$. Prove that

$$a_1 + \cdots + a_n - \frac{2}{a_1} - \cdots - \frac{2}{a_n} \geq n.$$

Proposed by Marin Chirciu, Piteşti, România

Solution by Daniel Lasaosa, Pamplona, Spain

Let $g(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n$, and define the region $\mathcal{R}$ in $\mathbb{R}^n$ determined by $g(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n = 2^n$ and $x_1, x_2, \dots, x_n \geq 1$. The problem is therefore equivalent to the minimization of

$$f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + a_n - \frac{2}{x_1} - \frac{2}{x_2} - \cdots - \frac{2}{x_n}$$

in $\mathcal{R}$, and finding that this minimum is at least n. Note first that any point in the boundary of $\mathcal{R}$ satisfies that $x_i = 1$ for some of the i's in $\{1, 2, \dots, n\}$. We may then use Lagrange's multiplier method, which ensures that a real constant λ exists such that at any given minimum, for each $i \in \{1, 2, \dots, n\}$ such that $x_i \neq 1$ at said minimum, we have

$$1 - \frac{2}{x_i^2} = \frac{\partial f}{\partial x_i} = \lambda \frac{\partial g}{\partial x_i} = \frac{2^n \lambda}{x_i}, \qquad 2^n \lambda = \frac{x_i^2 - 2}{x_i}.$$

Therefore, if $i \neq j \in \{1, 2, \dots, n\}$ satisfy that $x_i, x_j \neq 1$, we must have

$$\frac{x_i^2 - 2}{x_i} = \frac{x_j^2 - 2}{x_j}, \qquad (x_i x_j + 2)(x_i - x_j) = 0,$$

and since $x_i x_j + 2 > 3$, we have $x_i = x_j$. Therefore, at any minimum an integer $1 \leq m \leq n$ exists such that m of the x_i's have the same value k which is different than 1, and the remaining $n - m$ take value 1. Or, the common value of the m x_i's is $k = 2^{\frac{n}{m}}$, and $m = \frac{n \ln 2}{\ln k}$, for

$$f(x_1, x_2, \dots, x_n) \geq n \left(\frac{k \ln 2}{\ln k} - \frac{2 \ln 2}{k \ln k} - 1 + \frac{\ln 2}{\ln k} \right),$$

and it suffices to show that for all $k \geq 2$, we have

$$\frac{k}{\ln k} - \frac{2}{k \ln k} + \frac{1}{\ln k} \geq \frac{2}{\ln 2}.$$

Therefore, defining

$$h(k) = \frac{k^2 + k - 2}{k \ln k},$$

the problem is equivalent to showing that $h(k) \geq \frac{2}{\ln 2}$ for all $k \geq 2$. Now, note that $h(k)$ is of the order of $\frac{k}{\ln k}$ when k grows, which diverges, whereas any local minimum of $h(k)$ must satisfy

$$0 = h'(k) = (2k+1) k \ln k - (\ln k + 1)\left(k^2 + k - 2\right) = \left(k^2 + 2\right) \ln k - \left(k^2 + k - 2\right).$$

Now, it is well known that $\ln 2 > \frac{2}{3}$, or $\ln 4 > \frac{4}{3}$, whereas clearly $\ln 3 > 1$. Therefore, if $k > 4$, we have $h'(k) > \frac{k^2 - 3k + 10}{3} > \frac{k+10}{3} > 0$. At the same time, if $k \in (3, 4]$, we have $h'(k) > 4 - k \geq 0$. Finally, if $k \in [2, 3]$, we have $h'(2) > 6 \ln 2 - 4 > 0$, while $h''(k) = 2k \ln k + \frac{2}{k} - k - 1 > \frac{k+2}{3} - 1 \geq \frac{1}{3}$, and since $h''(k) > 0$ in $[2, 3]$ and $h'(2) > 0$, then $h'(k) > 0$ also in $[2, 3]$. It follows that $h(k)$ reaches its minimum when $k = 2$, and this minimum is indeed $\frac{2}{\ln 2}$. The conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland; S.Chandrasekhar, Chennai, India; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Chakib Belgani and Mahmoud Ezzaki, Morocco.

U502. Find all pairs (p, q) of primes such that pq divides

$$(20^p + 1)(7^q - 1).$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Corneliu Mănescu-Avram, Ploiești, Romania

Case 1: $p|20^p + 1$, $q|7^q - 1$. From the Fermat theorem we deduce that o divides $20 + 1 = 21 = 3 \cdot 7$ and q divides $7 - 1 = 6 = 2 \cdot 3$, whence the solutions $(3, 2), (3, 3), (7, 2), (7, 3)$.

Case 2: $p|20^p + 1$, $q \nmid 7^q - 1$.
As above, $p = 3, 7$. From $20^3 + 1 = 3^2 \cdot 7 \cdot 127, 20^7 + 1 = 3 \cdot 7^2 \cdot 827 \cdot 10529$, we find the solutions $(3, 7), (3, 127), (7, 7), (7, 827), (7, 10529)$.

Case 3: $p \nmid 20^p + 1, q|7^q - 1$. We have $q = 2, 3$. From $7^2 - 1 = 2^4 \cdot 3, 7^3 - 1 = 2 \cdot 3^2 \cdot 19$, we find the solutions $(2, 2), (2, 3), (19, 3)$.

Case 4: $p \nmid 20^p + 1, q \nmid 7^q - 1$. In this case, we have no solutions. Indeed, from $p|7^q - 1$ we deduce $q|p - 1$ and similarly, from $q|20^{2p} - 1$ we deduce $2p|q - 1$, therefore $q \leq p - 1$ and $2p \leq q - 1$, which is impossible.

Also solved by Albert Stadler, Herrliberg, Switzerland; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.

U503. Let $m < n$ be positive integers and let a and b be real numbers. It is known that for every positive real number c the polynomial

$$P_c(x) = bx^n - ax^m + a - b - c$$

has exactly m roots strictly inside the unit circle. Prove that the polynomial

$$Q(x) = mx^n + nx^m - m + n$$

has exactly $m - \gcd(m,n)$ roots lying strictly inside the unit circle.

Proposed by Navid Safaei, Sharif Institute of Technology, Tehran, Iran

Solution by the author

Assume that $z_1, ..., z_m$ are roots of $P_c(x)$ lying strictly inside the unit circle. As c tends to zero, it is possible that some of these roots lying on the unit circle. Now, we prove the following lemma.

Lemma: Let $b > a$, then the polynomial $bx^n - ax^m + a - b$ has exactly $\gcd(m,n)$ roots that lying on the unit circle.

Proof: Let $|z| = 1$ and $bz^n - az^m = b - a$, then bz^n is a point on the circle $|z| = b$, and az^m is a point on the circle az^m. Since the length of the segment between these points is $b - a$ and its argument is zero, we find that $m\arg(z) \equiv n\arg(z) \equiv 0 \pmod{2\pi}$. Hence, $\gcd(m,n)\arg(z) \equiv 0 \pmod{2\pi}$. That is, $z^{\gcd(m,n)} = 1$. On the other hand, if $z^{\gcd(m,n)} = 1$, then $bz^n - az^m = b - a$. This completes our proof.

Back to our problem, according to our problem, there are at most $\gcd(m,n)$ roots from $z_1, ..., z_m$ that could be lying on the unit circle. Therefore, at least $m - \gcd(m,n)$ roots inside the unit circle. Thus, it has at most $n - \gcd(n,m)$ roots outside the unit circle.

Now, consider the polynomial

$$x^n Q\left(\frac{1}{x}\right) = (n-m)x^n - nx^{n-m} + n - (n-m).$$

Again, it has at least $n - \gcd(n,m)$ roots strictly inside the unit circle. Hence, the polynomial $Q(x)$ has at least $n - \gcd(n,m)$ roots outside the unit circle. The equality case occurs! So, the polynomial $Q(x)$ has exactly $m - \gcd(m,n)$ roots inside the unit circle.

Remark: By Rouche's theorem, you can prove the following statement Let $n > m > 0$ be integers.
If $|B| > |A| + |C|$, then the polynomial $Az^n + Bz^m + C$ has exactly m zeros inside the unit circle.

U504. Evaluate

$$\int \frac{x^2+1}{(x^3+1)\sqrt{x}}dx$$

Proposed by Titu Andreescu, University of Texas a Dallas, USA

Solution by Alexandru Daniel Pirvuceanu, National "Traian" College, Drobeta-Turnu Severin, Romania
With the substitution $t=\sqrt{x}$, $dt=\frac{dx}{2\sqrt{x}}$, we have to compute

$$2\int \frac{t^4+1}{t^6+1}dt = 2\int \frac{t^4-t^2+1+t^2}{(t^2+1)(t^4-t^2+1)}dt = 2\int \frac{dt}{t^2+1} + 2\int \frac{t^2}{t^6+1}dt = 2\arctan t + \frac{2}{3}\arctan t^3 + C.$$

Hence,

$$\int \frac{x^2+1}{(x^3+1)\sqrt{x}}dx = 2\arctan\sqrt{x} + \frac{2}{3}\arctan(x\sqrt{x}) + C.$$

Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Suhas Sheikh, Indian Institute of Science, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Albert Stadler, Herrliberg, Switzerland; Adarsh Kumar, IIT Bombay, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; S.Chandrasekhar, Chennai, India; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania.

O499. For each positive integer d find the interval $I \subset \mathbb{R}$ of largest length such that for any choice of $a_0, a_1, \ldots, a_{2d-1} \in I$ the polynomial

$$x^{2d} + a_{2d-1}x^{2d-1} + \cdots + a_1 x + a_0$$

has no real root.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Daniel Lasaosa, Pamplona, Spain

Denote by $p(x)$ the proposed polynomial. Note that

$$\lim_{x\to\pm\infty} p(x) = +\infty,$$

or $p(x)$ has a real root iff there is a real r such that $p(r) \le 0$. Since $p(0) = a_0$ must be positive, I must contain only positive reals, or $p(r) > 0$ for all nonnegative real r, and for all negative real $-r$, we must have $p(-r) > 0$. Now, denoting respectively by $e(x)$ and $o(x)$ the even and odd parts of $p(x)$, for each negative real $-r$ we must have $p(-r) = e(r) - o(r) > 0$. Assume that $M > m > 0$ belong in I. Then, we may take $a_{2i+1} = M$ and $a_{2i} = m$ for $i = 0, 1, \ldots, d-1$, or

$$0 < p(-1) = e(1) - o(1) = 1 + dm - dM, \qquad M - m < \frac{1}{d},$$

and the length of I must clearly be at most $\frac{1}{d}$.

Consider $I = \left(1, 1 + \frac{1}{d}\right)$. For any negative real $-r$, we have $p(-r) = e(r) - o(r)$, where

$$e(r) > r^{2d} + r^{2d-2} + \cdots + 1, \qquad o(r) < \frac{d+1}{d}\left(r^{2d-1} + r^{2d-3} + \cdots + r\right),$$

and consequently

$$(r+1)p(-r) > r^{2d+1} + 1 - \frac{r^{2d} + r^{2d-1} + \cdots + r}{d},$$

or the problem reduces to proving that

$$d\left(r^{2d+1} + 1\right) \ge r^{2d-1} + r^{2d-3} + \cdots + r.$$

Now, for each positive integer $k = 1, 2, \ldots, 2d$, we have by the weighted AM-GM inequality that

$$\frac{kr^{2d+1}}{2d+1} + \frac{2d+1-k}{2d+1} \ge k^r,$$

with equality iff $r = 1$. Adding these $2d$ inequalities produces the desired result. It follows that for each positive integer d, the maximum length of I is $\frac{1}{d}$, and and example of such an I with length $\frac{1}{d}$ is $\left(1, 1 + \frac{1}{d}\right)$. Note that since at least one of the bounds, either for $e(r)$ or for $o(r)$, would still be strict, two other intervals which also satisfy the stated conditions and have length $\frac{1}{d}$ are $\left[1, 1 + \frac{1}{d}\right)$ and $\left(1, 1 + \frac{1}{d}\right]$, but not $\left[m, m + \frac{1}{d}\right]$ for any positive real m because -1 would be a root if all even coefficients are taken to be m and all odd coefficients are taken to be $m + \frac{1}{d}$.

Also solved by Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.

O500. In triangle ABC, $\angle A \le \angle B \le \angle C$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{7}{2} - \frac{r}{R}$$

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \le \frac{R}{r} + \frac{r}{R} + \frac{1}{2}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Titu Andreescu, USA and Albert Stadler, Switzerland

From the given conditions it follows that

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \le \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Hence,

$$2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \le \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3$$

and

$$2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \ge \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3$$

But (see solution to problem O489)

$$10 - \frac{2r}{R} \le (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge \frac{2R}{r} + \frac{2r}{R} + 4$$

and the conclusion follows.

Also solved by Ioannis D. Sfikas, Athens, Greece.

O501. Let x, y, z be real numbers such that $-1 \le x, y, z \le 1$ and $x + y + z + xyz = 0$. Prove that

$$x^2 + y^2 + z^2 + 1 \ge (x + y + z \pm 1)^2.$$

Proposed by Marius Stănean, Zalău, România

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that if x, y, z are either all nonnegative or all nonpositive, then $x + y + z + xyz$ is respectively nonnegative and nonpositive, being zero iff $x = y = z = 0$, which clearly results in equality in the proposed inequality. Note further that we may simultaneously invert the signs of x, y, z without altering the problem, or we may assume wlog that $x, y < 0 < z$, for $xyz > 0$ and $x + y + z < 0$. It then suffices to show that $xyz + xy + yz + zx \le 0$, or equivalently, denoting $u = -x$, $v = -y$ and since $z = -\frac{x+y}{1+xy} = \frac{u+v}{1+uv}$, it suffices to show that for all $0 \le u, v \le 1$, the following inequality holds:

$$(u + v)^2 \ge uv(1 + u)(1 + v), \qquad (u - v)^2 \ge uv(uv + u + v - 3),$$

which is clearly true since $0 \le uv, u, v \le 1$, or $uv + u + v \le 3$. The RHS is thus negative and the inequality holds strictly, unless either $uv = 0$ or $u = v = 1$. In the first case, the LHS is positive unless $u = v = 0$, and in the second case the LHS is clearly zero. The conclusion follows, and restoring generality, equality holds in the proposed inequality iff either $x = y = z = 0$, or (x, y, z) is a permutation of $(-1, -1, 1)$ and we choose the $-$ sign inside the squared bracket, or (x, y, z) is a permutation of $(-1, 1, 1)$ and we choose the $+$ sign inside the squared bracket.

Also solved by Albert Stadler, Herrliberg, Switzerland; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece.

O502. Let $ABCDE$ be a convex pentagon and let M be the midpoint of AE. Suppose that $\angle ABC + \angle CDE = 180^\circ$ and $AB \cdot CD = BC \cdot DE$. Prove that

$$\frac{BM}{DM} = \frac{AB \cdot CE}{DE \cdot AC}$$

Proposed by Khakimboy Egamberganov, ICTP, Trieste, Italy

Solution by the author

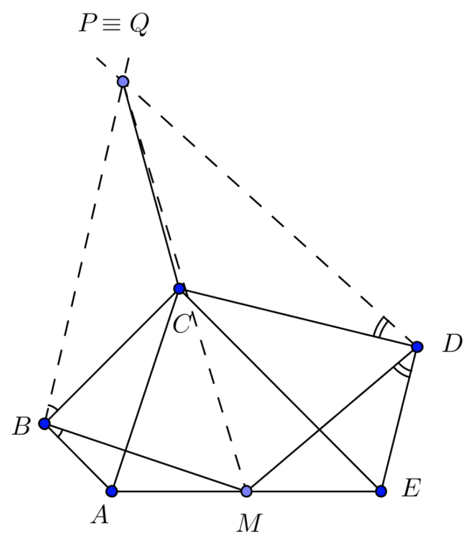
Let us define two transformations (homothetic transformations with rotation):

- A transformation $H_D^{k_1,\phi_1}$ (call it H_D for simplicity) with center D, the scalling coefficient is $k_1 = \frac{DC}{DE}$ and the rotation angle is $\phi_1 = \angle CDE$, clockwisely.
- A transformation $H_B^{k_2,\phi_2}$ (call it H_B for simplicity) with center D, the scalling coefficient is $k_2 = \frac{BA}{BC}$ and the rotation angle is $\phi_2 = \angle CBA$, clockwisely.

Now, if we take a transformation $H_O(X)$ with center O, coefficient k and angle ϕ clockwisely, it means that it sends a point X to some point Y (on the plane) and $OY = k \cdot OX, \angle XOY = \phi$ (the angle from X to Y clockwisely). Next, H_O^{-1} is a transformation with center O, the coefficient k^{-1} and the angle ϕ, counterclockwisely.

So, we have $k_1 \cdot k_2 = 1, \phi_1 + \phi_2 = 180^\circ$ and

$$H_D(E) = C(.), H_B^{-1}(A) = C(.) \Rightarrow H_B(H_D(E)) = A(.)$$



Suppose that $H_D(M) = P(.)$ and $H_B^{-1}(M) = Q(.)$. Then, we get $\triangle MED \longmapsto \triangle PCD$ by the transformation H_D and $\triangle MAB \longmapsto QCB$ by the transformation H_B^{-1}.. Therefore,

$$\angle PCD = \angle MED = \angle AED \text{ and } \angle QCB = \angle MAB = \angle EAB$$

and since $\angle EAB + \angle AED + \angle BCD = 540^\circ - (\angle ABC + \angle CDE) = 360^\circ$, we get that

$$\angle QCB + \angle PCD + \angle BCD = 360^\circ$$

Equivalently, we can say that the points C, Q, P are collinear. On the other hand, $AM = EM$ and $k_1 \cdot k_2 = 1$. By using the transformations H_D and H_B^{-1}, we can find that $CQ = CP$. Hence, $P \equiv Q$ (see the picture). Thus, $\triangle BMP \sim \triangle BAC$ and $\triangle DMP \sim \triangle DEC$,

$$PM = \frac{BM \cdot AC}{AB} \text{ and } PM = \frac{DM \cdot CE}{DE} \Rightarrow$$

$$\frac{BM}{DM} = \frac{AB \cdot CE}{DE \cdot AC}$$

Also solved by Taes Padhihary, Disha Delphi Public School, India.

O503. Prove that in any triangle ABC,

$$\left(\frac{a+b}{m_a+m_b}\right)^2+\left(\frac{b+c}{m_b+m_c}\right)^2+\left(\frac{c+a}{m_c+m_a}\right)^2\geq 4.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Since $m_am_b\leq\dfrac{2c^2+ab}{4}$ then $(m_a+m_b)^2=m_a^2+m_b^2+2m_am_b\leq$

$$\frac{2\left(b^2+c^2\right)-a^2}{4}+\frac{2\left(c^2+a^2\right)-b^2}{4}+2\cdot\frac{2c^2+ab}{4}=\frac{(a+b)^2+8c^2}{4}$$

and, therefore,

$$\sum\frac{(a+b)^2}{(m_a+m_b)^2}\geq\sum\frac{4\,(a+b)^2}{(a+b)^2+8c^2}.$$

By Cauchy Inequality

$$\sum\frac{(a+b)^2}{(a+b)^2+8c^2}=\sum\frac{(a+b)^4}{(a+b)^4+8c^2\,(a+b)^2}\geq\frac{\left(\sum(a+b)^2\right)^2}{\sum\left((a+b)^4+8c^2\,(a+b)^2\right)}$$

and we have

$\left(\sum(a+b)^2\right)^2-\sum\left((a+b)^4+8c^2\,(a+b)^2\right)=2\left(\sum(a+b)^2\,(c+a)^2-4\sum a^2\,(b+c)^2\right)=$

$$2\left(a^4+b^4+c^4+2a^3b+2ab^3+2b^3c+2bc^3+2a^3c+2ac^3-5a^2b^2-5b^2c^2-5c^2a^2\right)=$$

$$2\left(\left(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2\right)+2ab\,(a-b)^2+2ac\,(a-c)^2+2bc\,(b-c)^2\right)\geq 0.$$

Thus,

$$\left(\sum(a+b)^2\right)^2\geq\sum\left((a+b)^4+8c^2\,(a+b)^2\right)$$

and since $\sum\dfrac{(a+b)^2}{(a+b)^2+8c^2}\geq 1$, then $\sum\dfrac{(a+b)^2}{(m_a+m_b)^2}\geq 4$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Konstantina Rasvani,1st High School, Volos, Greece; Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

O504. Let G be a connected graph such that all degrees are at least 2 and there are no even cycles. Prove that G has a subgraph on all vertices such that the degree of each of them is 1 or 2. Prove that the conclusion doesn't hold if we drop the 'no even cycles' condition (A spanning subgraph of G is a subgraph which contains all vertices of G.)

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Solution by the author

We have to prove that we can select some edges such that each vertex is incident to 1 or 2 of them. Assume that we have chosen some edges such that k vertices are incident to 1 or 2 of them, and the rest to none. Pick v incident to none of the edges. We want to do some changes such that the k vertices remain incident to 1 or 2 of the edges (though not necessarily the same), while v will also be incident to 1 (and the remaining vertices to 0,1 or 2).

Assume this is not possible. We will call the vertices currently incident to i edges 'of type i' ($i = 0, 1, 2$). We can assume there is no selected edge between vertices of type 2 (otherwise, we can de-select it, and everything remains fine).

If v had a neighbour of types 0 or 1, we could select that edge - a contradiction. Hence it is incident to some v_1 of type 2. It follows that v_1 has a selected edge, say to v_2, which must be of type 1 (we assumed no edge is selected between vertices of type 2). Assume v_2 has a neighbour distinct from v_1, say v_3. If it is of type 0 or 1, we can de-select v_1v_2 and select vv_1 and v_2v_3, and make v of type 1, while keeping all other vertices fine - a contradiction. It follows that v_3 is of type 2. If it has a selected edge to a vertex not yet mentioned, say v_4, that will be of type 1.

Continuing this process, we get the path (in the original graph):

$$vv_1v_2\ldots v_r$$

such that exactly the following edges of the path are selected:

$$v_1v_2, v_3v_4, \ldots$$

and we have vertices of types:

$$2 : v_1, v_3, \ldots$$

$$1 : v_2, v_4, \ldots$$

We can assume that the path cannot be extended as we have done so far. We have two cases, based on the parity of r:

If $r = 2t + 1$, then v_r is of type 2. It then has a selected edge going to one of the vertices in the path. But all vertices of type 1 already have their selected edge in the path, so it will be a vertex of type 2 - a contradiction, as we assumed that no two vertices of type 2 are connected.

If $r = 2t$, v_r will be of type 1. It will have another edge (not selected) to a vertex in the path (here, we are using the fact that all degrees are at least 2). If it is to v_{2i+1}, some i, then $v_{2i+1}v_{2i+2}\ldots v_{2t}v_{2i+1}$ will be an even cycle - a contradiction. If it is to v_{2i}, some i, we can select $v_{2i}v_{2t}$, as both are of type 1, and change from selected to not selected and vice versa all edges

$$vv_1, v_1v_2, \ldots, v_{2i-1}v_{2i}$$

If we drop the 'no even cycle' condition, we can just pick the complete bipartite graph $K_{2,n}$ with $n \geq 5$, with vertex-sets V_1, V_2, $|V_1| = 2$,$|V_2| = n$. Assuming we could select edges as required, there would be one edge incident to each vertex of V_2, so at least n of them. But these will be incident to vertices in V_1, so one of them will have at least $n/2 > 2$ edges incident to it.