Junior problems

J499. Let $a, b, c, d$ be positive real numbers such that

\[ a(a - 1)^2 + b(b - 1)^2 + c(c - 1)^2 + d(d - 1)^2 = a + b + c + d. \]

Prove that

\[ (a - 1)^2 + (b - 1)^2 + (c - 1)^2 + (d - 1)^2 \leq 4. \]

Proposed by Adrian Andreescu, University of Texas at Austin, USA

First solution by Polyahebra, Polk State College, USA

The given condition is equivalent to $a^3 + b^3 + c^3 + d^3 = 2(a^2 + b^2 + c^2 + d^2)$ and the claimed inequality is equivalent to $a^2 + b^2 + c^2 + d^2 \leq 2(a + b + c + d)$. By the Cauchy-Schwarz inequality,

\[ (a^2 + b^2 + c^2 + d^2)^2 \leq (a^3 + b^3 + c^3 + d^3)(a + b + c + d) = 2(a^2 + b^2 + c^2 + d^2)(a + b + c + d), \]

and the proof is complete.

Second solution by Lilit Ghazanchyan, Vanadzor, Armenia

According to Sedrakyan-Engel-Titu inequality

\[ a + b + c + d = a(a - 1)^2 + b(b - 1)^2 + c(c - 1)^2 + d(d - 1)^2 = \]

\[ = \frac{(a(a - 1))^2}{a} + \frac{(b(b - 1))^2}{b} + \frac{(c(c - 1))^2}{c} + \frac{(d(d - 1))^2}{d} \geq \]

\[ \geq \frac{(a(a - 1) + b(b - 1) + c(c - 1) + d(d - 1))^2}{a + b + c + d} \]

from which we immediately infer the inequality

\[ (a + b + c + d)^2 \geq (a(a - 1) + b(b - 1) + c(c - 1) + d(d - 1))^2 \]

The latter gives

\[ a + b + c + d \geq a(a - 1) + b(b - 1) + c(c - 1) + d(d - 1), \]

which can be easily transformed into the desired inequality

\[ (a - 1)^2 + (b - 1)^2 + (c - 1)^2 + (d - 1)^2 \leq 4 \]

Also solved by Daniel Lasaosa, Pamplona, Spain; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ioan Viorel Codreanu, Satu-Mare, Maramures, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA.
J500. Let $a, b, c, d$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{5a^2 - 2a + 1} + \frac{1}{5b^2 - 2b + 1} + \frac{1}{5c^2 - 2c + 1} + \frac{1}{5d^2 - 2d + 1} \geq 1.$$ 

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Arkady Alt, San Jose, CA, USA

By replacing $(a, b, c, d)$ in the problem with $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right)$ we obtain equivalent setting of the original problem:

Prove that $\sum \frac{a^2}{a^2 - 2a + 5} \geq 1$ if $abcd = 1$.

By Cauchy Inequality

$$\sum_{cyc} \frac{a^2}{a^2 - 2a + 5} \geq \frac{(a + b + c + d)^2}{a^2 + b^2 + c^2 + d^2 - 2(a + b + c + d) + 20} \geq$$

$$(a + b + c + d)^2 \geq a^2 + b^2 + c^2 + d^2 - 2(a + b + c + d) + 20 \iff$$

$$ab + ac + ad + bc + bd + cd + a + b + c + d \geq 10$$

where latter inequality holds because by AM-GM Inequality $ab + ac + ad + bc + bd + cd \geq 6 \sqrt[6]{a^3b^3c^3d^3} = 6$ and $a + b + c + d \geq 4 \sqrt[4]{abcd} = 4$.

Also solved by Polyahedra, Polk State College, USA; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Mohamed Ali, Houari Boumedien School, Algeria; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Suhas Sheikh, Indian Institute of Science, India; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania.
J501. In a convex quadrilateral $ABCD$, $M$ and $N$ are the midpoints of diagonals $AC$ and $BD$, respectively. The intersection of the diagonals lies on segments $CM$ and $DN$, while points $P$ and $Q$ lie on segment $AB$ and satisfy

\[ \angle PMN = \angle BCD \quad \text{and} \quad \angle QNM = \angle ADC. \]

Prove that lines $PM$ and $QN$ meet at a point lying in line $CD$.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Polyahedra, Polk State College, USA

Suppose that $MN$ intersects $BC$, $CD$, and $DA$ at $J$, $K$, and $L$, respectively. Applying Menelaus’ theorem to triangles $ACD$ and $BCD$ with transversal $MN$, we get

\[
\frac{AL}{LD} \cdot \frac{KD}{KC} \cdot \frac{CM}{MA} = 1 \quad \text{and} \quad \frac{BJ}{JC} \cdot \frac{KC}{KD} \cdot \frac{DN}{NB} = 1.
\]

Hence, $AL/ LD = KC/KD = JC/BJ$, thus $AD/LD = BC/BJ$ as well. Applying Menelaus’ theorem to triangles $ALM$ and $BJN$ with transversal $CD$, we get

\[
\frac{AD}{LD} \cdot \frac{KL}{KM} \cdot \frac{MC}{AC} = 1 \quad \text{and} \quad \frac{BC}{JC} \cdot \frac{KJ}{KN} \cdot \frac{ND}{BD} = 1.
\]

Therefore, $KL/KM = 2LD/AD$ and $KJ/KN = 2JC/BC$, so

\[
\frac{KD}{KC} \cdot \frac{KM}{KL} \cdot \frac{KJ}{KN} = \frac{BJ}{JC} \cdot \frac{AD}{2LD} \cdot \frac{2JC}{BC} = 1.
\]

Now suppose that $PM$ intersects $CD$ at $E$. Then $C$, $E$, $M$, and $J$ are concyclic. By the power of a point, $KC \cdot KE = KM \cdot KJ$. Consequently, $KD \cdot KE = KD \cdot KM \cdot KJ/KC = KL \cdot KJ$, that is, $D$, $E$, $N$, and $L$ are concyclic as well. Hence, $E$, $N$, and $Q$ are collinear.

Also solved by Taes Padhihary, Disha Delphi Public School, India.
J502. Let $a, b, c$ be positive real numbers. Prove that
\[
\frac{a^3}{c(a^2 + bc)} + \frac{b^3}{a(b^2 + ca)} + \frac{c^3}{b(c^2 + ab)} \geq \frac{3}{2}.
\]

Proposed by Konstantinos Metaxas, Athens, Greece

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain
By changing the variables to $x = \frac{a}{c}$, $y = \frac{b}{a}$, $z = \frac{c}{b}$, the inequality reads as
\[
\frac{x^2}{(x + y)} + \frac{y^2}{(y + z)} + \frac{z^2}{(z + x)} \geq \frac{3}{2},
\]
where $xyz = 1$.

This inequality follows by the Cauchy-Schwarz inequality in Engel form and the AM-GM inequality:
\[
\frac{x^2}{(x + y)} + \frac{y^2}{(y + z)} + \frac{z^2}{(z + x)} \geq \frac{(x + y + z)^2}{2(x + y + z)} = \frac{2}{x + y + z} \geq \frac{3\sqrt[3]{xyz}}{2} = \frac{3}{2}.
\]

Also solved by Polyahedra, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Dumitru Barac, Sibiu, Romania; Adarsh Kumar, IIT Bombay, India; Taes Padhihary, Disha Delphi Public School, India; Ioan Viorol Coderanu, Satulung, Maramures, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece.
J503. Solve in positive integers the equation

\[ \min(x^4 + 8y, 8x + y^4) = (x + y)^2. \]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Since we may exchange \( x, y \) without altering the problem, we may assume wlog that \( 8x + y^4 = (x + y)^2 \), or \( (x + y - 4)^2 = y^4 - 8y + 16 \). Therefore, \( y^4 - 8y + 16 \) must be a perfect square. If \( y \geq 3 \), note that \( 2y^2 - 8y + 15 = 2(y-1)(y-3) + 9 > 0 \), or \( y^4 - 8y + 15 > (y^2 - 1)^2 \), whereas \( 8y - 16 > 0 \) for \( y^4 - 8y + 16 < (y^2)^2 \). Therefore, \( y^4 - 8y + 16 \) is a perfect square for positive integer \( y \) only when \( y \in \{1, 2\} \).

For \( y = 1 \), we have \( (x-3)^2 = 9 = 3^2 \), yielding either \( x = 0 \) which is not a positive integer, or \( x = 6 \), in which case \( x^4 + 8y > y^4 + 8x = 49 = (6 + 1)^2 \) is indeed a solution.

For \( y = 2 \), we have \( (x-2)^2 = 16 = 4^2 \), yielding either \( x = -2 \) which is not a positive integer, or \( x = 6 \), in which case \( x^4 + 8y > y^4 + 8x = 64 = (6 + 2)^2 \) is indeed a solution.

Restoring generality, all solutions are

\( (x,y) = (6,2), \quad (x,y) = (6,1), \quad (x,y) = (1,6), \quad (x,y) = (2,6). \)

Also solved by George Theodoropoulos, National Technical University of Athens, Athens, Greece; Polyhedra, Polk State College, USA; Dumitru Barac, Sibiu, Romania; Titu Žvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Ioannis D. Sfikas, Athens, Greece.
Let $ABC$ be a triangle with circumcenter $O$, $E$ be an arbitrary point on $AC$ and $F$ a point on $AB$ such that $B$ lies between $A$ and $F$. Let $K$ be the circumcenter of the triangle $AEF$. Denote by $D$ the intersection of lines $BC$ and $EF$, $M$ the intersection of lines $AK$ and $BC$, and $N$ the intersection of lines $AO$ and $EF$. Prove that points $A, D, M, N$ are concyclic.

Proposed by Mihai Micuțița, Oradea, România

First solution by Polyahedra, Polk State College, USA

Let $P$ be the second intersection point of the circumcircles of $ABC$ and $AEF$. By Miquel’s theorem, $P$ is on the circumcircles of $FBD$ and $ECD$ as well. Using directed angles and taking all equalities modulo $\pi$, we have

\[ \angle NDM = \angle FDC = \angle FDP + \angle PDC = \angle FBP + \angle PEC \]
\[ = -\angle PBC - \angle CBA - \angle FEP - \angle AEF = -\angle PAC - \frac{\angle COA}{2} - \angle FAP - \frac{\angle AKF}{2} \]
\[ = -\angle FAC - \left( \frac{\pi}{2} - \angle OAC \right) - \left( \frac{\pi}{2} - \angle FAK \right) = -\angle FAC + \angle OAC + \angle FAK = -\angle MAN, \]

thus $A, D, M, N$ are concyclic.
Second solution by Daniel Lasaosa, Pamplona, Spain

Note that

\[ \angle AND = 180^\circ - \angle ANE = \angle NAE + \angle AEN = \angle OAC + \angle AEF = 90^\circ + \angle AEF - \angle B. \]

Similarly,

\[ \angle AMD = 180^\circ - \angle AMB = \angle MBA + \angle MAB = \angle B + \angle KAF = \angle B + 90^\circ - \angle AEF. \]

Since \( \angle AND + \angle AMD = 180^\circ \), we conclude that \( AMDN \) is cyclic.

Note: The proof is conducted in the case where \( M, N \) are on opposite sides of line \( AD \). Certain choices of \( E, F \) may result on \( M, N \) being on the same side of \( AD \), in which case we may analogously prove that \( \angle AND = \angle AMD \), using that in this case angles \( \angle AND \) and \( \angle ANE \) are equal instead of adding up to \( 180^\circ \). Similarly, the proof relies on \( ABC \) and \( AEF \) being acute-angled triangles. If \( ABC \) is obtuse at \( B \), then \( O \) is on the opposite side of line \( AC \) with respect to \( B \), and \( E \) is inside segment \( DN \), or we have again \( \angle AND = \angle ANE \), but \( \angle NAE = \angle OAC = \angle B - 90^\circ \), and \( \angle AEN = 180^\circ - \angle AEF \), resulting once more in \( \angle AND = 90^\circ + \angle AEF - \angle B \). We may similarly treat the case where \( AEF \) is obtuse at \( E \).

Also solved by Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.
S499. Let $a$ and $b$ be distinct real numbers. Prove that $27ab\left(\sqrt[3]{a} + \sqrt[3]{b}\right)^3 = 1$ if and only if $27ab(a + b + 1) = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Let $\sqrt[3]{a} = x, \sqrt[3]{b} = y$. The problem may be restate as follows: Prove that $3xy(x + y) = 1$ if and only if $27x^3y^3(x^3 + y^3 + 1) = 1$. Now we have $27x^3y^3(x^3 + y^3 + 1) = 1$ is equivalent to one of the following

$$x^3 + y^3 + 1 = \frac{1}{(3xy)^3},$$

$$x^3 + y^3 + \left(\frac{-1}{3xy}\right)^3 - 3xy\left(\frac{-1}{3xy}\right) = 0,$$

$$x^3 + y^3 + z^3 - 3xyz = 0 \quad \text{(where } z = \frac{-1}{3xy}),$$

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0,$$

$$(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2] = 0.$$

Since $x \neq y$ so this equation is equivalent to

$$x + y + z = 0.$$

That is

$$x + y - \frac{1}{3xy} = 0.$$

This is exactly $3xy(x + y) = 1$. Therefore the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Albert stadler, Herrliberg, Switzerland; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Suhas Sheikh, Indian Institute of Science, India; Arkady Alt, San Jose, CA, USA.
S500. Let \( a, b, c \) be pairwise distinct real numbers. Prove that

\[
\left( \frac{a-b}{b-c} - 2 \right)^2 + \left( \frac{b-c}{c-a} - 2 \right)^2 + \left( \frac{c-a}{a-b} - 2 \right)^2 \geq 17.
\]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Albert Stadler, Herrliberg, Switzerland

Note that

\[
\left( \frac{a-b}{b-c} - 2 \right)^2 + \left( \frac{b-c}{c-a} - 2 \right)^2 + \left( \frac{c-a}{a-b} - 2 \right)^2 - 17 = \left( \frac{a^3 + b^3 + c^3 - 4a^2b - 4b^2c - 4c^2a + ab^2 + bc^2 + ca^2 + 6abc}{(b-c)(c-a)(a-b)} \right)^2 \geq 0.
\]

Second solution by Daniel Lasaosa, Pamplona, Spain

Since \( a - c \neq 0 \), the inequality is homogeneous in \( a, b, c \), and invariant under simultaneous exchange of the signs of \( a, b, c \), we may substitute \( a, b, c \) respectively by \( a - c, b - c \), and \( c - a \), so that \( a - b = 1 \), and we may define \( x = a - b \), and consequently \( b - c = 1 - x \), where clearly \( x \notin \{0, 1\} \). The inequality then rewrites as

\[
17 \leq \left( \frac{3x - 2}{1 - x} \right)^2 + (x - 3)^2 + \left( \frac{2x + 1}{x} \right)^2 = \frac{x^6 - 8x^5 + 35x^4 - 40x^3 + 10x^2 + 2x + 1}{(1 - x)^2 x^2},
\]

or equivalently after multiplying by the (clearly nonzero) denominator and rearranging terms,

\[
0 \leq x^6 - 8x^5 + 18x^4 - 6x^3 - 7x^2 + 2x + 1 = (x^3 - 4x^2 + x + 1)^2.
\]

The inequality therefore always holds. Note that equality occurs when \( x \) is one of the roots of \( x^3 - 4x^2 + x + 1 \), or equivalently, when \( y = 1 - x \) is one of the roots of \( y^3 + y^2 - 4y + 1 = 0 \). Denote therefore by \( r \) each one of the three roots of \( y^3 + y^2 - 4y + 1 = 0 \), and denoting by \( d \) the original value of \( a - c \), we find that \( a = c + d \), \( b = c + rd \), or equality follows if

\[
(a,b,c) = (c+d,c+rd,c),
\]

where \( c \) takes any real value, \( d \) takes any nonzero real value, and \( r \) is any one of the roots of \( y^3 + y^2 - 4x + 1 \).

Also solved by Ioannis D. Sfikas, Athens, Greece; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Titu Zvonaru, Comănești, Romania.
Solve the equation \( \lfloor x \rceil \{ 8x \} = 2x^2 \), where \( \lfloor a \rceil \) and \( \{ a \} \) are the greatest integers less than or equal to \( a \) and the fractional part of \( a \), respectively.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Alessandro Ventullo, Milan, Italy and Sarah B. Seales, Northern Arizona University, USA

Clearly, \( \lfloor x \rceil \geq 0 \). Let \( \lfloor x \rceil = n \) and \( \{ x \} = t \). The equation becomes

\[
2(n + t)^2 = \lfloor x \rceil \{ 8x \} \leq 8\lfloor x \rceil \{ x \} = 8nt,
\]

which gives \((n - t)^2 \leq 0\). It follows that \( n = t \). Since \( 0 \leq t < 1 \) and \( n \) is an integer, then \( n = 0 \) and so \( t = 0 \), i.e \( x = 0 \).

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Niciusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.
Find all positive integers $n$ such that

$$a + b + c | a^n + b^n + c^n - nabc$$

for all positive integers $a, b, c$.

Proposed by Oleg Muskarov, Sofia, Bulgaria

Solution by Dumitru Barac, Sibiu, Romania

We particularize $b = c = 1$, hence $a + 2 | a^n - na + 2$, for all positive integers $a$. From $a^n - na + 2 = (a^n - (-2)^n) - n(a + 2) + 2 + 2n + (-2)^n$, we deduce that $a + 2 | 2 + 2n + (-2)^n$, namely the number $2 + 2n + (-2)^n = 0$. We conclude that $n$ is an odd number, hence $2^n - 2n - 2 = 0, 2^{n-1} = n + 1$. Clearly, $n = 3$ is a solution and $n < 3$ is not. Indeed, for $n = 3$, we have

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca); (a + b + c).$$

We claim that $2^{n-1} > n + 1$, for all $n \geq 4$. The assertion is clearly true for $n = 4$. We suppose $2^{n-1} > n + 1$, and we deduce:

$$2^n = 2 \cdot 2^{n-1} > 2(n + 1) = 2n + 2 > n + 2.$$ 

Finally, only $n = 3$ is the solution of the problem.

Also solved by Albert Stadler, Herrliberg, Switzerland; Chakib Belgani and Mahmoud Ezzaki, Morocco; Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Taes Padhikary, Disha Delphi Public School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.
Solve in positive integers the equation
\[ 101x^3 - 2019xy + 101y^3 = 100. \]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain
Note first that \( 101( x^3 + y^3 - 20xy - 1) + xy + 1 = 0 \), or 101 divides \( xy + 1 \), and since \( xy > 0 \), we have \( xy + 1 \geq 101 \), for \( xy \geq 100 \). Moreover,
\[
1 \leq \frac{xy + 1}{101} = 1 + xy(20 - x - y) - (x + y)(x - y)^2 \leq 1 + xy(20 - x - y),
\]
or \( x + y \leq 20 \). But then the AM of \( x,y \) is at most 10, and their GM is at least 10, or by the AM-GM inequality we must have \( x = y = 10 \), which since it satisfies the proposed equation, it is clearly its only solution in positive integers.

Also solved by Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Dumitru Barac, Sibiu, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Titu Zvonaru, Comănești, Romania.
S504. Let $a \geq b \geq c \geq 0$ be real numbers such that $a + b + c = 3$. Prove that

$$ab^2 + bc^2 + ca^2 + \frac{3}{8}abc \leq \frac{27}{8}.$$  

*Proposed by An Zhenping, Xianyang Normal University, China*

Solution by Li Zhou, Polk State College, USA

By assumption, $a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2 = (a - b)(b - c)(a - c) \geq 0$. Therefore,

$$27 - 8(ab^2 + bc^2 + ca^2) - 3abc \geq (a + b + c)^3 - 4(ab^2 + bc^2 + ca^2 + a^2b + b^2c + c^2a) - 3abc$$

$$= a^3 + b^3 + c^3 - (ab^2 + bc^2 + ca^2 + a^2b + b^2c + c^2a) + 3abc$$

$$= a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b)$$

which is non-negative by Schur’s inequality.

*Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Kevin Soto Palacios Huarmey, Perú; Arkady Alt, San Jose, CA, USA.*
Undergraduate problems

U499. Let \( a, b, c \) be positive real numbers not greater than \( 2 \). The sequence \( (x_n)_{n \geq 0} \) is defined by \( x_0 = a \), \( x_1 = b \), \( x_2 = c \) and

\[
x_{n+1} = \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}
\]

for all \( n \geq 2 \). Prove that \( (x_n)_{n \geq 0} \) is convergent and find its limit.

Proposed by Mircea Becheanu, Canada and Nicolae Secelean, România

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that if \( 0 < x_{n-2}, x_{n-1}, x_n \leq 2 \), then \( 0 < x_{n+1} \leq \sqrt{2 + \sqrt{2 + 2}} = 2 \), or after trivial induction we have \( 0 < x_n \leq 2 \) for all \( n \geq 0 \). Note also that if \( x_n > 1 \), then \( x_{n+1} > \sqrt{x_n} > 1 \). Note next that if \( x_n > 2^{-2m} \) for some positive integer \( m \), then \( x_{n+1} > \sqrt{x_n} > 2^{-2m-1} \), or after trivial induction, \( x_{n+m} > \frac{1}{4} \) and \( x_{n+m} > \frac{1}{2} \), for \( x_{n+m+1} > \sqrt{\frac{1}{2} + \frac{1}{4}} = 1 \). Therefore, we eventually have \( x_n > 1 \) for all \( n \geq N \) for some integer \( N \).

Denote now \( \delta_n = 2 - x_n \), where clearly \( 0 \leq \delta_n < 2 \) for all positive integer \( n \), or for all \( n \geq N + 2 \) with \( N \) defined as above, we have

\[
\delta_{n+1} = 2 - x_{n+1} = \frac{4 - x_n - \sqrt{x_{n-1} + x_{n-2}}}{2 + \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}} \leq \frac{\delta_n}{2 + \sqrt{1 + \sqrt{2}}} + \frac{\delta_{n-1} + \delta_{n-2}}{(2 + \sqrt{1 + \sqrt{2}})(2 + \sqrt{2})} \leq \frac{6\delta_n + 2\delta_{n-1} + 2\delta_{n-2}}{21},
\]

where for proving the last inequality, it suffices to prove that \( \sqrt{1 + \sqrt{2}} > \frac{3}{2} \) and \( 2 + \sqrt{2} > 3 \). The latter is obvious from \( \sqrt{2} > 1 \), whereas the former is equivalent to \( \sqrt{2} > \frac{3}{2} \), also clearly true since \( 4\sqrt{2} = \sqrt{32} > \sqrt{25} = 5 \). It follows that \( 21\delta_{n+1} < 6\delta_n + 2\delta_{n-1} + 2\delta_{n-2} \), or

\[
21\delta_{n+1} + 8\delta_n + \frac{10\delta_{n-1}}{3} \leq \frac{2}{3} \left( 21\delta_n + 8\delta_{n-1} + \frac{10\delta_{n-2}}{3} \right) - \frac{2\delta_{n-2}}{9} \leq \frac{2}{3} \left( 21\delta_n + 8\delta_{n-1} + \frac{10\delta_{n-2}}{3} \right).
\]

Or, denoting \( \Delta_n = 21\delta_{n+1} + 8\delta_n + \frac{10\delta_{n-1}}{3} \), we have \( 0 \leq \Delta_n < \frac{2\Delta_n - 1}{3} \) for all \( n \geq N + 2 \), hence \( \lim_{n \to \infty} \Delta_n = 0 \), or since \( 0 \leq \delta_n \leq \frac{\Delta_n - 1}{21} \), we finally have \( \lim_{n \to \infty} \delta_n = 0 \), and \( \lim_{n \to \infty} x_n = 2 \).

Also solved by Albert Stadler, Herrliberg, Switzerland.
Evaluate

$$\lim_{n \to \infty} \tan \pi \sqrt{4n^2 + n}.$$  

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Joel Schlosberg, Bayside, NY, USA

For $r \in \mathbb{R}$ and $m \in \mathbb{N}$, arbitrarily large values

$$n = \frac{1}{2} \sqrt{\left(\frac{\pi m + \arctan r}{\pi}\right)^2 + \frac{1}{16} - \frac{1}{8}}$$

satisfy $\tan \pi \sqrt{4n^2 + n} = r$ if noninteger values of $n$ are allowed. To find a well-defined limit, assume that $n \in \mathbb{N}$.

For $x \in ((2n - \frac{1}{2})\pi, (2n + \frac{1}{2})\pi)$, tan $x$ is strictly increasing. Since

$$4n^2 + n < 4n^2 + n + \frac{1}{16} = (2n + 1/4)^2 < (2n + 1/2)^2,$$

$$\tan \pi \sqrt{4n^2 + n} < \tan(2n + 1/4) = \tan(\pi/4) = 1.$$  

For any $s \in (0, 1)$, if $n > \frac{s^2}{16(1-s)}$, then

$$4n^2 + n > 4n^2 + sn + \frac{s^2}{16} = (2n + s/4)^2 > (2n - 1/2)^2,$$

so

$$\tan \pi \sqrt{4n^2 + n} > \tan(2n + s/4) = \tan(\pi s/4).$$

Therefore,

$$\lim_{n \to \infty} \tan \pi \sqrt{4n^2 + n} \geq \tan(\pi s/4)$$

and so

$$\lim_{n \to \infty} \tan \pi \sqrt{4n^2 + n} \geq \lim_{s \to 1^-} \tan(\pi s/4) = \tan(\pi/4) = 1.$$  

By the squeeze theorem, $\lim_{n \to \infty} \tan \pi \sqrt{4n^2 + n} = 1$.

Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Ioannis D. Sfikas, Athens, Greece; Suhas Sheikh, Indian Institute of Science, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Afram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Goleșcu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; S. Chandrasekhar, Chennai, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA.
U501. Let \( a_1, a_2, \ldots, a_n \geq 1 \) be real numbers such that \( a_1 a_2 \cdots a_n = 2^n \). Prove that
\[
a_1 + \cdots + a_n - \frac{2}{a_1} - \cdots - \frac{2}{a_n} \geq n.
\]

Proposed by Marin Chirciu, Pitești, România

Solution by Daniel Lasasoa, Pamplona, Spain
Let \( g(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n \), and define the region \( \mathcal{R} \) in \( \mathbb{R}^n \) determined by \( g(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n = 2^n \) and \( x_1, x_2, \ldots, x_n \geq 1 \). The problem is therefore equivalent to the minimization of
\[
f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + a_n - \frac{2}{x_1} - \frac{2}{x_2} - \cdots - \frac{2}{x_n}
\]
in \( \mathcal{R} \), and finding that this minimum is at least \( n \). Note first that any point in the boundary of \( \mathcal{R} \) satisfies that \( x_i = 1 \) for some of the \( i \)’s in \( \{1, 2, \ldots, n\} \). We may then use Lagrange’s multiplier method, which ensures that a real constant \( \lambda \) exists such that at any given minimum, for each \( i \in \{1, 2, \ldots, n\} \) such that \( x_i \neq 1 \) at said minimum, we have
\[
1 - \frac{2}{x_i^2} = \frac{\partial f}{\partial x_i} = \lambda \frac{\partial g}{\partial x_i} = \frac{2^n \lambda}{x_i}, \quad 2^n \lambda = \frac{x_i^2 - 2}{x_i^2}.
\]
Therefore, if \( i \neq j \in \{1, 2, \ldots, n\} \) satisfy that \( x_i, x_j \neq 1 \), we must have
\[
\frac{x_i^2 - 2}{x_i} = \frac{x_j^2 - 2}{x_j}, \quad (x_i x_j + 2)(x_i - x_j) = 0,
\]
and since \( x_i x_j + 2 > 3 \), we have \( x_i = x_j \). Therefore, at any minimum an integer \( 1 \leq m \leq n \) exists such that \( m \) of the \( x_i \)’s have the same value \( k \) which is different than 1, and the remaining \( n - m \) take value 1. Or, the common value of the \( m \) \( x_i \)’s is \( k = 2^{\frac{n}{m}} \), and \( m = \frac{\ln n}{\ln k} \) for
\[
f(x_1, x_2, \ldots, x_n) \geq n \left( \frac{k \ln 2}{\ln k} - \frac{2 \ln 2}{k \ln k} - 1 + \frac{\ln 2}{\ln k} \right),
\]
and it suffices to show that for all \( k \geq 2 \), we have
\[
\frac{k}{\ln k} - \frac{2}{k \ln k} + \frac{1}{\ln k} \geq \frac{2}{\ln 2}.
\]
Therefore, defining
\[
h(k) = \frac{k^2 + k - 2}{k \ln k},
\]
the problem is equivalent to showing that \( h(k) \geq \frac{2}{\ln 2} \) for all \( k \geq 2 \). Now, note that \( h(k) \) is of the order of \( \frac{k}{\ln k} \) when \( k \) grows, which diverges, whereas any local minimum of \( h(k) \) must satisfy
\[
0 = h'(k) = (2k + 1)k \ln k - (\ln k + 1) (k^2 + k - 2) = \left( k^2 + 2 \right) \ln k - \left( k^2 + k - 2 \right).
\]
Now, it is well known that \( \ln 2 > \frac{2}{3} \), or \( \ln 4 > \frac{4}{3} \), whereas clearly \( \ln 3 > 1 \). Therefore, if \( k > 4 \), we have
\[
h'(k) > \frac{k^2 - 3k + 10}{3} = \frac{k + 10}{3} > 0.
\]
At the same time, if \( k \in (3, 4) \), we have \( h'(k) > 4 - k \geq 0 \). Finally, if \( k \in [2, 3] \), we have \( h'(2) > 6 \ln 2 - 4 > 0 \), while \( h''(k) = 2k \ln k + \frac{2}{k} - k - 1 > \frac{k + 2}{3} - 1 \geq \frac{1}{3} \), and since \( h''(k) > 0 \) in \([2, 3] \) and \( h''(2) > 0 \), then \( h'(k) > 0 \) also in \([2, 3] \). It follows that \( h(k) \) reaches its minimum when \( k = 2 \), and this minimum is indeed \( \frac{2}{\ln 2} \). The conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland; S.Chandrasekhar, Chennai, India; Corneliu Mănescu-Arramids, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Nicușor Zlotă, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Chakib Belgani and Mahmoud Ezzaki, Morocco.
U502. Find all pairs \((p, q)\) of primes such that \(pq\) divides 
\[
(20^p + 1)(7^q - 1).
\]

*Proposed by Alessandro Ventullo, Milan, Italy*

*Solution by Corneliu Mănescu-Avram, Ploiești, Romania*

**Case 1:** \( p | 20^p + 1, q | 7^q - 1 \). From the Fermat theorem we deduce that \( o \) divides \( 20 + 1 = 21 = 3 \cdot 7 \) and \( q \) divides \( 7 - 1 = 6 = 2 \cdot 3 \), whence the solutions \((3, 2), (3, 3), (7, 2), (7, 3)\).

**Case 2:** \( p | 20^p + 1, q \not| 7^q - 1 \).

As above, \( p = 3, 7 \). From \( 20^3 + 1 = 3^2 \cdot 7 \cdot 127, 20^7 + 1 = 3 \cdot 7^2 \cdot 827 \cdot 10529 \), we find the solutions \((3, 7), (3, 127), (7, 7), (7, 827), (7, 10529)\).

**Case 3:** \( p \not| 20^p + 1, q | 7^q - 1 \). We have \( q = 2, 3 \). From \( 7^2 - 1 = 2^4 \cdot 3, 7^3 - 1 = 2 \cdot 3^2 \cdot 19 \), we find the solutions \((2, 2), (2, 3), (19, 3)\).

**Case 4:** \( p \not| 20^p + 1, q \not| 7^q - 1 \). In this case, we have no solutions. Indeed, from \( p | 7^q - 1 \) we deduce \( q | p - 1 \) and similarly, from \( q | 20^2p - 1 \) we deduce \( 2p | q - 1 \), therefore \( q \leq p - 1 \) and \( 2p \leq q - 1 \), which is impossible.

*Also solved by Albert Stadler, Herrliberg, Switzerland; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.*
Let $m < n$ be positive integers and let $a$ and $b$ be real numbers. It is known that for every positive real number $c$ the polynomial

$$P_c(x) = bx^n - ax^m + a - b - c$$

has exactly $m$ roots strictly inside the unit circle. Prove that the polynomial

$$Q(x) = ma^n + nx^m - m + n$$

has exactly $m - \gcd(m, n)$ roots lying strictly inside the unit circle.

Proposed by Navid Safaei, Sharif Institute of Technology, Tehran, Iran

Solution by the author

Assume that $z_1, \ldots, z_m$ are roots of $P_c(x)$ lying strictly inside the unit circle. As $c$ tends to zero, it is possible that some of these roots lying on the unit circle. Now, we prove the following lemma.

**Lemma:** Let $b > a$, then the polynomial $bx^n - ax^m + a - b$ has exactly $\gcd(m, n)$ roots that lying on the unit circle.

**Proof:** Let $|z| = 1$ and $bz^n - az^m = b - a$, then $bz^n$ is a point on the circle $|z| = b$, and $az^m$ is a point on the circle $az^m$. Since the length of the segment between these points is $b - a$ and its argument is zero, we find that $m \arg(z) \equiv n \arg(z) \equiv 0 \pmod{2\pi}$. Hence, $\gcd(m, n) \arg(z) \equiv 0 \pmod{2\pi}$. That is, $z^{\gcd(m, n)} = 1$. On the other hand, if $z^{\gcd(m, n)} = 1$, then $bz^n - az^m = b - a$. This completes our proof.

Back to our problem, according to our problem, there are at most $\gcd(m, n)$ roots from $z_1, \ldots, z_m$ that could be lying on the unit circle. Therefore, at least $m - \gcd(m, n)$ roots inside the unit circle. Thus, it has at most $n - \gcd(n, m)$ roots outside the unit circle.

Now, consider the polynomial

$$x^n Q\left(\frac{1}{x}\right) = (n - m)x^n - nx^{n-m} + n - (n - m).$$

Again, it has at least $n - \gcd(n, m)$ roots strictly inside the unit circle. Hence, the polynomial $Q(x)$ has at least $n - \gcd(n, m)$ roots outside the unit circle. The equality case occurs! So, the polynomial $Q(x)$ has exactly $m - \gcd(m, n)$ roots inside the unit circle.

**Remark:** By Rouche’s theorem, you can prove the following statement Let $n > m > 0$ be integers. If $|B| > |A| + |C|$, then the polynomial $Az^n + Bz^m + C$ has exactly $m$ zeros inside the unit circle.
U504. Evaluate

\[
\int \frac{x^2 + 1}{(x^3 + 1) \sqrt{x}} \, dx
\]

Proposed by Titu Andreescu, University of Texas a Dallas, USA

Solution by Alexandru Daniel Pirvuceanu, National "Traian" College, Drobeta-Turnu Severin, Romania

With the substitution \( t = \sqrt{x}, \, dt = \frac{dx}{2 \sqrt{x}} \), we have to compute

\[
2 \int \frac{t^4 + 1}{t^6 + 1} \, dt = 2 \int \frac{t^4 - t^2 + 1 + t^2}{(t^2 + 1)(t^4 - t^2 + 1)} \, dt = 2 \int \frac{dt}{t^2 + 1} + 2 \int \frac{t^2}{t^6 + 1} \, dt = 2 \arctan t + \frac{2}{3} \arctan t^3 + C.
\]

Hence,

\[
\int \frac{x^2 + 1}{(x^3 + 1) \sqrt{x}} \, dx = 2 \arctan \sqrt{x} + \frac{2}{3} \arctan(x \sqrt{x}) + C.
\]

Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel La- saosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Suhas Sheikh, Indian Institute of Science, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Albert Stadler, Herrliberg, Switzerland; Adarsh Kumar, IIT Bombay, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Dumitru Bar- rac, Sibiu, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nguyen Viet Hung, Hanoi University of Science, Viet- nam; S.Chandrasekhar, Chennai, India; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania.
O499. For each positive integer $d$ find the interval $I \subset \mathbb{R}$ of largest length such that for any choice of $a_0, a_1, \ldots, a_{2d-1} \in I$ the polynomial

$$x^{2d} + a_{2d-1}x^{2d-1} + \cdots + a_1x + a_0$$

has no real root.

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Denote by $p(x)$ the proposed polynomial. Note that

$$\lim_{x \to \pm\infty} p(x) = +\infty,$$

or $p(x)$ has a real root iff there is a real $r$ such that $p(r) \leq 0$. Since $p(0) = a_0$ must be positive, $I$ must contain only positive reals, or $p(r) > 0$ for all nonnegative real $r$, and for all negative real $-r$, we must have $p(-r) > 0$. Now, denoting respectively by $e(x)$ and $o(x)$ the even and odd parts of $p(x)$, for each negative real $-r$ we must have $p(-r) = e(r) - o(r) > 0$. Assume that $M > m > 0$ belong in $I$. Then, we may take $a_{2i+1} = M$ and $a_{2i} = m$ for $i = 0, 1, \ldots, d - 1$, or

$$0 < p(-1) = e(1) - o(1) = 1 + dm - dM, \quad M - m < \frac{1}{d},$$

and the length of $I$ must clearly be at most $\frac{1}{d}$.

Consider $I = (1, 1 + \frac{1}{d})$. For any negative real $-r$, we have $p(-r) = e(r) - o(r)$, where

$$e(r) > r^{2d} + r^{2d-2} + \cdots + 1, \quad o(r) < \frac{d+1}{d} \left( r^{2d-1} + r^{2d-3} + \cdots + r \right),$$

and consequently

$$(r + 1)p(-r) > r^{2d+1} + 1 - \frac{r^{2d} + r^{2d-1} + \cdots + r}{d},$$

or the problem reduces to proving that

$$d \left( r^{2d+1} + 1 \right) \geq r^{2d-1} + r^{2d-3} + \cdots + r.$$ 

Now, for each positive integer $k = 1, 2, \ldots, 2d$, we have by the weighted AM-GM inequality that

$$\frac{kr^{2d+1}}{2d+1} \geq k^r,$$

with equality iff $r = 1$. Adding these $2d$ inequalities produces the desired result. It follows that for each positive integer $d$, the maximum length of $I$ is $\frac{1}{d}$, and and example of such an $I$ with length $\frac{1}{d}$ is $(1, 1 + \frac{1}{d})$. Note that since at least one of the bounds, either for $e(r)$ or for $o(r)$, would still be strict, two other intervals which also satisfy the stated conditions and have length $\frac{1}{d}$ are $[1, 1 + \frac{1}{d}]$ and $(1, 1 + \frac{1}{d}]$, but not $[m, m + \frac{1}{d}]$ for any positive real $m$ because $1$ would be a root if all even coefficients are taken to be $m$ and all odd coefficients are taken to be $m + \frac{1}{d}$.

*Also solved by Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.*
O500. In triangle $ABC$, $\angle A \leq \angle B \leq \angle C$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{7}{2} - \frac{r}{R}$$

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \leq \frac{R}{r} + \frac{r}{R} + \frac{1}{2}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Titu Andreescu, USA and Albert Stadler, Switzerland

From the given conditions it follows that

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \leq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

Hence,

$$2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \leq \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3$$

and

$$2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3$$

But (see solution to problem O489)

$$10 - \frac{2r}{R} \leq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{2R}{r} + \frac{2r}{R} + 4$$

and the conclusion follows.

Also solved by Ioannis D. Sfikas, Athens, Greece.
O501. Let \( x, y, z \) be real numbers such that \(-1 \leq x, y, z \leq 1\) and \( x + y + z + xyz = 0 \). Prove that
\[
x^2 + y^2 + z^2 + 1 \geq (x + y + z \pm 1)^2.
\]

*Proposed by Marius Stănean, Zalău, România*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note first that if \( x, y, z \) are either all nonnegative or all nonpositive, then \( x + y + z + xyz \) is respectively nonnegative and nonpositive, being zero iff \( x = y = z = 0 \), which clearly results in equality in the proposed inequality. Note further that we may simultaneously invert the signs of \( x, y, z \) without altering the problem, or we may assume wlog that \( x, y < 0 < z \), for \( xyz > 0 \) and \( x + y + z < 0 \). It then suffices to show that \( xyz + xy + yz + zx \leq 0 \), or equivalently, denoting \( u = -x \), \( v = -y \) and since \( z = -\frac{x+y}{1+xy} = \frac{u+v}{1+uv} \), it suffices to show that for all \( 0 \leq u, v \leq 1 \), the following inequality holds:
\[
(u + v)^2 \geq uv(1 + u)(1 + v),
\]
\[
(u - v)^2 \geq uv(uv + u + v - 3),
\]
which is clearly true since \( 0 \leq uv, u, v \leq 1 \), or \( uv + u + v \leq 3 \). The RHS is thus negative and the inequality holds strictly, unless either \( uv = 0 \) or \( u = v = 1 \). In the first case, the LHS is positive unless \( u = v = 0 \), and in the second case the LHS is clearly zero. The conclusion follows, and restoring generality, equality holds in the proposed inequality iff either \( x = y = z = 0 \), or \((x, y, z)\) is a permutation of \((-1, -1, 1)\) and we choose the \(-\) sign inside the squared bracket, or \((x, y, z)\) is a permutation of \((-1, 1, 1)\) and we choose the \(+\) sign inside the squared bracket.

*Also solved by Albert Stadler, Herrliberg, Switzerland; Pooya Esmail Akhondy, Atomic Energy High School, Tehran, Iran; Kevin Soto Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece.*
O502. Let $ABCDE$ be a convex pentagon and let $M$ be the midpoint of $AE$. Suppose that 
$\angle ABC + \angle CDE = 180^\circ$ and $AB \cdot CD = BC \cdot DE$. Prove that

$$BM = \frac{AB \cdot CE}{DE \cdot AC}$$

$$DM = \frac{DE \cdot AC}{AB}$$

Proposed by Khakimboy Egamberganov, ICTP, Trieste, Italy

Solution by the author

Let us define two transformations (homothetic transformations with rotation):

- A transformation $H_{D}^{k_1,\phi_1}$ (call it $H_D$ for simplicity) with center $D$, the scaling coefficient is $k_1 = \frac{DC}{DE}$ and the rotation angle is $\phi_1 = \angle CDE$, clockwise.
- A transformation $H_{B}^{k_2,\phi_2}$ (call it $H_B$ for simplicity) with center $D$, the scaling coefficient is $k_2 = \frac{BA}{BC}$ and the rotation angle is $\phi_2 = \angle CBA$, clockwise.

Now, if we take a transformation $H_{O}^{k,\phi}$ with center $O$, coefficient $k$ and angle $\phi$ clockwise, it means that it sends a point $X$ to some point $Y$ (on the plane) and $OY = k \cdotOX$, $\angle XOY = \phi$ (the angle from $X$ to $Y$ clockwise). Next, $H_{O}^{-1}$ is a transformation with center $O$, the coefficient $k^{-1}$ and the angle $\phi$, counterclockwisely.

So, we have $k_1 \cdot k_2 = 1, \phi_1 + \phi_2 = 180^\circ$ and

$$H_{D}(E) = C(\cdot),H_{B}^{-1}(A) = C(\cdot) \Rightarrow H_{B}(H_{D}(E)) = A(\cdot)$$

Suppose that $H_{D}(M) = P(\cdot)$ and $H_{B}^{-1}(M) = Q(\cdot)$. Then, we get $\triangle MED \mapsto \triangle PCD$ by the transformation $H_{D}$ and $\triangle MAB \mapsto \triangle QCB$ by the transformation $H_{B}^{-1}$. Therefore,

$$\angle PCD = \angle MED = \angle AED \text{ and } \angle QCB = \angle MAB = \angle EAB$$

and since $\angle EAB + \angle AED + \angle BCD = 540^\circ - (\angle ABC + \angle CDE) = 360^\circ$, we get that

$$\angle QCB + \angle PCD + \angle BCD = 360^\circ$$

Equivalently, we can say that the points $C, Q, P$ are collinear. On the other hand, $AM = EM$ and $k_1 \cdot k_2 = 1$. By using the transformations $H_{D}$ and $H_{B}^{-1}$, we can find that $CQ = CP$. Hence, $P = Q$ (see the picture). Thus, $\triangle BMP \sim \triangle BAC$ and $\triangle DMP \sim \triangle DEC$,

$$PM = \frac{BM \cdot AC}{AB} \text{ and } PM = \frac{DM \cdot CE}{DE} \Rightarrow$$

$$BM = \frac{AB \cdot CE}{DE \cdot AC}$$

Also solved by Taes Padhihary, Disha Delphi Public School, India.
O503. Prove that in any triangle $ABC$,

$$\left(\frac{a+b}{m_a+m_b}\right)^2 + \left(\frac{b+c}{m_b+m_c}\right)^2 + \left(\frac{c+a}{m_c+m_a}\right)^2 \geq 4.$$  

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**Solution by Arkady Alt, San Jose, CA, USA**

Since $m_am_b \leq \frac{2c^2+ab}{4}$ then $(m_a+m_b)^2 = m_a^2 + m_b^2 + 2m_am_b \leq$

$$\frac{2(b^2+c^2) - a^2}{4} + 2\left(\frac{c^2+a^2}4\right) - b^2 + 2\frac{2c^2+ab}{4} = \frac{(a+b)^2 + 8c^2}{4}$$

and, therefore,

$$\sum \frac{(a+b)^2}{(m_a+m_b)^2} \geq \sum \frac{4(a+b)^2}{(a+b)^2 + 8c^2}.$$  

By Cauchy Inequality

$$\sum \frac{(a+b)^2}{(a+b)^2 + 8c^2} = \sum \frac{(a+b)^4}{(a+b)^4 + 8c^2(a+b)^2} \geq \frac{(\sum (a+b)^2)^2}{\sum (a+b)^4 + 8c^2(a+b)^2}$$

and we have

$$\left(\sum (a+b)^2\right)^2 - \sum \left((a+b)^4 + 8c^2(a+b)^2\right) = 2\left(\sum (a+b)^2(c+a)^2 - 4\sum a^2(b+c)^2\right) =$$

$$2\left(a^4 + b^4 + c^4 + 2a^3b + 2ab^3 + 2b^3c + 2bc^3 + 2a^3c + 2ac^3 - 5a^2b^2 - 5b^2c^2 - 5c^2a^2\right) =$$

$$2\left((a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2) + 2ab(a-b)^2 + 2ac(a-c)^2 + 2bc(b-c)^2\right) \geq 0.$$  

Thus,

$$\left(\sum (a+b)^2\right)^2 \geq \sum \left((a+b)^4 + 8c^2(a+b)^2\right)$$

and since $\sum \frac{(a+b)^2}{(a+b)^2 + 8c^2} \geq 1$, then $\sum \frac{(a+b)^2}{(m_a+m_b)^2} \geq 4.$

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O504. Let \( G \) be a connected graph such that all degrees are at least 2 and there are no even cycles. Prove that \( G \) has a subgraph on all vertices such that the degree of each of them is 1 or 2. Prove that the conclusion doesn’t hold if we drop the ‘no even cycles’ condition (A spanning subgraph of \( G \) is a subgraph which contains all vertices of \( G \)).

**Solution by the author**

We have to prove that we can select some edges such that each vertex is incident to 1 or 2 of them. Assume that we have chosen some edges such that \( k \) vertices are incident to 1 or 2 of them, and the rest to none. Pick \( v \) incident to none of the edges. We want to do some changes such that the \( k \) vertices remain incident to 1 or 2 of the edges (though not necessarily the same), while \( v \) will also be incident to 1 (and the remaining vertices to 0,1 or 2).

Assume this is not possible. We will call the vertices currently incident to \( i \) edges ‘of type \( i \)’ (\( i = 0,1,2 \)). We can assume there is no selected edge between vertices of type 2 (otherwise, we can de-select it, and everything remains fine).

If \( v \) had a neighbour of types 0 or 1, we could select that edge - a contradiction. Hence it is incident to some \( v_1 \) of type 2. It follows that \( v_1 \) has a selected edge, say to \( v_2 \), which must be of type 1 (we assumed no edge is selected between vertices of type 2). Assume \( v_2 \) has a neighbour distinct from \( v_1 \), say \( v_3 \). If it is of type 0 or 1, we can de-select \( v_1 v_2 \) and select \( vv_1 \) and \( v_2 v_3 \), and make \( v \) of type 1, while keeping all other vertices fine - a contradiction. It follows that \( v_3 \) is of type 2. If it has a selected edge to a vertex not yet mentioned, say \( v_4 \), that will be of type 1.

Continuing this process, we get the path (in the original graph):

\[
v v_1 v_2 \ldots v_r
\]

such that exactly the following edges of the path are selected:

\[
v_1 v_2, v_3 v_4, \ldots
\]

and we have vertices of types:

\[
2 : v_1, v_3, \ldots
\]

\[
1 : v_2, v_4, \ldots
\]

We can assume that the path cannot be extended as we have done so far. We have two cases, based on the parity of \( r \):

If \( r = 2t + 1 \), then \( v_r \) is of type 2. It then has a selected edge going to one of the vertices in the path. But all vertices of type 1 already have their selected edge in the path, so it will be a vertex of type 2 - a contradiction, as we assumed that no two vertices of type 2 are connected.

If \( r = 2t \), \( v_r \) will be of type 1. It will have another edge (not selected) to a vertex in the path (here, we are using the fact that all degrees are at least 2). If it is to \( v_{2t+1} \), some \( i \), then \( v_{2i+1} v_{2i+2} \ldots v_{2t} v_{2t+1} \) will be an even cycle - a contradiction. If it is to \( v_{2i} \), some \( i \), we can select \( v_{2i} v_{2t} \), as both are of type 1, and change from selected to not selected and vice versa all edges

\[
v v_1, v_1 v_2, \ldots, v_{2i-1} v_{2i}
\]

If we drop the 'no even cycle' condition, we can just pick the complete bipartite graph \( K_{2,n} \) with \( n \geq 5 \), with vertex-sets \( V_1, V_2, |V_1| = 2, |V_2| = n \). Assuming we could select edges as required, there would be one edge incident to each vertex of \( V_2 \), so at least \( n \) of them. But these will be incident to vertices in \( V_1 \), so one of them will have at least \( n/2 > 2 \) edges incident to it.