Junior problems

J505. Solve the equation x

$$2x^3 + x\{x\} + 2\{x\}^3 = \frac{1}{108},$$

where $\{x\}$ denotes the fractional part of x.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Polyahedra, Polk State College, USA
Clearly, $x = 0, \pm 1$ are not solutions.

If $x > 1$, then $2x^3 + x\{x\} + 2\{x\}^3 > 2$; if $x < -1$, then $2x^3 + x\{x\} + 2\{x\}^3 < 0$.

If $0 < x < 1$, then $\{x\} = x$ and the equation becomes $\frac{1}{108}(12x - 1)(6x + 1)^2 = 0$, so $x = \frac{1}{12}$.

If $-1 < x < 0$, then $\{x\} = x + 1$ and the equation becomes $\frac{1}{108}(12x + 5)(36x^2 + 48x + 43) = 0$, so $x = -\frac{5}{12}$.

In conclusion, $\frac{1}{12}$ and $-\frac{5}{12}$ are the only solutions.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Titu Zvonaru, Comănești, Romania.

J506. Prove that any integer $n > 6$ can be written as $n = p + m$, where p is a prime less than $\frac{n}{2}$ and p does not divide m.

Proposed by Li Zhou, Polk State College, USA

First solution by Daniel Lasaosa, Pamplona, Spain

If n is not a multiple of 3, let p be the largest prime which is smaller than $\frac{n}{2}$, and let $m = n - p$. If p divides m, then p divides n. But $p < \frac{n}{2}$ and $p \neq \frac{n}{3}$, or $p \leq \frac{n}{4}$, absurd since by Bertrand's postulate, there exists at least one prime in $\left(\frac{n}{4}, \frac{n}{2}\right)$.

If n is a multiple of 3, let p be the largest prime which is smaller than $\frac{n}{3}$, and let $m = n - p$. If p divides m, then p divides n. But $p < \frac{n}{3}$ and by Bertrand's postulate $p > \frac{n}{6}$. Therefore, such a p exists and is coprime with $m = n - p$, which solves the problem, except when $n = 4p$ or when $n = 5p$. But in these two cases, and since 3 divides n and is coprime with $4, 5$, we must have $p = 3$, for either $n = 12$ or $n = 15$. But $12 = 5 + 7$ with 5 prime and $15 = 7 + 8$ with 7 prime. The conclusion follows.

Second solution by Polyahedra, Polk State College, USA

If n is odd, then $n \geq 7$ and we can take $p = 2$ and $m = n - 2$. Suppose that n is even. If 4 divides n, then $n \geq 8$, so $\frac{n}{2} - 1 \geq 3$ is odd and has an odd prime factor p. Since p does not divide n, p does not divide $m = n - p$. If 4 does not divide n, then $n \geq 10$, so $\frac{n}{2} - 2 \geq 3$ is odd and has an odd prime factor p. Again, since p does not divide n, p does not divide $m = n - p$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Joe Simons, Utah Valley University, Orem, UT, USA; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Michail Prousalidis, Evangeliki Model Junior High School of Smyrna, Athens, Greece; Mohamed Ali, Houari Boumedien School, Algeria.

J507. Consider a real number a,

$$b = \left(a^2 + 2a + 2\right)\left(a^2 - (1-\sqrt{3})a + 2\right)\left(a^2 + (1+\sqrt{3})a + 2\right)$$

and

$$c = \left(a^2 - 2a + 2\right)\left(a^2 + (1-\sqrt{3})a + 2\right)\left(a^2 - (1+\sqrt{3})a + 2\right)$$

Find a knowing that $b + c = 16$.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

First solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan

Expanding $b + c$, we obtain

$$\begin{aligned} & b + c = 16 \\ \Leftrightarrow\ & a^2(a^4 + 4(2+\sqrt{3})a^2 + 8(2+\sqrt{3})) = 0. \end{aligned}$$

Since $a^4 + 4(2+\sqrt{3})a^2 + 8(2+\sqrt{3}) > 0$, the solution is $a = 0$.

Second solution by Polyahedra, Polk State College, USA

Notice that

$$b, c > 0 \text{ and } a^{12} + 64 = bc \le \left(\frac{b+c}{2}\right)^2 = 64,$$

so $a = 0$.

Also solved by Arkady Alt, San Jose, CA, USA; Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Michail Prousalidis, Evangeliki Model Junior High School of Smyrna, Athens, Greece; Mohamed Ali, Houari Boumedien School, Algeria; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlosberg, Bayside, NY, USA; David E. Manes, Oneonta, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Daniel López-Aguayo, MSCI, Monterrey, Mexico; Titu Zvonaru, Comănești, Romania.

J508. Let a, b, c be positive numbers such that $a + b + c + 2 = abc$. Prove that

$$(1 + ab)(1 + bc)(1 + ca) \geq 125.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the given condition and the AM-GM inequality we have

$$abc = a + b + c + 2 \geq 4\sqrt[4]{2abc}.$$

It follows that

$$abc \geq 8.$$

Now we use again the AM-GM inequality to obtain

$$1 + ab = 1 + \frac{ab}{4} + \frac{ab}{4} + \frac{ab}{4} + \frac{ab}{4} \geq 5\sqrt[5]{\left(\frac{ab}{4}\right)^4}.$$

Similarly,

$$1 + bc \geq 5\sqrt[5]{\left(\frac{bc}{4}\right)^4},$$

$$1 + ca \geq 5\sqrt[5]{\left(\frac{ca}{4}\right)^4}.$$

Which yields to

$$(1 + ab)(1 + bc)(1 + ca) \geq 125\sqrt[5]{\left(\frac{abc}{8}\right)^8} \geq 125.$$

The equality holds for $a = b = c = 2$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Mohamed Ali, Houari Boumedien School, Algeria; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; An Nguyen Huu Bui, High School For The Gifted, VNUHCM, Ho Chi Minh City, Vietnam; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Titu Zvonaru, Comănești, Romania.

J509. Find the least 4-digit prime of the form $6k-1$ that divides $8^{1010}11^{2020}+1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Polyahedra, Polk State College, USA

It is easy to check that 1013 is the least 4-digit prime of the form $6k-1$. Also, $\left(8^2 11^4\right)^{505}+1$ is divisible by $8^2 11^4+1=5^2\cdot 37\cdot 1013$.

Second solution by Joel Schlosberg, Bayside, NY, USA

For any such prime p,

$$(8^{505}11^{1010})^2 \equiv -1 \pmod{p}.$$

so -1 is a quadratic residue of p, which is well-known to necessitate that $p\equiv 1 \pmod 4$. Since $p\equiv -1 \pmod 6$, $p\equiv 5 \pmod{12}$.

Since $1001=83\cdot 12+5=7\cdot 11\cdot 13$ is composite, the smallest possible value of p is $1001+12=1013$. By any of the standard primaility tests (or simply testing that none of the primes $2,3,5,\ldots,23,29,31=\lfloor\sqrt{1013}\rfloor$ are divisors), 1013 is prime; since

$$8^2 11^4 = 925\cdot 1013-1,$$

$$8^{1010}11^{2020}=(8^2 11^4)^{505}\equiv(-1)^{505}=-1 \pmod{1013}$$

so the smallest 4-digit prime $\equiv -1 \pmod 6$ dividing $8^{1010}11^{2020}+1$ is 1013.

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; David E. Manes, Oneonta, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania.

J510. Let a, b, c be positive real numbers. Prove that

$$(1+a)(1+b)(1+c) \geq \left(1+\frac{2ab}{a+b}\right)\left(1+\frac{2bc}{b+c}\right)\left(1+\frac{2ca}{c+a}\right)$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that by the AM-GM inequality, $\frac{a+b}{2} \geq \sqrt{ab}$ with equality iff $a = b$, and similarly for the cyclic permutations of a, b, c, or

$$\left(1+\frac{2ab}{a+b}\right)\left(1+\frac{2bc}{b+c}\right)\left(1+\frac{2ca}{c+a}\right) \leq \left(1+\sqrt{ab}\right)\left(1+\sqrt{bc}\right)\left(1+\sqrt{ca}\right),$$

with equality iff $a = b = c$, and it suffices to show that

$$a+b+c+ab+bc+ca \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}+a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}.$$

Now, using again that $\frac{a+b}{2} \geq \sqrt{ab}$ and its cyclic permutations, we clearly have $a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}$ with equality iff $a = b = c$, and noting that $\frac{ca+ab}{2} \geq a\sqrt{bc}$ and its cyclic permutations, we also have $ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$, with equality iff $ab = bc = ca$, ie iff $a = b = c$. The conclusion follows, equality holds iff $a = b = c$.

Also solved by Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Mohamed Ali, Houari Boumedien School, Algeria; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Idamia Abdelhamid, Groupe Scolaire Berrada, Casablanca, Morocco; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Mihály Bencze, Braşov, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Shashwata Roy, Mumbai, India; Kevin Soto, Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania.

Senior problems

S505. Find k such that a triangle with sides a, b, c is right if and only if

$$\sqrt[6]{a^6+b^6+c^6+3a^2b^2c^2} = k\max\{a,b,c\}$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan
Assume that $c^2 = a^2+b^2$, then

$$\begin{aligned}&\sqrt[6]{a^6+b^6+c^6+3a^2b^2c^2} = k\max\{a,b,c\}\\ &\Leftrightarrow a^6+b^6+(a^2+b^2)^3+3a^2b^2(a^2+b^2) = k^6(a^2+b^2)^3\\ &\Leftrightarrow (k^6-2)(a^2+b^2)^3 = 0,\end{aligned}$$

we obtain $k=\sqrt[6]{2}$. Conversely, we assume that $k=\sqrt[6]{2}$ and $c > a, b$, then

$$\begin{aligned}&\sqrt[6]{a^6+b^6+c^6+3a^2b^2c^2} = k\max\{a,b,c\}\\ &\Leftrightarrow a^6+b^6+c^6+3a^2b^2c^2 = 2c^6\\ &\Leftrightarrow (c^2-(a^2+b^2))(c^4+(a^2+b^2)c^2+a^4+b^4-a^2b^2) = 0.\end{aligned}$$

Since $c^4+(a^2+b^2)c^2+a^4+b^4-a^2b^2 > 0$, we obtain $c^2-(a^2+b^2)=0$. Therefore, the solution is $k=\sqrt[6]{2}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Arkady Alt, San Jose, CA, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania.

S506. Let x, y, z, t be real numbers, $0 \le x, y, z, t \le 1$, such that

$$(1-x)(1-y)(1-z)(1-t) = xyzt.$$

Prove that

$$x^2 + y^2 + z^2 + t^2 \ge 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

If any one out of x, y, z, t is 0, then the RHS of the proposed condition is 0, or so is the LHS, and at least one out of x, y, z, t is also equal to 1, and the proposed inequality is trivially true, with equality iff (x, y, z, t) is a permutation of $(1, 0, 0, 0)$. Or it suffices to prove the desired result when $0 < x, y, z, t < 1$, which we will assume henceforth to hold.

Denote $a = \frac{1}{x} - 1$, $b = \frac{1}{y} - 1$, $c = \frac{1}{z} - 1$ and $d = \frac{1}{t} - 1$, and note that the proposed condition rewrites as $abcd = 1$. Note further that $x = \frac{1}{1+a}$, $y = \frac{1}{1+b}$, $z = \frac{1}{1+c}$ and $t = \frac{1}{1+d}$, and that the proposed inequality rewrites as

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \ge 1.$$

We now note that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \ge \frac{1}{1+ab}.$$

Indeed, this is equivalent to

$$1 + a^3b + ab^3 \ge 2ab + a^2b^2, \qquad ab(a-b)^2 + (ab-1)^2 \ge 0,$$

clearly true and with equality iff $a = b = 1$. It then follows, using that $abcd = 1$, that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \ge \frac{1}{1+ab} + \frac{1}{1+cd} = \frac{1}{1+ab} + \frac{ab}{ab+1} = 1.$$

Equality holds iff $a = b = c = d = 1$, or equivalently iff $x = y = z = t = \frac{1}{2}$.

The conclusion follows, equality holds iff either $x = y = z = t = \frac{1}{2}$ or (x, y, z, t) is a permutation of $(1, 0, 0, 0)$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; Daniel Pascuas, Universitat de Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Maalav Mehta, Prakash Higher Secondary School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece.

S507. If a, b, c are real numbers such that $ax^2 + bx + c \geq 0$ for all real numbers x, prove that $4a^3 - b^3 + 4c^3 \geq 0$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Oana Prajitura, College at Brockport, SUNY, USA

If $a = 0$ the condition becomes $bx + c \geq 0$ for all real numbers x and thus $b = 0$ and $c \geq 0$

This implies that

$$4a^3 - b^3 + 4c^3 = 4c^3 \geq 0$$

If $a \neq 0$ then the quadratic function must have negative discriminant and positive leading coefficient. Thus, $a > 0$ and

$$b^2 - 4ac \leq 0 \iff 4ac \geq b^2 \iff ac \geq \frac{b^2}{4} \geq 0$$

Since $ac \geq 0$ and $a > 0$ we conclude that $c \geq 0$.

If $b \leq 0$ then

$$4a^3 + 4c^3 \geq 0 \geq b^3.$$

If $b > 0$ then

$$4a^3 + 4c^3 = 4(a^3 + c^3) \geq 8\sqrt{a^3c^3} = 8\sqrt{ac}^3 \geq 8\sqrt{\frac{b^2}{4}}^3 = b^3.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

S508. Prove that in any triangle ABC,

$$\left(\frac{h_a}{\ell_a}\right)^2+\left(\frac{h_b}{\ell_b}\right)^2+\left(\frac{h_c}{\ell_c}\right)^2-2\frac{h_a}{\ell_a}\frac{h_b}{\ell_b}\frac{h_c}{\ell_c}=1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Since $\angle(h_a,l_a)=\frac{|B-C|}{2}$, $\angle(h_b,l_b)=\frac{|C-A|}{2}$, $\angle(h_c,l_c)=\frac{|A-B|}{2}$, we have $\frac{h_a}{l_a}=\cos\frac{|B-C|}{2}$, $\frac{h_b}{l_b}=\cos\frac{|C-A|}{2}$ and $\frac{h_c}{l_c}=\cos\frac{|A-B|}{2}$ and we will prove the given equality in the equivalent form

$$\cos^2\frac{A-B}{2}+\cos^2\frac{B-C}{2}+\cos^2\frac{C-A}{2}-2\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}=1,$$

where $\frac{A-B}{2}+\frac{B-C}{2}+\frac{C-A}{2}=0$, using the following lemma.

Lemma: If $x+y+z=0$, then $\cos^2 x+\cos^2 y+\cos^2 z-2\cos x\cos y\cos z=1$.

Proof: We have $z=-(x+y)$, giving

$$\cos z=\cos(x+y).$$

Squaring both sides, we find that

$$\begin{aligned}
\cos^2 z &= \cos^2(x+y)\\
&= (\cos x\cos y-\sin x\sin y)^2\\
&= \cos^2 x\cos^2 y+\sin^2 x\cos^2 y-2\cos x\cos y\sin x\sin y\\
&= \cos^2 x\cos^2 y+\left(1-\cos^2 x\right)\left(1-\cos^2 y\right)-2\cos x\cos y\sin x\sin y\\
&= 1-\cos^2 x-\cos^2 y+2\cos x\cos y\underbrace{(\cos x\cos y-\sin x\sin y)}_{=\cos z}
\end{aligned}$$

and

$$\cos^2 x+\cos^2 y+\cos^2 z-2\cos x\cos y\cos z=1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Arkady Alt, San Jose, CA, USA; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Titu Zvonaru, Comănești, Romania.

S509. Solve in integers the equation

$$2(xy+2)^2 - 6(x+y)^2 = (x+y-1)^3 - 6.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Li Zhou, Polk State College, USA

Considering the equation modulo 2 we see that $x+y-1$ must be even. Thus x and y have opposite parity. By symmetry in x and y we may assume that $x = 2m$ and $y = 2n+1$. Then the equation reduces to

$$(m+n+1)^3 - (2mn+m+1)^2 = 1.$$

It is well known that $u^3 - v^2 = 1$ has only the trivial solution $(u,v) = (1,0)$, so $m = -n$, and thus

$$0 = -2n^2 - n + 1 = (1-2n)(1+n).$$

Therefore, $(x,y) = (2,-1)$ or $(-1,2)$.

Also solved by Arkady Alt, San Jose, CA, USA; Martín Lupin, IDRA Secondary School, Argentina; Ioannis D. Sfikas, Athens, Greece.

S510. Consider an array of 49 consecutive integers whose median is a perfect square. Prove that the sum of the cubes of the 49 integers can be written as a sum of four perfect squares two of which are equal.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Let the 49 integers be

$$n^2-24,\ldots,n^2-1,n^2,n^2+1,\ldots,n^2+24.$$

The sum of their cubes is

$$49n^6+6n^2(1^2+2^2+\cdots+24^2),$$

as the term in n^4 has coefficient zero and the free term is zero as well.

Since

$$1^2+2^2+\cdots+24^2=\frac{24(24+1)(2\times 24+1)}{6}=70^2,$$

the sum is equal to

$$(7n^3)^2+(2\times 70n)^2+(70n)^2+(70n)^2,$$

as desired.

Also solved by Albert Stadler, Herrliberg, Switzerland; Li Zhou, Polk State College, USA; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Joel Schlosberg, Bayside, NY, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Satvik Dasariraju; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania; Daniel López-Aguayo, MSCI, Monterrey, Mexico.

Undergraduate problems

U505. Let K be a field. Prove that the polynomial

$$X^n + X^2Y + XY + XY^2 + Y^n$$

is irreducible in the ring $K[X,Y]$, for all $n \geq 2$.

Proposed by Mircea Becheanu, Montreal, Canada

Solution by the author

Let's denote $F(X,Y) = X^n + X^2Y + XY + XY^2 + Y^n$. Consider the ring homomorphism $\varphi : K[X,Y] \longrightarrow K[X,Y]$ which is identity on K and $\varphi(X) = X$, $\varphi(Y) = XY$. We mention that φ is one to one because for any monomial X^aY^b we have $\varphi(X^aY^b) = X^{a+b}Y^b$ and $\varphi(K[X,Y])$ consists in all polynomials for which every nonzero monomial cX^aY^b which appears in it, has the property $a \geq b$.

We have

$$\varphi(F(X,Y)) = F(X,XY) = X^n + X^3Y + X^2Y + X^3Y^2 + X^nY^n =$$
$$X^2[X^{n-2}(Y^n + 1) + XY(Y+1) + Y].$$

For $n > 2$, the polynomial $X^{n-2}(Y^n+1)+XY(Y+1)+Y$ is irreducible in $K[X,Y]$ by Eisenstein criterion, by considering it as a polynomial in X. For $n = 2$, the polynomial $XY(Y+1) + Y^n + Y + 1$ is irreducible because it is of degree 1 in X and it is not divisible by a polynomial in Y. Therefore, the equality

$$\varphi(F(X,Y)) = X^2[X^{n-2}(Y^n + 1) + XY(Y+1) + Y] \tag{1}$$

represents the splitting in irreducible factors of $\varphi(F(X,Y))$.

Assume now that $F(X,Y)$ splits in nonconstant factors

$$F(X,Y) = P(X,Y)Q(X,Y).$$

Then

$$\varphi(F(X,Y)) = \varphi(P(X,Y))\varphi(Q(X,Y)),$$

that is $\varphi(F(X,Y))$ splits in two nonconstant factors which are polynomials in $\varphi(K[X,Y])$. Because $K[X,Y]$ is a UFD, every such splitting is obtained by combining factors in its splitting in irreducible factors.
From (1) it is clear that this is a contradiction.

Also solved by Li Zhou, Polk State College, USA; Prajnanaswaroopa S, Amrita University, Coimbatore, India.

U506. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(1+x) = 1 + f(x) \qquad \text{and} \qquad f\left(\frac{1}{x}\right) = \frac{1}{f(x)}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

First solution by Li Zhou, Polk State College, USA

Clearly $f(x) = x$ is a solution. We show it is the only one. First, $f(1) = f(1/1) = 1/f(1)$, so $f(1) = 1$. By induction using $f(1+x) = 1 + f(x)$, we get $f(m) = m$ for all $m \in \mathbb{N}$. So it suffices to show $f(x) = x$ for all $x \in (0,1)$. Now it is well known that any such x has a simple continued-fraction expansion

$$x = \cfrac{1}{x_1 + \frac{1}{x_2 + \frac{1}{\ddots}}} = [x_1, x_2, \ldots],$$

where $x_i \in \mathbb{N}$ for all $i \geq 1$ (non-terminating case) or $x_i \in \mathbb{N}$ for $1 \leq i \leq n$ and $x_i = 0$ for all $i > n$ (terminating case).

For terminating $x = [x_1, x_2, \ldots, x_n]$, an easy induction using both functional equations yields

$$f(x) = \frac{1}{f([x_1, \ldots, x_n])} = \cdots = [f(x_1), f(x_2), \ldots, f(x_n)] = [x_1, x_2, \ldots, x_n] = x.$$

If $x = [x_1, x_2, \ldots]$ is non-terminating, then the expansion is unique and the functional equations force $f(x)$ to have the unique expansion $[f(x_1), f(x_2), \ldots] = [x_1, x_2, \ldots] = x$ as well. This completes the proof.

Second solution by Daniel Lasaosa, Pamplona, Spain

Note first that taking $x = 1$ in the second condition and since f takes only positive values, we obtain $f(1) = 1$, which after trivial induction using the first condition results in $f(n) = n$ for all positive integer n. Note further that if $n < x < n+1$ we have $n < f(x) < n+1$, since otherwise we would have either $f(x-n) = f(x) - n < 0$ in contradiction with f taking only positive values, or $f(x-n) = f(x) - n \geq 1$, for $\frac{1}{x-n} > 1$ and $f\left(\frac{1}{x-n} - 1\right) = \frac{1}{f(x-n)} - 1 \leq 0$, reaching a contradiction again. Therefore, $|f(x) - x| < 1$.

Assume that $f(x)$ is not the identity, or a supremum $0 < s \leq 1$ exists such that $|f(x) - x| \leq s$ for all positive real x. Therefore, a positive real δ exists such that $s \geq \delta > \sqrt{s + \frac{1}{4}} - \frac{1}{2}$, since either s is a maximum and x exists such that $|f(x) - x| = s$, or s is not a maximum and real values of x exist such that $|f(x) - x|$ is less than s but arbitrarily close to s. Note that this is also possible becase $\sqrt{s + \frac{1}{4}} - \frac{1}{2} < s$ is equivalent to $s^2 > 0$, which is clearly true. Now, taking such an x, we consider two cases:

Case 1: $f(x) = x + \delta$. If $x > 1$, let $m = \lfloor x \rfloor$, or after trivial induction using the first condition we have $f(x) = f(x-m) + m$, hence substitution of $x - m$ by x yields $f(x) = x + \delta$ for some real $0 < x < 1$. Note first that $x + \delta < 1$, since otherwise we would have $\frac{1}{x} > 1$ and $f(x) > 1$, or $f\left(\frac{1}{x} - 1\right) = \frac{1}{f(x)} - 1 < 0$, in contradiction with f taking only positive values. It follows that $x < 1 - \delta$, and

$$\frac{1}{x} - f\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{f(x)} = \frac{1}{x} - \frac{1}{x+\delta} = \frac{\delta}{x(x+\delta)} > \frac{\delta}{1-\delta} > \delta + \delta^2 > s,$$

absurd since it contradicts that s is the supremum of $|f(x) - x|$.

Case 2: $f(x) = x - \delta$. As before, subtracting $m = \lfloor x \rfloor$ we find that we may assume that $0 < x < 1$, or $x - \delta < 1 - \delta$, and

$$f\left(\frac{1}{x}\right) - \frac{1}{x} = \frac{1}{f(x)} - \frac{1}{x} = \frac{1}{x-\delta} - \frac{1}{x} = \frac{\delta}{x(x-\delta)} > \frac{\delta}{1-\delta} > \delta + \delta^2 > s,$$

absurd again.

We conclude that the only possible solution is $f(x) = x$, which trivially satisfies the proposed conditions.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Pascuas, Universitat de Barcelona, Spain; M.A.Prasad, Mumbai, India; Ioannis D. Sfikas, Athens, Greece.

U507. Evaluate

$$\int_{-1/3}^{1} \frac{1}{2x+\sqrt{x^2+x+2}}dx$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Donaldo Garrido-Islas, Instituto Tecnológico y de Estudios Superiores de Monterrey, Monterrey, Mexico and Daniel López-Aguayo, MSCI, Monterrey, Mexico

We perform the so-called Euler's substitution. Let $\sqrt{x^2+x+2} = t-x$, hence $x = \dfrac{t^2-2}{1+2t}$.

Therefore, $dx = \dfrac{2(t^2+t+2)}{(1+2t)^2}$. Also, note that the lower limit is now 1 and the upper is 3. A little algebra shows that the integrand equals

$$\frac{2(t^2+t+2)}{(t+1)(2t+1)(3t-2)}.$$

Since the latter expression is a proper rational function, we apply partial fractions to obtain:

$$\int_{1}^{3} \frac{2(t^2+t+2)}{(t+1)(2t+1)(3t-2)}dt = \int_{1}^{3}\left(\frac{4}{5(t+1)} - \frac{2}{2t+1} + \frac{8}{5(3t-2)}\right)dt$$

Computing the indefinite integral yields

$$\frac{4}{5}\ln|t+1| - \ln|2t+1| + \frac{8}{15}\ln|3t-2|$$

Evaluating from $t=1$ to $t=3$ gives $\dfrac{4\ln(2)}{5} + \ln(3) - \dfrac{7\ln(7)}{15}$, and we are done.

Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Daniel Pascuas, Universitat de Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Ioannis D. Sfikas, Athens, Greece.

U508. For positive integer n, let $S_1, S_2, \dots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \dots n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is $m_{ij} = |S_i \cup S_j|$. Find the determinant of $M = (m_{ij})$.

Proposed by Li Zhou, Polk State College, USA

Solution by the author

If $n = 1$, then $M = [1]$ and $\det(M) = 1$. Suppose $n \geq 2$. Interchanging S_i and S_j corresponds to switching the ith and jth columns and rows, thus leaves the the determinant invariant. Hence, for $n = 2$, with $S_1 = \{1\}$, $S_2 = \{2\}$, and $S_3 = \{1, 2\}$, we get

$$\det(M) = \det \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} = 2.$$

Now consider $n \geq 3$. Let $S_j = \{1\}$, $S_k = \{2, \dots, n\}$ (the complement of S_j), $S_p = \{2\}$, and $S_q = \{1, 3, \dots, n\}$ (the complement of S_p). Then for any $i = 1, \dots, 2^n - 1$, $|S_i \cap S_j| + |S_i \cap S_k| = |S_i|$, thus by PIE,

$$\begin{aligned} m_{ij} + m_{ik} &= |S_i \cup S_j| + |S_i \cup S_k| \\ =&|S_i| + 1 - |S_i \cap S_j| + |S_i| + (n-1) - |S_i \cap S_k| = n + |S_i|. \end{aligned}$$

Likewise, $m_{ip} + m_{iq} = n + |S_i|$ for all $i = 1, \dots, 2^n - 1$. Since these column operations (adding the kth to the jth column and adding the qth to the pth column) lead to two identical columns, we conclude that $\det(M) = 0$ for all $n \geq 3$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joel Schlosberg, Bayside, NY, USA; M.A.Prasad, Mumbai, India.

U509. Prove that for any $x > 1$, the following inequalities hold.

$$\log\left(\frac{1+x^2}{x^2-2x+2}\right)^{\frac{1}{2x-1}} < \arctan(x) - \arctan(x-1) < \log\left(\frac{1+x^2}{x^2-2x+2}\right)^{\frac{1}{2(x-1)}}.$$

Proposed by Besfort Shala, University of Primorska, Slovenia

Solution by Li Zhou, Polk State College, USA

For any $x > 1$,

$$\log\frac{x^2+1}{x^2-2x+2} - 2(x-1)\left(\arctan x - \arctan(x-1)\right) = \int_{x-1}^{x} \frac{2t-2(x-1)}{t^2+1}dt > 0,$$

establishing the right inequality. On the other hand,

$$\begin{aligned}
&(2x-1)\left(\arctan x - \arctan(x-1)\right) - \log\frac{x^2+1}{x^2-2x+2} = \int_{x-1}^{x} \frac{2x-1-2t}{t^2+1}dt \\
&= \int_{x-1}^{(2x-1)/2} \frac{2x-1-2t}{t^2+1}dt + \int_{(2x-1)/2}^{x} \frac{2x-1-2t}{t^2+1}dt \\
&= \int_{x-1}^{(2x-1)/2} \frac{2x-1-2t}{t^2+1}dt - \int_{x-1}^{(2x-1)/2} \frac{2x-1-2u}{(2x-1-u)^2+1}du \\
&= \int_{x-1}^{(2x-1)/2} (2x-1-2t)\left(\frac{1}{t^2+1} - \frac{1}{(2x-1-t)^2+1}\right)dt > 0,
\end{aligned}$$

establishing the left inequality.

We also notice that the problem would have been more elusive if the "footprint" $\arctan(x) - \arctan(x-1)$ were covered up by $\arctan\frac{1}{x^2-x+1}$.

Also solved by Arkady Alt, San Jose, CA, USA; Daniel Lasaosa, Pamplona, Spain; Daniel Pascuas, Universitat de Barcelona, Spain; Oana Prajitura, College at Brockport, SUNY, USA; Albert Stadler, Herrliberg, Switzerland.

U510. Evaluate

$$\int_0^\pi \frac{x\sin x}{2021+4\sin^2 x}dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Abdelouahed Hamdi, Qatar University, Doha, Qatar

We use the change of variable: $x=\pi-t \Longrightarrow dx=-dt$ and we get:

$$\begin{aligned}
I &= \int_0^\pi \frac{x\,\sin(x)}{2021+4\sin^2(x)} \\
&= \int_\pi^0 \frac{(\pi-t)\,\sin(\pi-t)}{2021+4\sin^2(\pi-t)}\,(-dt) \\
&= \int_0^\pi \frac{(\pi-t)\,\sin(t)}{2021+4\sin^2(t)}\,dt \\
&= \int_0^\pi \frac{\pi\sin(t)}{2021+4\sin^2(t)}\,dt - \int_0^\pi \frac{t\,\sin(t)}{2021+4\sin^2(t)}\,dt \\
&= \int_0^\pi \frac{\pi\sin(t)}{2021+4\sin^2(t)}\,dt - I \\
2I &= \pi\int_0^\pi \frac{\sin(t)}{2021+4(1-\cos^2(t))}\,dt \\
2I &= \pi\int_0^\pi \frac{\sin(t)}{2025-4\cos^2(t)}\,dt
\end{aligned}$$

Let $u=-\cos(t) \Longrightarrow du=\sin(t)\,dt$

$$\begin{aligned}
2I &= \pi\int_{-1}^1 \frac{1}{2025-4u^2}\,du \quad \text{the integrand is an even function} \\
2I &= 2\pi\int_0^1 \frac{1}{2025-4u^2}\,du = \frac{\pi}{2}\int_0^1 \frac{du}{a^2-u^2}, \quad a=45/2 \\
2I &= \frac{\pi}{2}\int_0^1 \frac{du}{(a-u)(a+u)} = \frac{\pi}{4a}\Big[\int_0^1 \frac{du}{a-u} + \int_0^1 \frac{du}{a+u}\Big] \\
2I &= \frac{\pi}{4a}\Big[\ln\Big|\frac{a+u}{a-u}\Big|\Big]_0^1 = \frac{\pi}{4a}\ln\Big|\frac{a+1}{a-1}\Big| \\
I &= \frac{\pi}{180}\ln(47/43).
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel Pascuas, Universitat de Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Matthew Too, The College at Brockport, SUNY, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Sergio Esteban Muñoz, Universidad de Buenos Aires, Argentina; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Titu Zvonaru, Comănești, Romania; Donaldo Garrido-Islas, Instituto Tecnológico y de Estudios Superiores de Monterrey, Monterrey, Mexico and Daniel López-Aguayo, MSCI, Monterrey, Mexico.

O505. Let a, b, c, d be positive real numbers such that

$$a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$\frac{3(a^2+b^2+c^2+d^2)}{a+b+c+d}+1\geq a+b+c+d.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

We have the following identity

$$\begin{aligned}a^3+b^3+c^3+d^3 &=(a+b+c+d)^3-3(a+b+c+d)\sum ab+3(abc+bcd+cda+dab)\\ &=(a+b+c+d)\left[(a+b+c+d)^2-3\sum ab+3abcd\right].\end{aligned}$$

Using this, the our inequality can be rewrite as

$$\frac{a^3+b^3+c^3+d^3}{a+b+c+d}+\frac{a+b+c+d}{2}\geq 3abcd,$$

or

$$a^3+b^3+c^3+d^3+\frac{(a+b+c+d)^2}{2}\geq 3(abc+bcd+cd+dab),$$

or, after we homogenize

$$a^3+b^3+c^3+d^3+\sqrt{\frac{abcd(a+b+c+d)^5}{4(abc+bcd+cda+dab)}}\geq 3(abc+bcd+cda+dab).$$

But, by AM-GM Inequality and Maclaurin's Inequality

$$\begin{aligned}\sqrt{\frac{abcd(a+b+c+d)^5}{4(abc+bcd+cda+dab)}} &=\sqrt{\frac{abcd(a+b+c+d)^7}{4(abc+bcd+cda+dab)(a+b+c+d)^2}}\\ &\geq\sqrt{\frac{4^4a^2b^2c^2d^2\cdot 4^2(abc+bcd+cda+dab)}{4(abc+bcd+cda+dab)(a+b+c+d)^2}}\\ &=\frac{32abcd}{a+b+c+d}.\end{aligned}$$

So, it remains to prove that

$$a^3+b^3+c^3+d^3+\frac{32abcd}{a+b+c+d}\geq 3(abc+bcd+cda+dab),$$

which is a well-known inequality.

Also solved by Kevin Soto, Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland.

O506. Let a be a nonnegative integer. Find all pairs (x,y) of nonnegative integers such that

$$\left(a^2+1\right)\left(x^3-2axy+y^3\right)=a^2-xy.$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that if the RHS is positive, then its absolute value is at most a^2, whereas the absolute value of the LHS is a multiple of a^2+1. Therefore, the RHS must be a nonpositive multiple of a^2+1, or a nonnegative integer k exists such that $a^2-xy=-k\left(a^2+1\right)$. Then, $xy=a^2(k+1)+k$, and $x^3+y^3=2axy-k=2a^3(k+1)+2ak-k$, and since a,x,y are nonnegative, the AM-GM inequality applied to x^3,y^3,a^3 yields

$$a^3(2k+3)+2ak-k=x^3+y^3+a^3\geq 3axy=a^3(3k+3)+3ak, \qquad k\left(a^3+a+1\right)\leq 0.$$

However, since $a^3+a+1>0$, it follows that for the AM-GM to hold we need $k=0$, in which case equality occurs, yielding $x=y=a$. Substitution in the proposed equation indeed yields both sides being equal to 0, or this is the only possible solution, and it holds for any nonnegative integer a.

Also solved by Martín Lupin, IDRA Secondary School, Argentina; Ioannis D. Sfikas, Athens, Greece; Li Zhou, Polk State College, USA.

O507. Let $a, b, c, d > 0$ and $a^4 + b^4 + c^4 + d^4 = 4$. Prove that

$$\frac{a^2b}{a^4+b^3+c^2+d} + \frac{b^2c}{b^4+c^3+d^2+a} + \frac{c^2d}{c^4+d^3+a^2+b} + \frac{d^2a}{d^4+a^3+b^2+c} \le \frac{16}{(a+b+c+d)^2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

Using Cauchy-Schwarz Inequality we get

$$(a^4+b^3+c^2+d)\left(\frac{1}{a^2}+\frac{1}{b}+1+d\right) \ge (a+b+c+d)^2 = 16,$$

Namely,

$$\frac{1}{a^4+b^3+c^2+d} \le \frac{1}{16}\left(\frac{1}{a^2}+\frac{1}{b}+1+d\right)$$

Which yields to

$$\frac{a^2b}{a^4+b^3+c^2+d} \le \frac{1}{16}(b+a^2+a^2b+a^2bd)$$

Similarly for other permutations. Summing up four of them and applying the quaternion mean inequality results in

$$\frac{a^2b}{a^4+b^3+c^2+d} + \frac{b^2c}{b^4+c^3+d^2+a} + \frac{c^2d}{c^4+d^3+a^2+b} + \frac{d^2a}{d^4+a^3+b^2+c} \le \frac{1}{16}\left(\sum a + \sum a^2 + \sum a^2b + \sum a^2bd\right)$$

Now,

$$\sum a^2 \le \sqrt{4\sum a^4} = 4, \sum a \le \sqrt{4\sum a^2} = 4,$$

$$\sum a^2b \le \frac{1}{4}\sum\left(a^4+a^4+b^4+1\right) = \frac{3}{4}\sum a^4 + 1 = 4,$$

$$\sum a^2bd \le \frac{1}{4}\sum\left(a^4+a^4+b^4+d^4\right) = \sum a^4 = 4,$$

and the conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland.

O508. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{b(a+5c)^2}+\frac{b}{c(b+5a)^2}+\frac{c}{a(c+5b)^2} \geq \frac{1}{4(\sqrt{a}+\sqrt{b}+\sqrt{c})}.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
Applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\frac{a}{b(a+5c)^2}+\frac{b}{c(b+5a)^2}+\frac{c}{a(c+5b)^2} &= \frac{a^2}{ab(a+5c)^2}+\frac{b^2}{bc(b+5a)^2}+\frac{c^2}{ca(c+5b)^2} \\ &\geq \frac{\left(\frac{a}{a+5c}+\frac{b}{b+5a}+\frac{c}{c+5b}\right)^2}{ab+bc+ca}.\end{aligned}$$

On the other hand, the Cauchy-Schwarz inequality also gives us

$$\begin{aligned}\frac{a}{a+5c}+\frac{b}{b+5a}+\frac{c}{c+5b} &\geq \frac{(a+b+c)^2}{a(a+5c)+b(b+5a)+c(c+5b)} \\ &= \frac{(a+b+c)^2}{(a+b+c)^2+3(ab+bc+ca)} \\ &\geq \frac{(a+b+c)^2}{2(a+b+c)^2} \\ &= \frac{1}{2}.\end{aligned}$$

Combining these two inequalities we get

$$\frac{a}{b(a+5c)^2}+\frac{b}{c(b+5a)^2}+\frac{c}{a(c+5b)^2} \geq \frac{1}{4(ab+bc+ca)}.$$

Therefore, it suffices to show that

$$ab+bc+ca \leq \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

This is equivalent to

$$9=(a+b+c)^2 \leq a^2+b^2+c^2+2(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

But this is follows from the following

$$\begin{aligned}a^2+\sqrt{a}+\sqrt{a} &\geq 3a, \\ b^2+\sqrt{b}+\sqrt{b} &\geq 3b, \\ c^2+\sqrt{c}+\sqrt{c} &\geq 3c.\end{aligned}$$

The equality holds if and only if $a=b=c=1$.

Also solved by Albert Stadler, Herrliberg, Switzerland; An Nguyen Huu Bui, High School For The Gifted, VNUHCM, Ho Chi Minh City, Vietnom; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú.

O509. Prove that for any positive real numbers a, b, c

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{27\left(a^3+b^3+c^3\right)}{(a+b+c)^3}+\frac{21}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Assuming $a+b+c=1$ (due homogeneity of the inequality) and denoting $p := ab+bc+ca, q := abc$ we obtain

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{27\left(a^3+b^3+c^3\right)}{(a+b+c)^3}-\frac{21}{4}=\frac{p}{q}-27(1+3q-3p)-\frac{21}{4}.$$

Noting that $p=ab+bc+ca \leq \frac{(a+b+c)^2}{3}=\frac{1}{3}$ then, denoting $t := \sqrt{1-3p}$, we obtain $p=\frac{1-t^2}{3}$, where $t \in [0,1)$.

Since the criterion of solvability of Vieta's system

$$\left\{\begin{array}{c} a+b+c=1 \\ ab+bc+ca=p=\dfrac{1-t^2}{3} \\ abc=q \end{array}\right|$$

in real a, b, c is

$$\frac{(1-2t)(1+t)^2}{27} \leq q \leq \frac{(1+2t)(1-t)^2}{27},$$

then

$$\frac{p}{q}-27(1+3q-3p)-\frac{21}{4} \geq \frac{\frac{1-t^2}{3}}{\frac{(1+2t)(1-t)^2}{27}}-27\left(1+3\cdot\frac{(1+2t)(1-t)^2}{27}-3\cdot\frac{1-t^2}{3}\right)=$$

$$\frac{9(t+1)}{(1-t)(2t+1)}-3\left(2t^3+6t^2+1\right)-\frac{21}{4}=\frac{3\left(4t^3+14t^2+5t+1\right)(2t-1)^2}{4(2t+1)(1-t)} \geq 0$$

where equality occurs iff $t=\frac{1}{2} \iff p=\frac{1}{4},\ q=\frac{(1+2\cdot(1/2))(1-(1/2))^2}{27}=\frac{1}{54}$.

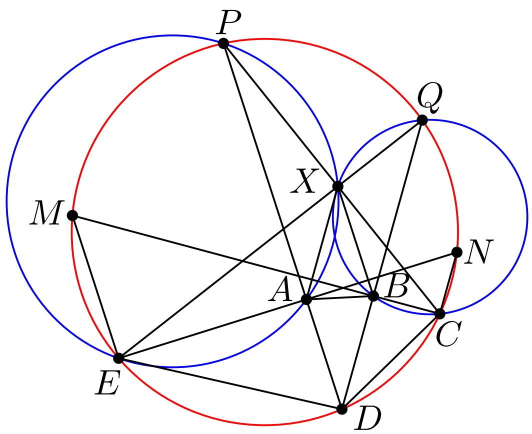
From cubic equation $x^3-x^2+\frac{1}{4}x-\frac{1}{54}=0 \iff \frac{1}{108}(3x-2)(6x-1)^2=0$, we obtain $a=b=\frac{1}{6}, c=\frac{2}{3}$.

Also solved by Albert Stadler, Herrliberg, Switzerland; An Nguyen Huu Bui, High School For The Gifted, VNUHCM, Ho Chi Minh City, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

O510. Let $ABCDE$ be a convex pentagon with $\angle BCD = \angle ADE$ and $\angle BDC = \angle AED$. The circumcircle of triangle CDE meets lines DA and DB for the second time at points P and Q, respectively. Lines CP and QE intersect at X. Prove that $ADBX$ is a parallelogram.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Li Zhou, Polk State College, USA



Suppose that CB and EA intersect the circumcircle of CDE again at M and N, respectively. Since $\angle MCD = \angle PDE$, $EM \parallel DP$, so $\angle EQD = \angle MCP$. Therefore, $BCQX$ is cyclic, thus

$$\angle XBQ = \angle PCQ = \angle PDQ$$

Hence, $XB \parallel PD$. Likewise, $XA \parallel QD$, completing the proof.

Also solved by Martín Lupin, IDRA Secondary School, Argentina.