Junior problems

J505. Solve the equation $x^2 x + 2 x^3 + 2 \{x\}^3 = \frac{1}{108}$,

where $\{x\}$ denotes the fractional part of $x$.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Polyahedra, Polk State College, USA

Clearly, $x = 0, \pm 1$ are not solutions.

If $x > 1$, then $2x^3 + x\{x\} + 2\{x\}^3 > 2$; if $x < -1$, then $2x^3 + x\{x\} + 2\{x\}^3 < 0$.

If $0 < x < 1$, then $\{x\} = x$ and the equation becomes $\frac{1}{108}(12x - 1)(6x + 1)^2 = 0$, so $x = \frac{1}{12}$.

If $-1 < x < 0$, then $\{x\} = x + 1$ and the equation becomes $\frac{1}{108}(12x + 5)(36x^2 + 48x + 43) = 0$, so $x = -\frac{5}{12}$.

In conclusion, $\frac{1}{12}$ and $-\frac{5}{12}$ are the only solutions.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Isko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Titu Zvonaru, Comănești, Romania.
J506. Prove that any integer \( n > 6 \) can be written as \( n = p + m \), where \( p \) is a prime less than \( \frac{n}{2} \) and \( p \) does not divide \( m \).

Proposed by Li Zhou, Polk State College, USA

First solution by Daniel Lasaosa, Pamplona, Spain
If \( n \) is not a multiple of 3, let \( p \) be the largest prime which is smaller than \( \frac{n}{2} \), and let \( m = n - p \). If \( p \) divides \( m \), then \( p \) divides \( n \). But \( p < \frac{n}{2} \) and \( p \neq \frac{n}{3} \), or \( p \leq \frac{n}{4} \), absurd since by Bertrand’s postulate, there exists at least one prime in \( \left( \frac{n}{4}, \frac{n}{2} \right) \).

If \( n \) is a multiple of 3, let \( p \) be the largest prime which is smaller than \( \frac{n}{3} \), and let \( m = n - p \). If \( p \) divides \( m \), then \( p \) divides \( n \). But \( p < \frac{n}{3} \) and by Bertrand’s postulate \( p > \frac{n}{6} \). Therefore, such a \( p \) exists and is coprime with \( m = n - p \), which solves the problem, except when \( n = 4p \) or when \( n = 5p \). But in these two cases, and since 3 divides \( n \) and is coprime with 4, 5, we must have \( p = 3 \), for either \( n = 12 \) or \( n = 15 \). But \( 12 = 5 + 7 \) with 5 prime and \( 15 = 7 + 8 \) with 7 prime. The conclusion follows.

Second solution by Polyahedra, Polk State College, USA
If \( n \) is odd, then \( n \geq 7 \) and we can take \( p = 2 \) and \( m = n - 2 \). Suppose that \( n \) is even. If 4 divides \( n \), then \( n \geq 8 \), so \( \frac{n}{2} - 1 \geq 3 \) is odd and has an odd prime factor \( p \). Since \( p \) does not divide \( n \), \( p \) does not divide \( m = n - p \). If 4 does not divide \( n \), then \( n \geq 10 \), so \( \frac{n}{2} - 2 \geq 3 \) is odd and has an odd prime factor \( p \). Again, since \( p \) does not divide \( n \), \( p \) does not divide \( m = n - p \).

Also solved by Albert Stadler, Herrliberg, Switzerland; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitesti, Romania; Joe Simons, Utah Valley University, Orem, UT, USA; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Michail Prousalidis, Evangeliki Model Junior High School of Smyrna, Athens, Greece; Mohamed Ali, Houari Boumediens School, Algeria.
J507. Consider a real number \( a \),

\[
b = (a^2 + 2a + 2) \left( a^2 - (1 - \sqrt{3})a + 2 \right) \left( a^2 + (1 + \sqrt{3})a + 2 \right)
\]

and

\[
c = (a^2 - 2a + 2) \left( a^2 + (1 - \sqrt{3})a + 2 \right) \left( a^2 - (1 + \sqrt{3})a + 2 \right)
\]

Find \( a \) knowing that \( b + c = 16 \).

Proposed by Adrian Andreescu, University of Texas at Austin, USA

First solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan

Expanding \( b + c \), we obtain

\[
b + c = 16
\]

\[
\iff a^2(a^4 + 4(2 + \sqrt{3})a^2 + 8(2 + \sqrt{3})) = 0.
\]

Since \( a^4 + 4(2 + \sqrt{3})a^2 + 8(2 + \sqrt{3}) > 0 \), the solution is \( a = 0 \).

Second solution by Polyahedra, Polk State College, USA

Notice that

\[
b, c > 0 \text{ and } a^{12} + 64 = bc \leq \left( \frac{b + c}{2} \right)^2 = 64,
\]

so \( a = 0 \).

Also solved by Arkady Alt, San Jose, CA, USA; Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Michail Prousalidis, Evangeliki Model Junior High School of Smyrna, Athens, Greece; Mohamed Ali, Houari Boumedien School, Algeria; Albert Stadler, Herrliberg, Switzerland; Cornelia Romanescu-Avram, Ploiesti, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlossberg, Bayside, NY, USA; David E. Manes, Oneonta, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Daniel López-Aguayo, MSCI, Monterrey, Mexico; Titu Zvonaru, Comănești, Romania.
J508. Let \(a, b, c\) be positive numbers such that \(a + b + c + 2 = abc\). Prove that

\[(1 + ab)(1 + bc)(1 + ca) \geq 125.\]

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the given condition and the AM-GM inequality we have

\[abc = a + b + c + 2 \geq 4 \sqrt[4]{2abc}.\]

It follows that

\[abc \geq 8.\]

Now we use again the AM-GM inequality to obtain

\[1 + ab = 1 + \frac{ab}{4} + \frac{ab}{4} + \frac{ab}{4} + \frac{ab}{4} \geq 5 \sqrt[5]{\left(\frac{ab}{4}\right)^4}.\]

Similarly,

\[1 + bc \geq 5 \sqrt[5]{\left(bc\right)^4},\]
\[1 + ca \geq 5 \sqrt[5]{\left(ca\right)^4}.\]

Which yields to

\[(1 + ab)(1 + bc)(1 + ca) \geq 125 \sqrt[8]{\left(\frac{abc}{8}\right)^8} \geq 125.\]

The equality holds for \(a = b = c = 2\).

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martin Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Mohamed Ali, Houari Boumedien School, Algeria; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; An Nguyen Huu Bui, High School For The Gifted, VNUHCM, Ho Chi Minh City, Vietnam; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Albert Stalder, Herrliberg, Switzerland; Prajnanaswaroop S, Amrita University, Coimbatore, India; Titu Zvonaru, Comănești, Romania.
J509. Find the least 4-digit prime of the form $6k - 1$ that divides $8^{1010}11^{2020} + 1$.

_proposed by Titu Andreescu, University of Texas at Dallas, USA_

First solution by Polyahedra, Polk State College, USA
It is easy to check that 1013 is the least 4-digit prime of the form $6k - 1$. Also, $(8^211^4)^{505} + 1$ is divisible by $8^211^4 + 1 = 5^2 \cdot 37 \cdot 1013$.

Second solution by Joel Schlosberg, Bayside, NY, USA
For any such prime $p$,

$$(8^{505}11^{1010})^2 \equiv -1 \pmod{p},$$

so $-1$ is a quadratic residue of $p$, which is well-known to necessitate that $p \equiv 1 \pmod{4}$. Since $p \equiv -1 \pmod{6}$, $p \equiv 5 \pmod{12}$.

Since $1001 = 83 \cdot 12 + 5 = 7 \cdot 11 \cdot 13$ is composite, the smallest possible value of $p$ is $1001 + 12 = 1013$. By any of the standard primality tests (or simply testing that none of the primes $2, 3, 5, \ldots, 23, 29, 31 = \sqrt{1013}$ are divisors), 1013 is prime; since

$$8^211^4 = 925 \cdot 1013 - 1,$$

$$8^{1010}11^{2020} = (8^211^4)^{505} \equiv (-1)^{505} = -1 \pmod{1013}$$

so the smallest 4-digit prime $\equiv -1 \pmod{6}$ dividing $8^{1010}11^{2020} + 1$ is 1013.

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; David E. Manes, Oneonta, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania.
J510. Let $a, b, c$ be positive real numbers. Prove that

$$(1 + a)(1 + b)(1 + c) \geq \left(1 + \frac{2ab}{a + b}\right)\left(1 + \frac{2bc}{b + c}\right)\left(1 + \frac{2ca}{c + a}\right)$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that by the AM-GM inequality, $\frac{a + b}{2} \geq \sqrt{ab}$ with equality iff $a = b$, and similarly for the cyclic permutations of $a, b, c$, or

$$\left(1 + \frac{2ab}{a + b}\right)\left(1 + \frac{2bc}{b + c}\right)\left(1 + \frac{2ca}{c + a}\right) \leq \left(1 + \sqrt{ab}\right)\left(1 + \sqrt{bc}\right)\left(1 + \sqrt{ca}\right),$$

with equality iff $a = b = c$, and it suffices to show that

$$a + b + c + ab + bc + ca \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca} + a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$ 

Now, using again that $\frac{a + b}{2} \geq \sqrt{ab}$ and its cyclic permutations, we clearly have $a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$ with equality iff $a = b = c$, and noting that $\frac{ca + ab}{2} \geq a\sqrt{bc}$ and its cyclic permutations, we also have $ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$, with equality iff $ab = bc = ca$, ie iff $a = b = c$. The conclusion follows, equality holds iff $a = b = c$.

Also solved by Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Mohamed Ali, Houari Boumedien School, Algeria; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Idamia Abdelhamid, Groupe Scolaire Berrada, Casablanca, Morocco; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Mihály Benze, Brașov, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicușor Zlota, Traian Vuia Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, USA; Ioannis D. Sfikas, Athens, Greece; Shashwata Roy, Mumbai, India; Kevin Soto, Palacios Huarmey, Perú; Titu Zvonaru, Comânești, Romania.
Senior problems

S505. Find $k$ such that a triangle with sides $a, b, c$ is right if and only if

$$\sqrt[6]{a^6 + b^6 + c^6 + 3a^2b^2c^2} = k \max\{a, b, c\}$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan

Assume that $c^2 = a^2 + b^2$, then

$$\sqrt[6]{a^6 + b^6 + c^6 + 3a^2b^2c^2} = k \max\{a, b, c\}$$
$$\iff a^6 + b^6 + (a^2 + b^2)^3 + 3a^2b^2(a^2 + b^2) = k^6(a^2 + b^2)^3$$
$$\iff (k^6 - 2)(a^2 + b^2)^3 = 0,$$

we obtain $k = \sqrt[6]{2}$. Conversely, we assume that $k = \sqrt[6]{2}$ and $c > a, b$, then

$$\sqrt[6]{a^6 + b^6 + c^6 + 3a^2b^2c^2} = k \max\{a, b, c\}$$
$$\iff a^6 + b^6 + c^6 + 3a^2b^2c^2 = 2c^6$$
$$\iff (c^2 - (a^2 + b^2))(c^4 + (a^2 + b^2)c^2 + a^4 + b^4 - a^2b^2) = 0.$$

Since $c^4 + (a^2 + b^2)c^2 + a^4 + b^4 - a^2b^2 > 0$, we obtain $c^2 - (a^2 + b^2) = 0$. Therefore, the solution is $k = \sqrt[6]{2}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Arkady Alt, San Jose, CA, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania.
S506. Let \( x, y, z, t \) be real numbers, \( 0 \leq x, y, z, t \leq 1 \), such that

\[
(1 - x)(1 - y)(1 - z)(1 - t) = xyzt.
\]

Prove that

\[
x^2 + y^2 + z^2 + t^2 \geq 1.
\]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

If any one out of \( x, y, z, t \) is 0, then the RHS of the proposed condition is 0, or so is the LHS, and at least one out of \( x, y, z, t \) is also equal to 1, and the proposed inequality is trivially true, with equality iff \( (x, y, z, t) \) is a permutation of \( (1, 0, 0, 0) \). Or it suffices to prove the desired result when \( 0 < x, y, z, t < 1 \), which we will assume henceforth to hold.

Denote \( a = \frac{1}{x} - 1 \), \( b = \frac{1}{y} - 1 \), \( c = \frac{1}{z} - 1 \) and \( d = \frac{1}{t} - 1 \), and note that the proposed condition rewrites as \( abcd = 1 \). Note further that \( x = \frac{1}{1+a} \), \( y = \frac{1}{1+b} \), \( z = \frac{1}{1+c} \) and \( t = \frac{1}{1+d} \), and that the proposed inequality rewrites as

\[
\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} + \frac{1}{(1 + d)^2} \geq 1.
\]

We now note that

\[
\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} \geq \frac{1}{1 + ab}.
\]

Indeed, this is equivalent to

\[
1 + a^3b + ab^3 \geq 2ab + a^2b^2, \quad ab(a - b)^2 + (ab - 1)^2 \geq 0,
\]

clearly true and with equality iff \( a = b = 1 \). It then follows, using that \( abcd = 1 \), that

\[
\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} + \frac{1}{(1 + d)^2} \geq \frac{1}{1 + ab} + \frac{1}{1 + cd} = \frac{1}{1 + ab} + \frac{ab}{ab + 1} = 1.
\]

Equality holds iff \( a = b = c = d = 1 \), or equivalently iff \( x = y = z = t = \frac{1}{2} \).

The conclusion follows, equality holds iff either \( x = y = z = t = \frac{1}{2} \) or \( (x, y, z, t) \) is a permutation of \( (1, 0, 0, 0) \).

Also solved by Albert Stadler, Herrliberg, Switzerland; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Martín Lupin, IDRA Secondary School, Argentina; Daniel Pascuas, Universitat de Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Maalav Mehta, Prakash Higher Secondary School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece.
S507. If $a, b, c$ are real numbers such that $ax^2 + bx + c \geq 0$ for all real numbers $x$, prove that $4a^3 - b^3 + 4c^3 \geq 0$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Oana Prajitura, College at Brockport, SUNY, USA*

If $a = 0$ the condition becomes $bx + c \geq 0$ for all real numbers $x$ and thus $b = 0$ and $c \geq 0$.

This implies that

$$4a^3 - b^3 + 4c^3 = 4c^3 \geq 0$$

If $a \neq 0$ then the quadratic function must have negative discriminant and positive leading coefficient. Thus, $a > 0$ and

$$b^2 - 4ac \leq 0 \iff 4ac \geq b^2 \iff ac \geq \frac{b^2}{4} \geq 0$$

Since $ac \geq 0$ and $a > 0$ we conclude that $c \geq 0$.

If $b \leq 0$ then

$$4a^3 + 4c^3 \geq 0 \geq b^3.$$  

If $b > 0$ then

$$4a^3 + 4c^3 = 4(a^3 + c^3) \geq 8\sqrt{a^3c^3} = 8\sqrt{ac}^3 \geq 8\sqrt{\frac{b^2}{4}} = b^3.$$  

*Also solved by Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Pascuas, Universitat de Barcelona, Spain; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Ieko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.*
S508. Prove that in any triangle $ABC$,

$$\left(\frac{h_a}{l_a}\right)^2 + \left(\frac{h_b}{l_b}\right)^2 + \left(\frac{h_c}{l_c}\right)^2 - 2\frac{h_a h_b h_c}{l_a l_b l_c} = 1.$$ 

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Since $\angle (h_a, l_a) = \frac{\angle B - \angle C}{2}$, $\angle (h_b, l_b) = \frac{\angle C - \angle A}{2}$, and $\angle (h_c, l_c) = \frac{\angle A - \angle B}{2}$, we have $\frac{h_a}{l_a} = \cos\frac{\angle B - \angle C}{2}$, $\frac{h_b}{l_b} = \cos\frac{\angle C - \angle A}{2}$, and $\frac{h_c}{l_c} = \cos\frac{\angle A - \angle B}{2}$ and we will prove the given equality in the equivalent form

$$\cos^2\frac{A - B}{2} + \cos^2\frac{B - C}{2} + \cos^2\frac{C - A}{2} - 2\cos\frac{A - B}{2}\cos\frac{B - C}{2}\cos\frac{C - A}{2} = 1,$$

where $\frac{A - B}{2} + \frac{B - C}{2} + \frac{C - A}{2} = 0$, using the following lemma.

**Lemma:** If $x + y + z = 0$, then $\cos^2 x + \cos^2 y + \cos^2 z - 2 \cos x \cos y \cos z = 1$.

**Proof:** We have $z = -(x + y)$, giving $\cos z = \cos (x + y)$.

Squaring both sides, we find that

$$\cos^2 z = \cos^2 (x + y) = (\cos x \cos y - \sin x \sin y)^2 = \cos^2 x \cos^2 y + \sin^2 x \cos^2 y - 2 \cos x \cos y \sin x \sin y = \cos^2 x \cos^2 y + (1 - \cos^2 x)(1 - \cos^2 y) - 2 \cos x \cos y \sin x \sin y = 1 - \cos^2 x - \cos^2 y + 2 \cos x \cos y (\cos x \cos y - \sin x \sin y) = \cos z$$

and

$$\cos^2 x + \cos^2 y + \cos^2 z - 2 \cos x \cos y \cos z = 1.$$ 

Also solved by Daniel Lasaosa, Pamplona, Spain; Martín Lupin, IDRA Secondary School, Argentina; Marin Chirciu, Colegiul National Zinca Golescu, Pitești, Romania; Arkady Alt, San Jose, CA, USA; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland; Telemachus Baltzavias, Kerameies Junior High School, Kefalonia, Greece; Titu Zvonaru, Comănești, Romania.
S509. Solve in integers the equation

\[ 2(xy + 2)^2 - 6(x + y)^2 = (x + y - 1)^3 - 6. \]

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Li Zhou, Polk State College, USA

Considering the equation modulo 2 we see that \( x + y - 1 \) must be even. Thus \( x \) and \( y \) have opposite parity. By symmetry in \( x \) and \( y \) we may assume that \( x = 2m \) and \( y = 2n + 1 \). Then the equation reduces to

\[ (m + n + 1)^3 - (2mn + m + 1)^2 = 1. \]

It is well known that \( u^3 - v^2 = 1 \) has only the trivial solution \((u,v) = (1,0)\), so \( m = -n \), and thus

\[ 0 = -2n^2 - n + 1 = (1 - 2n)(1 + n). \]

Therefore, \((x,y) = (2,-1) \) or \((-1,2)\).

Also solved by Arkady Alt, San Jose, CA, USA; Martín Lupin, IDRA Secondary School, Argentina; Ioannis D. Sfikas, Athens, Greece.
S510. Consider an array of 49 consecutive integers whose median is a perfect square. Prove that the sum of the cubes of the 49 integers can be written as a sum of four perfect squares two of which are equal.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by the author*

Let the 49 integers be

\[ n^2 - 24, \ldots, n^2 - 1, n^2, n^2 + 1, \ldots, n^2 + 24. \]

The sum of their cubes is

\[ 49n^6 + 6n^2(1^2 + 2^2 + \cdots + 24^2), \]

as the term in \( n^4 \) has coefficient zero and the free term is zero as well.

Since

\[ 1^2 + 2^2 + \cdots + 24^2 = \frac{24(24 + 1)(2 \times 24 + 1)}{6} = 70^2, \]

the sum is equal to

\[ (7n^3)^2 + (2 \times 70n)^2 + (70n)^2 + (70n)^2, \]

as desired.

*Also solved by Albert Stadler, Herrliberg, Switzerland; Li Zhou, Polk State College, USA; Martín Lupin, IDRA Secondary School, Argentina; David Park, Peddie School, Hightstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Joel Schlosberg, Bayside, NY, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Satvik Dasariraju; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania; Daniel López-Aguayo, MSCI, Monterrey, Mexico.*

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Undergraduate problems

U505. Let $K$ be a field. Prove that the polynomial

$$X^n + X^2Y + XY + XY^2 + Y^n$$

is irreducible in the ring $K[X,Y]$, for all $n \geq 2$.

Proposed by Mircea Becheanu, Montreal, Canada

Solution by the author

Let’s denote $F(X,Y) = X^n + X^2Y + XY + XY^2 + Y^n$. Consider the ring homomorphism $\varphi : K[X,Y] \rightarrow K[X,Y]$ which is identity on $K$ and $\varphi(X) = X$, $\varphi(Y) = XY$. We mention that $\varphi$ is one to one because for any monomial $X^aY^b$ we have $\varphi(X^aY^b) = X^{a+b}Y^b$ and $\varphi(K[X,Y])$ consists in all polynomials for which every nonzero monomial $cX^aY^b$ which appears in it, has the property $a \geq b$.

We have

$$\varphi(F(X,Y)) = F(X,XY) = X^n + X^3Y + X^2Y^2 + X^nY^n = X^2[X^{n-2}(Y^n + 1) + XY(Y + 1) + Y].$$

For $n > 2$, the polynomial $X^{n-2}(Y^n + 1) + XY(Y + 1) + Y$ is irreducible in $K[X,Y]$ by Eisenstein criterion, by considering it as a polynomial in $X$. For $n = 2$, the polynomial $XY(Y + 1) + Y^n + Y + 1$ is irreducible because it is of degree 1 in $X$ and it is not divisible by a polynomial in $Y$. Therefore, the equality

$$\varphi(F(X,Y)) = X^2[X^{n-2}(Y^n + 1) + XY(Y + 1) + Y] \quad (1)$$

represents the splitting in irreducible factors of $\varphi(F(X,Y))$.

Assume now that $F(X,Y)$ splits in nonconstant factors

$$F(X,Y) = P(X,Y)Q(X,Y).$$

Then

$$\varphi(F(X,Y)) = \varphi(P(X,Y))\varphi(Q(X,Y)),$$

that is $\varphi(F(X,Y))$ splits in two nonconstant factors which are polynomials in $\varphi(K[X,Y])$. Because $K[X,Y]$ is a UFD, every such splitting is obtained by combining factors in its splitting in irreducible factors. From (1) it is clear that this is a contradiction.

Also solved by Li Zhou, Polk State College, USA; Prajnanaswaroopa S, Amrita University, Coimbatore, India.
U506. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(1 + x) = 1 + f(x) \quad \text{and} \quad f\left(\frac{1}{x}\right) = \frac{1}{f(x)}.$$ 

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

*First solution by Li Zhou, Polk State College, USA*

Clearly $f(x) = x$ is a solution. We show it is the only one. First, $f(1) = f(1/1) = 1/f(1)$, so $f(1) = 1$. By induction using $f(1 + x) = 1 + f(x)$, we get $f(m) = m$ for all $m \in \mathbb{N}$. So it suffices to show $f(x) = x$ for all $x \in (0, 1)$. Now it is well known that any such $x$ has a simple continued-fraction expansion

$$x = \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \cdots}}} = [x_1, x_2, \ldots],$$

where $x_i \in \mathbb{N}$ for all $i \geq 1$ (non-terminating case) or $x_i \in \mathbb{N}$ for $1 \leq i \leq n$ and $x_i = 0$ for all $i > n$ (terminating case).

For terminating $x = [x_1, x_2, \ldots, x_n]$, an easy induction using both functional equations yields

$$f(x) = \frac{1}{f([x_1, \ldots, x_n])} = \cdots = [f(x_1), f(x_2), \ldots, f(x_n)] = [x_1, x_2, \ldots, x_n] = x.$$

If $x = [x_1, x_2, \ldots]$ is non-terminating, then the expansion is unique and the functional equations force $f(x)$ to have the unique expansion $[f(x_1), f(x_2), \ldots] = [x_1, x_2, \ldots] = x$ as well. This completes the proof.
Second solution by Daniel Lasaosa, Pamplona, Spain

Note first that taking \( x = 1 \) in the second condition and since \( f \) takes only positive values, we obtain \( f(1) = 1 \), which after trivial induction using the first condition results in \( f(n) = n \) for all positive integer \( n \). Note further that if \( n < x < n + 1 \) we have \( n < f(x) < n + 1 \), since otherwise we would have either \( f(x - n) = f(x) - n < 0 \) in contradiction with \( f \) taking only positive values, or \( f(x - n) = f(x) - n \geq 1 \), for \( \frac{1}{x - n} > 1 \) and \( f(\frac{1}{x - n} - 1) = \frac{1}{f(x - n)} - 1 \leq 0 \), reaching a contradiction again. Therefore, \( |f(x) - x| < 1 \).

Assume that \( f(x) \) is not the identity, or a supremum \( 0 < s \leq 1 \) exists such that \( |f(x) - x| \leq s \) for all positive real \( x \). Therefore, a positive real \( \delta \) exists such that \( s \geq \delta > \sqrt{s + \frac{1}{4} - \frac{1}{2}} \), since either \( s \) is a maximum and \( x \) exists such that \( |f(x) - x| = s \), or \( s \) is not a maximum and real values of \( x \) exist such that \( |f(x) - x| \) is less than \( s \) but arbitrarily close to \( s \). Note that this is also possible because \( \sqrt{s + \frac{1}{4} - \frac{1}{2}} < s \) is equivalent to \( s^2 > 0 \), which is clearly true. Now, taking such an \( x \), we consider two cases:

Case 1: \( f(x) = x + \delta \). If \( x > 1 \), let \( m = \lfloor x \rfloor \), or after trivial induction using the first condition we have \( f(x) = f(x - m) + m \), hence substitution of \( x - m \) by \( x \) yields \( f(x) = x + \delta \) for some real \( 0 < x < 1 \). Note first that \( x + \delta < 1 \), since otherwise we would have \( \frac{1}{x} > 1 \) and \( f(x) > 1 \), or \( f(\frac{1}{x} - 1) = \frac{1}{f(x)} - 1 < 0 \), in contradiction with \( f \) taking only positive values. It follows that \( x < 1 - \delta \), and

\[
\frac{1}{x} - f\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{f(x)} = \frac{1}{x} - \frac{1}{x + \delta} = \frac{\delta}{x(x + \delta)} > \frac{\delta}{1 - \delta} > \delta + \delta^2 > s,
\]

absurd since it contradicts that \( s \) is the supremum of \( |f(x) - x| \).

Case 2: \( f(x) = x - \delta \). As before, subtracting \( m = \lfloor x \rfloor \) we find that we may assume that \( 0 < x < 1 \), or \( x - \delta < 1 - \delta \), and

\[
f\left(\frac{1}{x}\right) - \frac{1}{x} = \frac{1}{f(x)} - \frac{1}{x} = \frac{1}{x - \delta} - \frac{1}{x} = \frac{\delta}{x(x - \delta)} > \frac{\delta}{1 - \delta} > \delta + \delta^2 > s,
\]

absurd again.

We conclude that the only possible solution is \( f(x) = x \), which trivially satisfies the proposed conditions.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Pascuas, Universitat de Barcelona, Spain; M.A. Prasad, Mumbai, India; Ioannis D. Sfikas, Athens, Greece.
U507. Evaluate
\[ \int_{-1/3}^{1} \frac{1}{2x + \sqrt{x^2 + x + 2}} \, dx \]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Donaldo Garrido-Islas, Instituto Tecnológico y de Estudios Superiores de Monterrey, Monterrey, Mexico and Daniel López-Aguayo, MSCI, Monterrey, Mexico

We perform the so-called Euler’s substitution. Let \( \sqrt{x^2 + x + 2} = t - x \), hence \( x = \frac{t^2 - 2}{1 + 2t} \). Therefore, \( dx = \frac{2(t^2 + t + 2)}{(1 + 2t)^2} \). Also, note that the lower limit is now 1 and the upper is 3. A little algebra shows that the integrand equals
\[ \frac{2(t^2 + t + 2)}{(t + 1)(2t + 1)(3t - 2)}. \]
Since the latter expression is a proper rational function, we apply partial fractions to obtain:
\[ \int_{1}^{3} \frac{2(t^2 + t + 2)}{(t + 1)(2t + 1)(3t - 2)} \, dt = \int_{1}^{3} \left( \frac{4}{5(t + 1)} - \frac{2}{2t + 1} + \frac{8}{5(3t - 2)} \right) \, dt \]
Computing the indefinite integral yields
\[ \frac{4}{5} \ln|t + 1| - \ln|2t + 1| + \frac{8}{15} \ln|3t - 2| \]

Evaluating from \( t = 1 \) to \( t = 3 \) gives \( \frac{4 \ln(2)}{5} + \ln(3) - \frac{7 \ln(7)}{15} \), and we are done.

Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Daniel Pascuas, Universitat de Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Marin Chiriciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Ioannis D. Sfikas, Athens, Greece.
U508. For positive integer $n$, let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let $M$ be the $(2^n - 1) \times (2^n - 1)$ matrix whose $(i, j)$ entry is $m_{ij} = |S_i \cup S_j|$. Find the determinant of $M = (m_{ij})$.

Proposed by Li Zhou, Polk State College, USA

Solution by the author

If $n = 1$, then $M = [1]$ and det($M$) = 1. Suppose $n \geq 2$. Interchanging $S_i$ and $S_j$ corresponds to switching the $i$th and $j$th columns and rows, thus leaves the determinant invariant. Hence, for $n = 2$, with $S_1 = \{1\}$, $S_2 = \{2\}$, and $S_3 = \{1, 2\}$, we get

$$
\text{det}(M) = \text{det} \begin{bmatrix}
1 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 2
\end{bmatrix} = 2.
$$

Now consider $n \geq 3$. Let $S_j = \{1\}, S_k = \{2, \ldots, n\}$ (the complement of $S_j$), $S_p = \{2\}$, and $S_q = \{1, 3, \ldots, n\}$ (the complement of $S_p$). Then for any $i = 1, \ldots, 2^n - 1$, $|S_i \cap S_j| + |S_i \cap S_k| = |S_i|$, thus by PIE,

$$
m_{ij} + m_{ik} = |S_i \cup S_j| + |S_i \cup S_k| = |S_i| + 1 - |S_i \cap S_j| + |S_i| + (n - 1) - |S_i \cap S_k| = n + |S_i|.
$$

Likewise, $m_{ip} + m_{iq} = n + |S_i|$ for all $i = 1, \ldots, 2^n - 1$. Since these column operations (adding the $k$th to the $j$th column and adding the $q$th to the $p$th column) lead to two identical columns, we conclude that det($M$) = 0 for all $n \geq 3$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joel Schlosberg, Bayside, NY, USA; M.A. Prasad, Mumbai, India.
U509. Prove that for any \( x > 1 \), the following inequalities hold.

\[
\log\left(\frac{1 + x^2}{x^2 - 2x + 2}\right)^{\frac{1}{x-1}} < \arctan(x) - \arctan(x-1) < \log\left(\frac{1 + x^2}{x^2 - 2x + 2}\right)^{\frac{1}{2(x-1)}}.
\]

Proposed by Besfort Shala, University of Primorska, Slovenia

Solution by Li Zhou, Polk State College, USA

For any \( x > 1 \),

\[
\log\frac{x^2 + 1}{x^2 - 2x + 2} - 2(x - 1) (\arctan x - \arctan(x-1)) = \int_{x-1}^{x} \frac{2t - 2(x - 1)}{t^2 + 1} \, dt > 0,
\]
establishing the right inequality. On the other hand,

\[
(2x - 1) (\arctan x - \arctan(x-1)) - \log\frac{x^2 + 1}{x^2 - 2x + 2} = \int_{x-1}^{x} \frac{2x - 1 - 2t}{t^2 + 1} \, dt
\]

\[
= \int_{x-1}^{(2x-1)/2} \frac{2x - 1 - 2t}{t^2 + 1} \, dt + \int_{(2x-1)/2}^{x} \frac{2x - 1 - 2t}{t^2 + 1} \, dt
\]

\[
= \int_{x-1}^{(2x-1)/2} \frac{2x - 1 - 2t}{t^2 + 1} \, dt - \int_{x-1}^{(2x-1)/2} \frac{2x - 1 - 2u}{(2x - 1 - u)^2 + 1} \, du
\]

\[
= \int_{x-1}^{(2x-1)/2} (2x - 1 - 2t) \left( \frac{1}{t^2 + 1} - \frac{1}{(2x - 1 - t)^2 + 1} \right) \, dt > 0,
\]
establishing the left inequality.

We also notice that the problem would have been more elusive if the “footprint” \( \arctan(x) - \arctan(x-1) \) were covered up by \( \arctan\frac{1}{x^2 - x + 1} \).

Also solved by Arkady Alt, San Jose, CA, USA; Daniel Lasaosa, Pamplona, Spain; Daniel Pascuas, Universitat de Barcelona, Spain; Oana Prajitura, College at Brockport, SUNY, USA; Albert Stadler, Herrliberg, Switzerland.
U510. Evaluate
\[ \int_0^\pi \frac{x \sin x}{2021 + 4 \sin^2 x} \, dx. \]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Abdelouahed Hamdi, Qatar University, Doha, Qatar

We use the change of variable: \( x = \pi - t \implies dx = -dt \) and we get:

\[
I = \int_0^\pi \frac{x \sin(x)}{2021 + 4 \sin^2(x)} \, dx
\]

\[
= \int_0^\pi \frac{(\pi - t) \sin(\pi - t)}{2021 + 4 \sin^2(\pi - t)} \, (-dt)
\]

\[
= \int_0^\pi \frac{(\pi - t) \sin(t)}{2021 + 4 \sin^2(t)} \, dt
\]

\[
= \int_0^\pi \frac{\pi \sin(t)}{2021 + 4 \sin^2(t)} \, dt - \int_0^\pi \frac{t \sin(t)}{2021 + 4 \sin^2(t)} \, dt
\]

\[
= \int_0^\pi \frac{\pi \sin(t)}{2021 + 4 \sin^2(t)} \, dt - I
\]

\[
2I = \pi \int_0^\pi \frac{1}{2021 + 4(1 - \cos^2(t))} \, dt
\]

\[
2I = \pi \int_0^\pi \frac{1}{2025 - 4 \cos^2(t)} \, dt
\]

Let \( u = -\cos(t) \implies du = \sin(t) \, dt \)

\[
2I = \pi \int_{-1}^1 \frac{1}{2025 - 4u^2} \, du \quad \text{the integrand is an even function}
\]

\[
2I = 2\pi \int_0^1 \frac{1}{2025 - 4u^2} \, du = \pi \int_0^1 \frac{1}{a^2 - u^2} \, du = \frac{\pi}{2} \left[ \int_0^1 \frac{1}{a^2 - u^2} \, du + \int_0^1 \frac{1}{a + u} \, du \right]
\]

\[
2I = \frac{\pi}{2} \int_{-1}^1 \frac{\ln |a + u|}{a - u} \, du = \frac{\pi}{4a} \left[ \frac{\ln |a + 1|}{a - 1} - \frac{\ln |a - 1|}{a + 1} \right]
\]

\[
I = \frac{\pi}{180} \ln(47/43).
\]

Also solved by Daniel Lasaosa, Pamplona, Spain; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel Pascuas, Universitat de Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Matthew Too, The College at Brockport, SUNY, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Sergio Esteban Muñoz, Universidad de Buenos Aires, Argentina; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland; Telemachus Baltsavias, Kerameius Junior High School, Kephalonia, Greece; Titu Zvonaru, Comănești, Romania; Donaldo Garrido-Islas, Instituto Tecnológico y de Estudios Superiores de Monterrey, Monterrey, Mexico and Daniel López-Aguayo, MSCI, Monterrey, Mexico.
Olympiad problems

O505. Let $a$, $b$, $c$, $d$ be positive real numbers such that

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$ 

Prove that

$$3\left(\frac{a^2 + b^2 + c^2 + d^2}{a + b + c + d}\right) + 1 \geq a + b + c + d.$$ 

*Proposed by Marius Stănean, Zalău, Romania*

*Solution by the author*

We have the following identity

$$a^3 + b^3 + c^3 + d^3 = (a + b + c + d)^3 - 3(a + b + c + d) \sum ab + 3(abc + bcd + cda + dab)$$

$$= (a + b + c + d) \left[ (a + b + c + d)^2 - 3 \sum ab + 3abcd \right].$$

Using this, the our inequality can be rewrite as

$$\frac{a^3 + b^3 + c^3 + d^3}{a + b + c + d} + \frac{a + b + c + d}{2} \geq 3abcd,$$

or

$$a^3 + b^3 + c^3 + d^3 + \frac{(a + b + c + d)^2}{2} \geq 3(abc + bcd + cd + dab),$$

or, after we homogenize

$$a^3 + b^3 + c^3 + d^3 + \sqrt[4]{\frac{abcd(a + b + c + d)^5}{4(abc + bcd + cda + dab)}} \geq 3(abc + bcd + cda + dab).$$

But, by AM-GM Inequality and Maclaurin’s Inequality

$$\sqrt[4]{\frac{abcd(a + b + c + d)^5}{4(abc + bcd + cda + dab)}} \geq \sqrt[7]{\frac{abcd(a + b + c + d)^7}{4(abc + bcd + cda + dab)(a + b + c + d)^2}}$$

$$\geq \sqrt[4]{\frac{4^4 a^2 b^2 c^2 d^2 \cdot 4^2 (abc + bcd + cda + dab)}{4(abc + bcd + cda + dab)(a + b + c + d)^2}}$$

$$= \frac{32abcd}{a + b + c + d}.$$

So, it remains to prove that

$$a^3 + b^3 + c^3 + d^3 + \frac{32abcd}{a + b + c + d} \geq 3(abc + bcd + cda + dab),$$

which is a well-known inequality.

*Also solved by Kevin Soto, Palacios Huarmey, Perú; Albert Stadler, Herrliberg, Switzerland.*
O506. Let \( a \) be a nonnegative integer. Find all pairs \((x, y)\) of nonnegative integers such that
\[
(a^2 + 1) (x^3 - 2axy + y^3) = a^2 - xy.
\]

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that if the RHS is positive, then its absolute value is at most \( a^2 \), whereas the absolute value of the LHS is a multiple of \( a^2 + 1 \). Therefore, the RHS must be a nonpositive multiple of \( a^2 + 1 \), or a nonnegative integer \( k \) exists such that \( a^2 - xy = -k(a^2 + 1) \). Then, \( xy = a^2(k + 1) + k \), and \( x^3 + y^3 = 2axy - k = 2a^3(k + 1) + 2ak - k \), and since \( a, x, y \) are nonnegative, the AM-GM inequality applied to \( x^3, y^3, a^3 \) yields
\[
a^3(2k + 3) + 2ak - k = x^3 + y^3 + a^3 \geq 3axy = a^3(3k + 3) + 3ak, \quad k \left( a^3 + a + 1 \right) \leq 0.
\]

However, since \( a^3 + a + 1 > 0 \), it follows that for the AM-GM to hold we need \( k = 0 \), in which case equality occurs, yielding \( x = y = a \). Substitution in the proposed equation indeed yields both sides being equal to \( 0 \), or this is the only possible solution, and it holds for any nonnegative integer \( a \).

Also solved by Martín Lupin, IDRA Secondary School, Argentina; Ioannis D. Sfikas, Athens, Greece; Li Zhou, Polk State College, USA.
O507. Let $a, b, c, d > 0$ and $a^4 + b^4 + c^4 + d^4 = 4$. Prove that

\[
\frac{a^2b}{a^4 + b^3 + c^2 + d} + \frac{b^2c}{b^4 + c^3 + d^2 + a} + \frac{c^2d}{c^4 + d^3 + a^2 + b} + \frac{d^2a}{d^4 + a^3 + b^2 + c} \leq \frac{16}{(a + b + c + d)^2}.
\]

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

Using Cauchy-Schwarz Inequality we get

\[
(a^4 + b^3 + c^2 + d)\left(\frac{1}{a^2} + \frac{1}{b} + 1 + d\right) \geq (a + b + c + d)^2 = 16,
\]

Namely,

\[
\frac{1}{a^4 + b^3 + c^2 + d} \leq \frac{1}{16} \left(\frac{1}{a^2} + \frac{1}{b} + 1 + d\right)
\]

Which yields to

\[
\frac{a^2b}{a^4 + b^3 + c^2 + d} \leq \frac{1}{16}(b + a^2 + a^2b + a^2bd)
\]

Similarly for other permutations. Summing up four of them and applying the quaternion mean inequality results in

\[
\frac{a^2b}{a^4 + b^3 + c^2 + d} + \frac{b^2c}{b^4 + c^3 + d^2 + a} + \frac{c^2d}{c^4 + d^3 + a^2 + b} + \frac{d^2a}{d^4 + a^3 + b^2 + c} \leq \frac{1}{16} \left(\sum a + \sum a^2 + \sum a^2b + \sum a^2bd\right)
\]

Now,

\[
\sum a^2 \leq \sqrt{4\sum a^4} = 4, \sum a \leq \sqrt{4\sum a^2} = 4,
\]

\[
\sum a^2b \leq \frac{1}{4}\sum (a^4 + a^4 + b^4 + 1) = \frac{3}{4}\sum a^4 + 1 = 4,
\]

\[
\sum a^2bd \leq \frac{1}{4}\sum (a^4 + a^4 + b^4 + d^4) = \sum a^4 = 4,
\]

and the conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland.
Let $a, b, c$ be positive real numbers such that $a + b + c = 3$. Prove that
\[
\frac{a}{b(a + 5c)^2} + \frac{b}{c(b + 5a)^2} + \frac{c}{a(c + 5b)^2} \geq \frac{1}{4(\sqrt{a} + \sqrt{b} + \sqrt{c})}.
\]

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

**Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam**

Applying the Cauchy-Schwarz inequality we obtain
\[
\frac{a}{b(a + 5c)^2} + \frac{b}{c(b + 5a)^2} + \frac{c}{a(c + 5b)^2} = \frac{a^2}{ab(a + 5c)^2} + \frac{b^2}{bc(b + 5a)^2} + \frac{c^2}{ca(c + 5b)^2} \geq \left(\frac{a + b + c}{ab + bc + ca}\right)^2.
\]

On the other hand, the Cauchy-Schwarz inequality also gives us
\[
\frac{a}{a + 5c} + \frac{b}{b + 5a} + \frac{c}{c + 5b} \geq \frac{(a + b + c)^2}{a(a + 5c) + b(b + 5a) + c(c + 5b)}
\]
\[
= \frac{(a + b + c)^2}{(a + b + c)^2 + 3(ab + bc + ca)}
\]
\[
\geq \frac{(a + b + c)^2}{2(a + b + c)^2}
\]
\[
= \frac{1}{2}.
\]

Combining these two inequalities we get
\[
\frac{a}{b(a + 5c)^2} + \frac{b}{c(b + 5a)^2} + \frac{c}{a(c + 5b)^2} \geq \frac{1}{4(ab + bc + ca)}.
\]

Therefore, it suffices to show that
\[
ab + bc + ca \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.
\]

This is equivalent to
\[
9 = (a + b + c)^2 \leq a^2 + b^2 + c^2 + 2(\sqrt{a} + \sqrt{b} + \sqrt{c}).
\]

But this follows from the following
\[
a^2 + \sqrt{a} + \sqrt{a} \geq 3a,
\]
\[
b^2 + \sqrt{b} + \sqrt{b} \geq 3b,
\]
\[
c^2 + \sqrt{c} + \sqrt{c} \geq 3c.
\]

The equality holds if and only if $a = b = c = 1$.

Also solved by Albert Stadler, Herrliberg, Switzerland; An Nguyen Huu Bui, High School For The Gifted, VNUHCM, Ho Chi Minh City, Vietnam; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Kevin Soto, Palacios Huarmey, Perú.
O509. Prove that for any positive real numbers $a, b, c$

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \frac{27(a^3 + b^3 + c^3)}{(a + b + c)^3} + \frac{21}{4}.$$ 

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Assuming $a + b + c = 1$ (due homogeneity of the inequality) and denoting $p := ab + bc + ca, q := abc$ we obtain

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \frac{27(a^3 + b^3 + c^3)}{(a + b + c)^3} - \frac{21}{4} = \frac{p}{q} - 27(1 + 3q - 3p) - \frac{21}{4}.$$

Noting that $p = ab + bc + ca \leq \left( \frac{a + b + c}{3} \right)^2 = \frac{1}{3}$ then, denoting $t := \sqrt{1 - 3p}$, we obtain $p = \frac{1 - t^2}{3}$, where $t \in [0, 1]$.

Since the criterion of solvability of Viete’s system

\[
\begin{cases}
    a + b + c = 1 \\
    ab + bc + ca = p = \frac{1 - t^2}{3} \\
    abc = q
\end{cases}
\]

in real $a, b, c$ is

$$\frac{(1 - 2t)(1 + t)^2}{27} \leq q \leq \frac{(1 + 2t)(1 - t)^2}{27},$$

then

$$\frac{p}{q} - 27(1 + 3q - 3p) - \frac{21}{4} \geq \frac{1 - t^2}{3} \frac{1}{(1 + 2t)(1 - t)^2} - 27 \left( 1 + 3 \cdot \frac{(1 + 2t)(1 - t)^2}{27} - 3 \cdot \frac{1 - t^2}{3} \right) =$$

$$\frac{9(t + 1)}{(1 - t)(2t + 1)} - 3 \left( 2t^3 + 6t^2 + 1 \right) - \frac{21}{4} = \frac{3 \left( 4t^3 + 14t^2 + 5t + 1 \right)}{4(2t + 1)(1 - t)} \geq 0$$

where equality occurs iff $t = \frac{1}{2} \iff p = \frac{1}{4}, q = \frac{(1 + 2 \cdot (1/2)) (1 - (1/2))^2}{27} = \frac{1}{54}$.

From cubic equation $x^3 - x^2 + \frac{1}{4}x - \frac{1}{54} = 0 \iff \frac{1}{108} (3x - 2) (6x - 1)^2 = 0$, we obtain $a = b = \frac{1}{6}, c = \frac{2}{3}$.

Also solved by Albert Stadler, Herrliberg, Switzerland; An Nguyen Huu Bui, High School For The Gifted, VNUHCM, Ho Chi Minh City, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.
O510. Let \( ABCDE \) be a convex pentagon with \( \angle BCD = \angle ADE \) and \( \angle BDC = \angle AED \). The circumcircle of triangle \( CDE \) meets lines \( DA \) and \( DB \) for the second time at points \( P \) and \( Q \), respectively. Lines \( CP \) and \( QE \) intersect at \( X \). Prove that \( ADBX \) is a parallelogram.

\( \text{Proposed by Waldemar Pompe, Warsaw, Poland} \)

\( \text{Solution by Li Zhou, Polk State College, USA} \)

Suppose that \( CB \) and \( EA \) intersect the circumcircle of \( CDE \) again at \( M \) and \( N \), respectively. Since \( \angle MCD = \angle PDE \), \( EM \parallel DP \), so \( \angle EQD = \angle MCP \). Therefore, \( BCQX \) is cyclic, thus

\[ \angle XQD = \angle MCP = \angle PDQ \]

Hence, \( XB \parallel PD \). Likewise, \( XA \parallel QD \), completing the proof.

\( \text{Also solved by Martín Lupin, IDRA Secondary School, Argentina.} \)