On mixtilinear incircles

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Abstract

In this article, I explore the rich and stunning configuration of mixtilinear incircles. I present a wide range of results (encompassing the typical ones and other not well-known facts), solve some example problems and provide many exercises to the reader. Basic knowledge of inversion (particularly, $\sqrt{bc}$ inversion), homothety for circles and harmonic bundles is strongly recommended.

I am heavily indebted to AoPS users `math_pi_rate` and `enhanced` for their valuable suggestions. Without their help, this article would not have been completed.

Contents

1. Preliminaries

Definition 1.

Let ABC be a triangle. The A-mixtilinear incircle (ω_A) is the circle internally tangent to the circumcircle of $\triangle ABC$ also touching sides AB and CA.

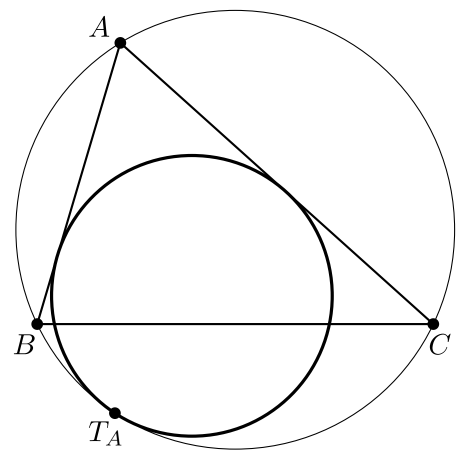

Figure 1.1. The A-mixtilinear incircle.

Naturally, there also exist mixtilinear incircles for vertices B (say ω_B) and C (say ω_C). Note that we use the article "the". Indeed, there is only one mixtilinear incircle for each vertex of $\triangle ABC$.

Lemma 1.

Given a triangle ABC, there exists a unique mixtilinear incircle corresponding to each vertex.

Let us provide a method of construction of the A-mixtilinear incircle and show it actually works.

Construction. Let I be the incenter of $\triangle ABC$. The line through I perpendicular to AI meets AB and AC at D and E, respectively. Denote by M the midpoint of $\widehat{BAC}$ and T_A the point where the ray MI intersects the circumcircle of $\triangle ABC$. The A-mixtilinear incircle coincides with the circumcircle of $\triangle DT_AE$.

Proof. Perform an inversion centered at A with radius $\sqrt{AB \cdot AC}$ followed by a reflection across AI. This transformation swaps B and C, (ABC) and BC, I and the A-excenter I_A. It is easy to prove that it also sends D and E to the contact points of the A-excircle with AC and AB, say D' and E', respectively. Denote by F and M' the point where the A-excircle touches BC and the foot of the external bisector of $\angle BAC$, respectively. Points M and M' are interchanged. Since $M'AFI_A$ is clearly cyclic, its circumcircle is sent to MI, so T_A and F are mapped to each other. We conclude that the A-excircle is mapped to the circumcircle of $\triangle DT_AE$; therefore, (DT_AE) must touch AB and AC and be internally tangent to (BAC) at T_A. The uniqueness holds immediately. □

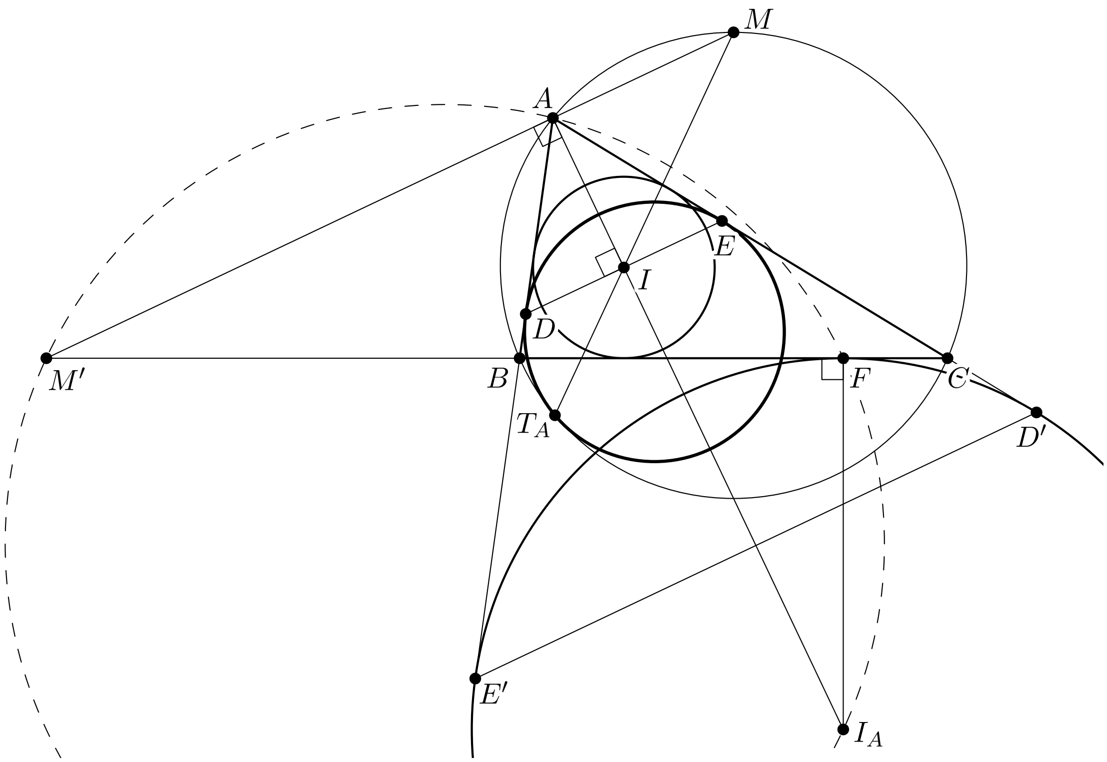

Figure 1.2. Proving the existence and uniqueness of ω_A via inversion.

We must remark two situations. Aside the uniqueness of ω_A, we have just inferred that its tangency point with (BAC), the incenter of $\triangle ABC$ and the midpoint of $\widehat{BAC}$ are collinear; in other words, T_AI bisects arc $\widehat{BAC}$. It is extremely important to recognize such a collinearity! Moreover, the incenter is the midpoint of the segment formed by the contact points of ω_A and sides AB, AC. Both facts are tremendously useful. Let us establish them separately.

Lemma 2.

The line joining the contact point of ω_A with (BAC) and the incenter of $\triangle ABC$ intersects $\widehat{BAC}$ at its midpoint.

Lemma 3.

The incenter of $\triangle ABC$ is the midpoint of the segment joining the tangency points of ω_A with sides AB and AC.

The previous proof was a bit tough! In the next section, we check that there is a more natural way to obtain the A-mixtilinear incircle.

2. A more natural approach

A first crucial fact is introduced below.

Lemma 4.

Let Ω and ω be two circles tangent at D, so that ω is in the interior of Ω. A chord AB of Ω is drawn in such a way AB is tangent to ω at C; then, DC bisects arc $\widehat{AB}$ not containing D.

Proof. Define O to be the center of Ω. Observe that D is the exsimilicenter of Ω and ω; then, if DC meets Ω again at M, the tangent to Ω through M (say ℓ) must be parallel to AB. Since $OM \perp \ell$, we deduce $OM \perp AB$, i.e. OM is the perpendicular bisector of $\overline{AB}$, which gives us the desired conclusion. □

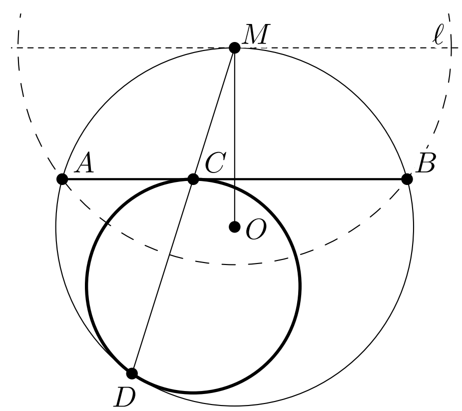

Figure 2.1. Line DC bisects arc $\widehat{AMB}$.

There is a more instructive proof via inversion[1]. Additionally, we can provide an easier argument for lemma 3.

Second proof of lemma 3. Let $X = \overline{T_AD} \cap (ABC), X \neq T_A$ and $Y = \overline{T_AE} \cap (ABC), Y \neq T_A$. By lemma 4, X and Y are the midpoints of arcs $\widehat{BXA}$ and $\widehat{CYA}$, therefore CX and BY intersect at I. Applying Pascal's theorem to the hexagon T_AXCABY we conclude that E, I and D are collinear. Since $AE = AD$ and $\angle DAI = \angle EAI$, line AI is the perpendicular bisector of $\overline{DE}$, from which the result follows immediately. □

Let us find some equal angles and isosceles trapezoids.

Lemma 5.

Lines AT_A and MT_A are isogonal conjugates of $\triangle XT_AY$.

Proof. It is clear that $AM \parallel DE$ due to $\angle IAM = \angle AID = 90°$. Recall that X and Y are the respective circumcenters of triangles AIB and AIC, so $XY \perp AI$ and then $XY \parallel AM$, which gives $\widehat{AX} = \widehat{MY}$, thus $\angle AT_AX = \angle YT_AM$. □

[1]Have a look at exercise 6.1.

This previous fact allows us to derive another proof of lemma 2.

Second proof of lemma 2. We have just found that $\angle AT_AX = \angle YT_AM$, i.e. $\angle AT_AD = \angle ET_AM$. Note that AT_A is a symmedian of $\triangle DT_AE$, whence $\angle AT_AD = \angle ET_AI$; in other words, $\angle ET_AM = \angle ET_AI$. Since M and I lie on the same side respect to T_AE, points M, I and T_A must be located on the same line. □

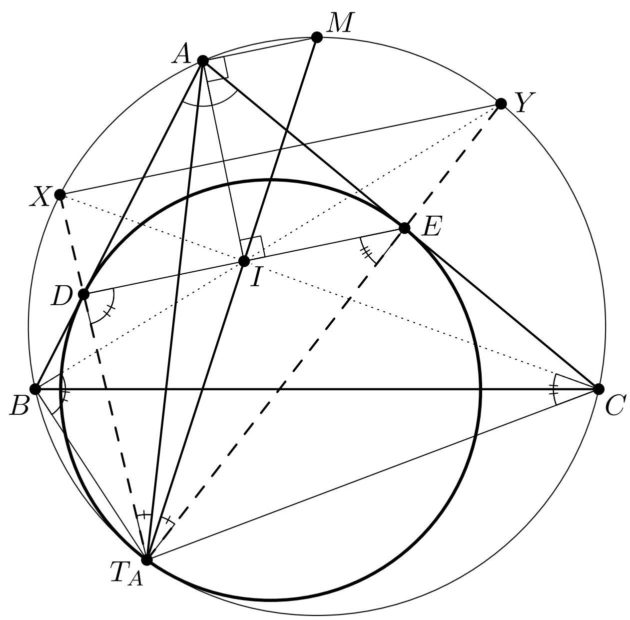

Figure 2.2. I is the midpoint of $\overline{DE}$ and T_AI bisects arc $\widehat{BAC}$.

The proof of lemma 5 indicates that $AXYM$ is an isosceles trapezoid. Furthermore, we have $\widehat{MX} = \widehat{AY} = \widehat{YC}$ and $\widehat{MY} = \widehat{AX} = \widehat{XB}$, so,

Lemma 6.

Quadrilaterals $AXYM$, $CYMX$ and $BXMY$ are isosceles trapezoids.

By using lemmas 2 and 6, we may also obtain that $\angle IT_AE = \angle MT_AY = \angle ACX = \angle ICE$ and similarly, $\angle IT_AD = \angle IBD$, thereby,

Lemma 7.

Quadrilaterals BT_AID and CT_AIE are cyclic.

Let us say more about BT_AID and CT_AIE. In fact,

$$\angle ICY = \frac{1}{2}\left(\widehat{AY} + \widehat{XA}\right) = \frac{1}{2}\left(\widehat{CY} + \widehat{YM}\right) = \angle CT_AI$$

Analogously, $\angle BT_AI = \angle IBX$. Taking into account that $YI = YC$ and $XI = XB$, we deduce that,

Lemma 8.

Lines BI and CY are tangent to the circumcircle of CT_AIE, whereas CI and BX are tangent to the circumcircle of BT_AID.

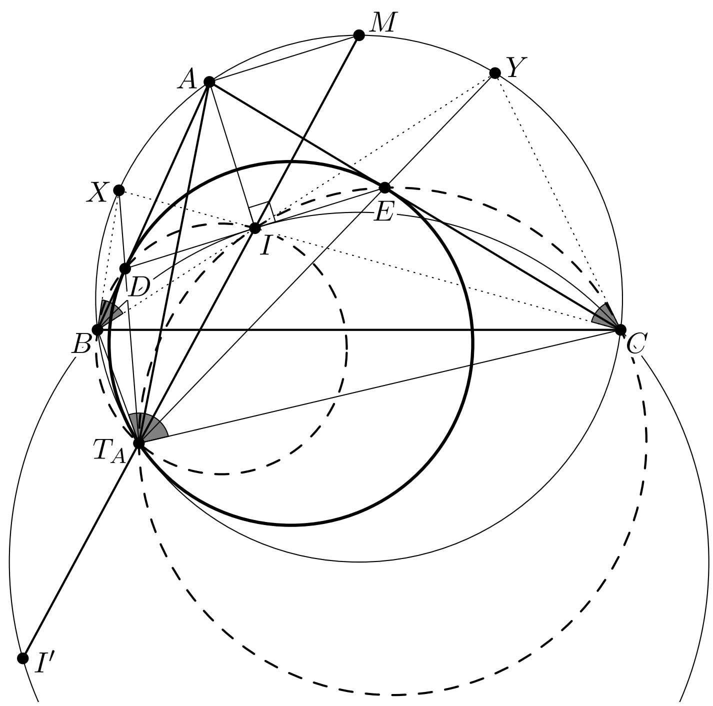

Figure 2.3. Cyclic quadrilaterals and isosceles trapezoids.

This result means that,

Lemma 9.

Quadrilaterals BT_AID and CT_AIE are harmonic.

The above fact implies that $B(X, I; D, T_A)$ is a harmonic pencil. Projecting from B to (ABC), we realize that,

Lemma 10.

Quadrilateral AXT_AY is harmonic.

According to lemma 8, we obtain that $\angle BIT_A = \angle ICT_A$ and $\angle T_ABI = \angle T_AIC$, then $\triangle BT_AI \sim \triangle IT_AC$, in other words,

Lemma 11.

The tangency point of ω_A and (BAC) is the center of spiral similarity taking $\overline{BI}$ to $\overline{IC}$.

Together with lemma 5 of [1], the aforesaid property leads us to derive the following claim:

Lemma 12.

Let I' be the symmetric of I with respect to T_A. So, I' lies on the circumcircle of $\triangle BIC$.

It is straightforward to ensure that MB and MC are tangent to (BIC); thus, it actually occurs that $I'BIC$ is harmonic as well.

Let K and L be the points where the incircle touches AB and AC, respectively. Let $R = \overline{CI} \cap \overline{KL}$ and $S = \overline{BI} \cap \overline{KL}$. Then, we have,

Lemma 13.

Line T_AI passes through the midpoint N of segment RS.

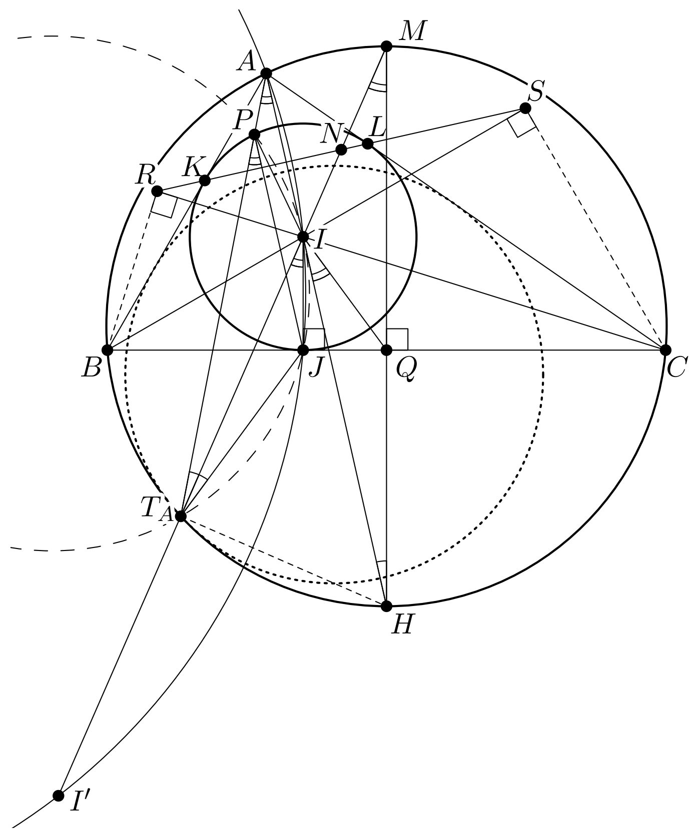

Figure 2.4. Another midpoint which lies on T_AI.

Proof. It is routine to show that $\angle BRC = 90° = \angle BSC$ (in fact, it is a simple task to obtain that $\measuredangle RKB = \measuredangle RIB$ and $\measuredangle CLS = \measuredangle CIS$ in terms of directed angles, so $BRKI$ and $CSLI$ are cyclic quadrilaterals and the result follows), providing that $BRSC$ is cyclic. Since T_A carries $\overline{BI}$ to $\overline{IC}$ by lemma 12, it is known that IT_A must be a symmedian of $\triangle BIC$; therefore, being $\overline{RS}$ and $\overline{BC}$ antiparallel, we conclude that N, I and T_A are collinear. ☐

Define J as the point where the incircle touches BC, and I' as constructed in lemma 12. We get that,

Lemma 14.

Point T_A is the center of spiral similarity mapping AI to IJ.

Proof. Let $H = \overline{AI} \cap (ABC)$, $H \neq A$ and Q the midpoint of $\overline{BC}$. As consequence of lemma 3, we know that $\angle IT_AH = 90°$. Since H is the center of (BIC), we have that IJ and IH are isogonals of $\triangle BIC$. Thus, because IT_A and IQ are isogonals, too, we infer that $\angle JIQ = \angle T_AIH$; therefore, $\triangle T_AIH \sim \triangle JIQ$, i.e. I carries $\overline{T_AH}$ to $\overline{JQ}$, so it also maps $\overline{T_AJ}$ to $\overline{HQ}$, implying that $\triangle T_AIJ \sim \triangle HIQ$. But

$$HI^2 = HC^2 = HQ \cdot HM$$

hence $\triangle HIQ \sim \triangle HMI$ and we clearly have that $\triangle T_AIA \sim \triangle HMI$. We conclude that $\triangle T_AIJ \sim \triangle T_AAI$. The result follows. □

As an immediate result of this previous fact, we deduce that

Lemma 15.

Points A, I, J and I' all lie on the same circle.

We invoke this last result in the example section.

3. More interesting properties

We have said a lot about the connection between the A-mixtilinear incircle and the incenter itself. Oppositely, we have disregarded other important points and connections. We solve this issue in the actual section.

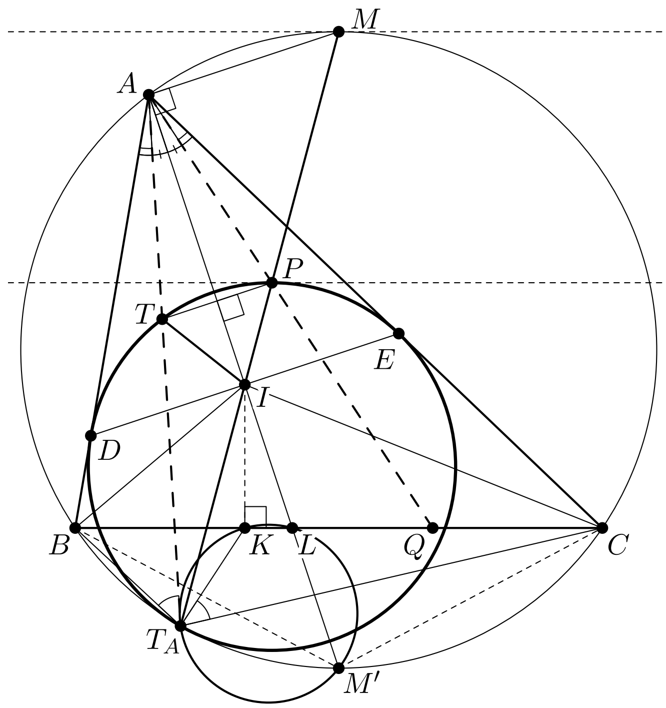

Figure 3.1. AT_A and AQ are isogonal conjugates of $\triangle ABC$, as AT_A and KT_A are of $\triangle BT_AC$.

Lemma 16.

Denote by Q the contact point of the A-excircle and BC. Then, $\angle BAT_A = \angle QAC$, i.e. AT_A and AQ are isogonal with respect to $\triangle ABC$.

First proof. Recall the proof of lemma 2. We concluded that Q and T_A are images of each other under the composition of inversion at A with radius $\sqrt{AB \cdot AC}$ and reflection across AI; hence, $\angle BAT_A = \angle QAC$, as desired. $\square$

Second proof. Let T and P be the second intersection points of AT_A with ω_A and MT_A with ω_A, respectively. Given that T_A is the exsimilicenter of ω_A and (BAC), we know $TP \parallel AM \therefore IA \perp TP$. Because the circumcenter of ω_A lies on AI, we infer that AI is the perpendicular bisector of $\overline{TP}$, which gives us

$$\angle BAT_A = \angle BAT = \angle BAI - \angle TAI = \angle CAI - \angle PAI = \angle CAP$$

Define ℓ_P and ℓ_M as the tangents to ω_A, (BAC) passing through P, M, respectively. Note that $\ell_P \parallel \ell_M$ and $\ell_M \parallel BC$, thus $\ell_P \parallel BC$. Since A is the exsimilicenter of A-excircle and ω_A, this implies that A, P, Q are collinear, and consecutively, $\angle BAT_A = \angle PAC = \angle QAC$. $\square$

Lemma 17.

Denote by K the common point of the incircle of $\triangle ABC$ and BC. Lines AT_A and KT_A are isogonal conjugates of $\triangle BT_AC$, i.e. $\angle BT_AK = \angle AT_AC$.

Proof. Evidently, $\angle BT_AA = \angle BCA$, which together with the previous claim yields that $\triangle BAT_A \sim \triangle QAC$, then, $\frac{AT_A}{BT_A} = \frac{AC}{QC}$, but recall that $BK = CQ$ (K and Q are the reflections of each other across the midpoint of $\overline{BC}$), so $\frac{AT_A}{BT_A} = \frac{AC}{BK}$. Taking into account that $\angle T_ABK = \angle T_ABC = \angle T_AAC$, the latter implies $\triangle BT_AK \sim \triangle AT_AC$, hence $\angle BT_AK = \angle AT_AC$, as required. □

Let L and M' the points where the ray AI meets BC and (BAC), respectively. The similarity we obtained before provides $\angle BKT_A = \angle ACT_A = \angle AM'T_A = \angle LM'T_A$; thus,

Lemma 18.

The intersection points of the ray AI with BC and (BAC), the contact point of the incircle with BC and the common point of ω_A with (BAC) (i.e. points L, M', K and T_A) are on a same circumference.

Let $R = \overline{MT_A} \cap \overline{BC}$. We have the fact below:

Lemma 19.

Lines CD, AR and BE are concurrent.

Proof. By Ceva's theorem, we just need to prove that $\frac{BD}{DA} \cdot \frac{AE}{EC} \cdot \frac{CR}{RB} = 1$ which reduces to show that $\frac{BR}{CR} = \frac{BD}{CE}$. Recalling proposition 9, we obtain $BT_A \cdot DI = BD \cdot IT_A$ and $IE \cdot CT_A = EC \cdot IT_A$, so

$$\frac{BD}{CE} = \frac{DI}{IE} \cdot \frac{BT_A}{IT_A} \cdot \frac{IT_A}{CT_A} = \frac{BT_A}{CT_A} = \frac{BR}{CR}$$

where we make use of the results 2 and 3. The proof is complete. □

Lemma 20.

Lines DE, BC and T_AM' concur.

Proof. Let $T = \overline{T_AM'} \cap \overline{BC}$. Since $\angle BT_AR = \angle RT_AC$ and $\angle RT_AM' = 90°$, we find that

$$(T, R; B, C) = -1$$

By proposition 19, DE must pass through T as well. □

It is worth mentioning that (RT_AT) is the T_A-Apollonius circle of $\triangle BT_AC$.

Lemma 21.

The common tangent of ω_A and (BAC) passes through the midpoint of $\overline{TR}$.

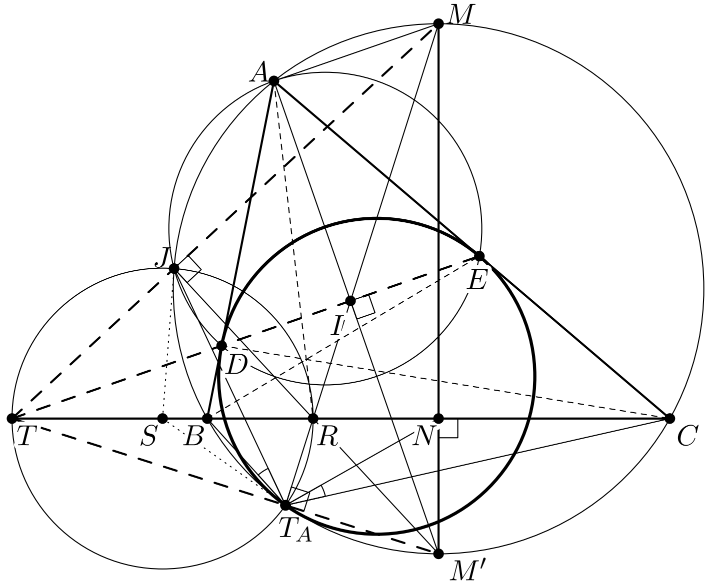

Figure 3.2. A bunch of concurrent lines.

Proof. Let S be the midpoint of $\overline{TR}$ (and also the circumcenter of $\triangle RT_AT$). It is easy to see that

$$\angle ST_AM = \angle ST_AR = \angle SRT_A = \frac{1}{2}\left(\widehat{BT_A} + \widehat{MC}\right) = \frac{1}{2}\left(\widehat{BT_A} + \widehat{MB}\right) = \angle MCT_A$$

which ensures that ST_A is tangent to (BAC) at T_A (and tangent to ω_A as well). □

Observe that R is the orthocenter of $\triangle TM'M$; in such a way, if $J = \overline{RM'} \cap (BAC),\ J \neq M'$, we certainly conclude that,

Lemma 22.

Points $T,\ J,\ M$ are collinear. In other words, lines TM and $M'R$ meet each other at (BAC).

This allows us to improve upon lemma 20, because we now know $MJ,\ DE,\ BC$ and T_AM' concur at T. Moreover, it implies that J is the second point of intersection of (RT_AT) and (BAC). Indeed, being (RT_AT) the T_A-Apollonius circle of $\triangle BT_AC$, it turns out JBT_AC is a harmonic quadrilateral, so JT_A is the reflection of the median T_AN of the triangle BT_AC across T_AR.

Lemma 23.

Quadrilateral JBT_AC is harmonic, therefore, lines T_AJ and T_AN are isogonal with respect to $\triangle BT_AC$.

Denote as J' the second point of intersection of (AED) and (ABC), so J' is the center of spiral similarity taking $\overline{DE}$ to $\overline{BC}$ and then it also carries $\overline{DI}$ to $\overline{BN}$, i.e. $\triangle J'ID \sim \triangle J'NB$, which gives $\angle J'IT = \angle J'NT$, thus $J'INT$ is cyclic; but note that $JINM'T$ is a inscribed pentagon, therefore $J = J'$.

Lemma 24.

Point J is the second point of intersection of the circles (AED) and ABC.

4. More than one mixtilinear incircle

In this section, we briefly discuss some facts related to two and three mixtilinear incircles. Let us denote by T_C and T_B the tangency points of ω_C and ω_B with (BAC), respectively. Suppose that ω_C and ω_B touch $\overline{BC}$ at $D,\ E$ respectively.

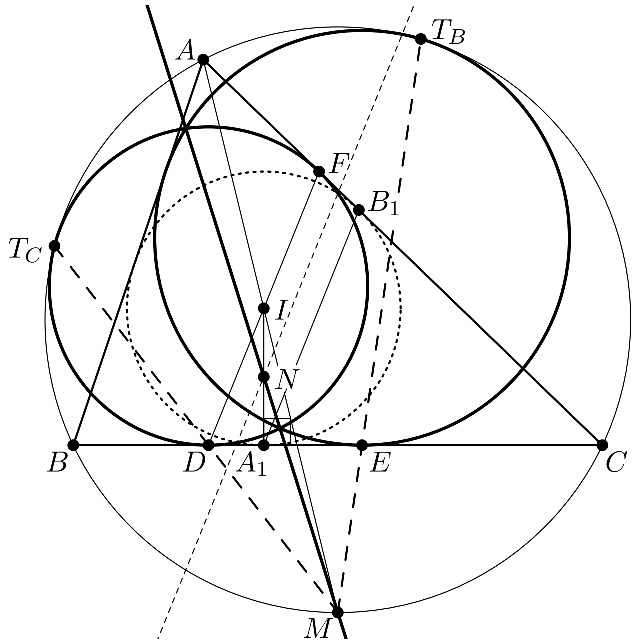

Figure 4.1. The radical axis of the $B,\ C$-mixtilinear incircles.

Lemma 25.

Denote by N the midpoint of the inradius perpendicular to BC and M the midpoint of arc $\widehat{BC}$ not containing A in (BAC). Line MN is the radical axis of ω_B and ω_C.

Proof. Let ℓ be the radical axis of ω_B and ω_C. In line with lemma 4, T_CD and T_BE meet at M. See that $\angle MT_BC = \angle MBC = \angle MCB = \angle ECM$, then MC is tangent to (T_BEC). Analogously, MB touches (T_CDB), so,

$$MD \cdot MT_C = MB^2 = MC^2 = ME \cdot MT_B$$

i.e. M has equal powers with respect to ω_B and ω_C, thus $M \in \ell$.

On the other hand, define $A_1,\ B_1$ and F as the contact points of the incircle with $BC,\ AC$ and the tangency point of ω_C with CA, respectively. It is straightforward to convince ourselves that the radical axis of the incircle and ω_C is the midline of the isosceles trapezoid FB_1A_1D parallel to FD and B_1A_1, so it must pass through N. Analogously, the radical axis of the incircle and ω_B passes through N; therefore, N is the radical center of the incircle, ω_B and ω_C, thus $N \in \ell$[1]. □

The last proof also implies that T_CT_BED is cyclic. Being D and E antihomologous points, it is a simple task to show that T_C and T_B are antihomologous with respect to the exsimilicenter of ω_B and ω_C;

[1] Based on [2]

of course, this leads to conclude that the intersection point of T_CT_B and BC is the exsimilicenter of ω_B and ω_C.

Lemma 26.

The contact point of ω_A and (ABC), T_A, lies on the circumcircle of $\triangle DME$.

Proof. Referring to lemma 2, we already know that $\angle IT_AM = 90°$; additionally, if BC and T_AM meet at T we have that $\angle TIM = 90°$ in accordance with proposition 20, then,

$$MD \cdot MT_C = MB^2 = MI^2 = MT_A \cdot MT$$

which means that T_CDT_AT is cyclic. Since T_CDET_B is cyclic, too, we obtain $\angle DT_AM = \angle TT_CD = \angle T_BED$ and the result follows. □

Applying the radical axis theorem to (T_CDET_B), $(DEMT_A)$ and (BAC) we derive that T_CT_B passes through T; hence, T is the exsimilicenter of ω_B and ω_C.

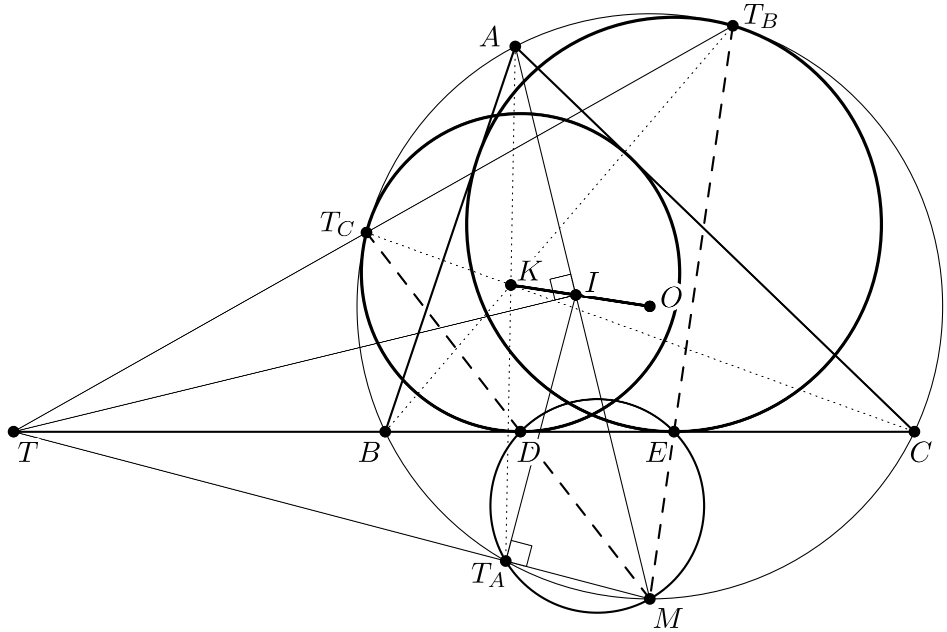

Figure 4.2. The exsimilicenter of the incircle and (BAC) is the point where AT_A, BT_B, CT_C concur.

Lemma 27.

The intersection point of the line perpendicular to AI passing through I and BC is the exsimilicenter of ω_B and ω_C. Moreover, the line joining the tangency points of ω_B and ω_C with (BAC), and the line formed by the contact point of ω_A with (BAC) and the midpoint of arc $\widehat{BC}$ opposite to A pass through this exsimilicenter.

According to Monge's theorem applied to circles ω_A, (ABC) and the incircle of $\triangle ABC$, we infer that the exsimilicenter of (ABC) and the incircle lies on AT_A. The same situation must occur for the lines BT_B and CT_C; therefore, recalling lemma 16 we discover that:

Lemma 28.

The lines joining each vertex and the tangency point of its corresponding mixtilinear incircle with (BAC) and the line passing through the incenter and circumcenter of $\triangle ABC$ concur at the exsimilicenter of its incircle and circumcircle, which turns out to be the isogonal point of the Nagel point of $\triangle ABC$.

In our context, OI, CT_C, BT_B and AT_A concur at K, the isogonal point of the Nagel point of $\triangle ABC$. Finally, let us examine what happens with the radical center of ω_A, ω_B and ω_C.

Lemma 29.

Let S be the radical center of ω_A, ω_B and ω_C. Then, S lies on OI and divides $\overline{OI}$ into a $-2R : r$ ratio, where the lengths are directed.

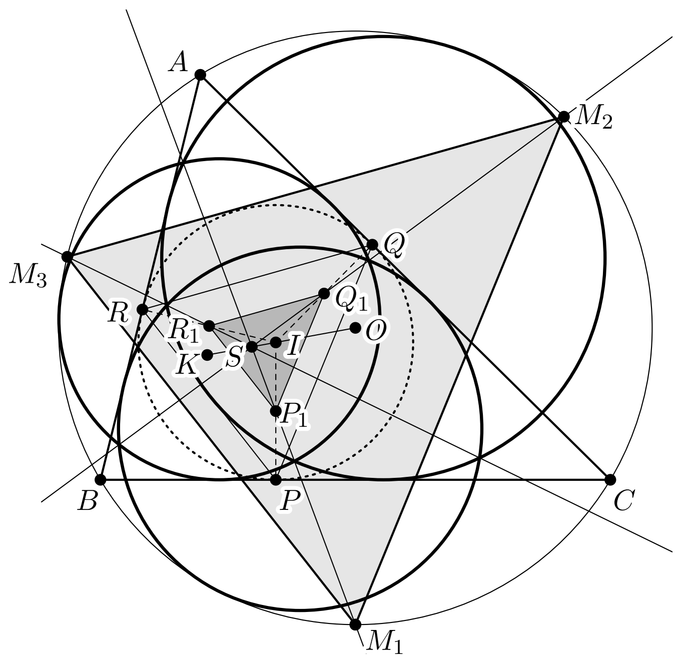

Figure 4.3. The radical center of the three mixtilinear incircles lies on OI.

Proof. Define M_1, M_2 and M_3 to be the midpoints of $\widehat{BC}$, $\widehat{CA}$ and $\widehat{AB}$ not containing A, B and C, respectively; PQR the tangential triangle of $\triangle ABC$ as shown above; P_1, Q_1 and R_1 the midpoints of $\overline{IP}$, $\overline{IQ}$ and $\overline{IR}$. By lemma 25, M_1P_1, M_2Q_1 and M_3R_1 concur at S. Triangles $\triangle M_1M_2M_3$ and $\triangle P_1Q_1R_1$ are homothetic with center S. Because I is the circumcenter of $\triangle P_1Q_1R_1$ as O is of $\triangle M_1M_2M_3$, we conclude that S, I, O are collinear. Since the circumradius of $\triangle P_1Q_1R_1$ is $\frac{r}{2}$, we ultimately get $OS : SI = R : -\frac{r}{2} = -2R : r$. □

This set of properties is absolutely not exhaustive but enough for our purposes here. You can find various "new" coincidences on your own while inquiring into what we have addressed so far.

5. Examples

Let us solve some example problems to illustrate how to use the propositions we have just discussed.

5.1. 2013 European Girls' Mathematical Olympiad, P5

Example 1.

(EGMO 2013, P5) Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.

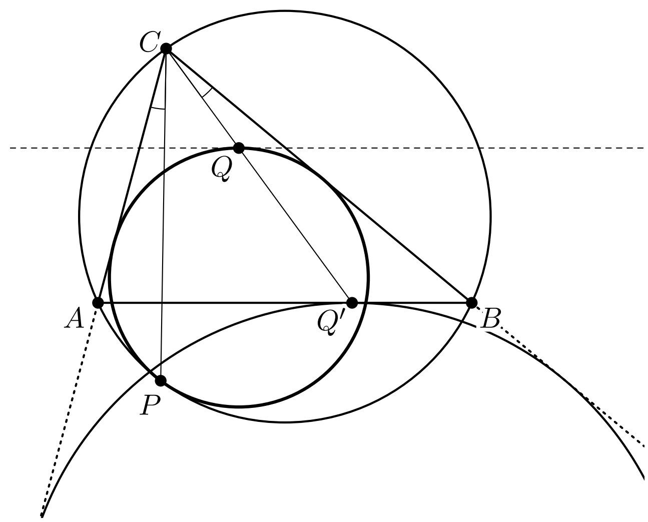

Figure 5.1. Lemma 13 overkills the fifth problem of EGMO 2013.

Solution. Let Q' be the point of tangency of the C-excircle and AB. By definition of Q and being C the exsimilicenter of ω and the C-excircle, we have that $C,\ Q,\ Q'$ are collinear. In line with proposition 16, we get

$$\angle ACP = \angle Q'CB = \angle QCB$$

as required. $\square$

5.2. 1999 International Mathematical Olympiad Shortlist, G8

Example 2.

(IMO 1999 SL, G8) Points $A,\ B,\ C$ divide the circumcircle Ω of the triangle ABC into three arcs. Let X be a variable point on the arc AB, and let $O_1,\ O_2$ be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects Ω in a fixed point.

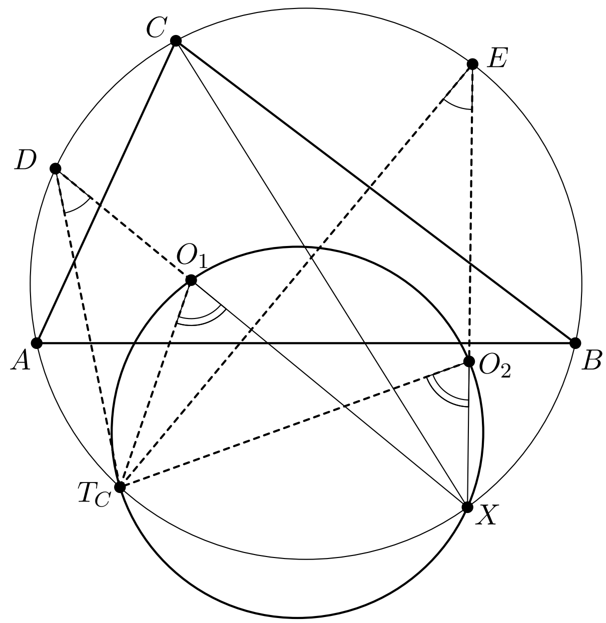

Figure 5.2. A beautiful concyclicity involving T_C.

Solution. We prove that the tangency point of the C-mixtilinear incircle and Ω (say T_C) is the required fixed point. Let $D = \overline{O_1X} \cap \Omega,\ D \neq X$ and $E = \overline{O_2X} \cap \Omega,\ E \neq X$. By lemma 10, we know that CDT_CE is harmonic, so

$$\frac{DC}{DT_C} = \frac{EC}{ET_C}$$

But we know that D and E are the circumcenters of $\triangle CO_1A$ and $\triangle CO_2B$, respectively; therefore,

$$\frac{DO_1}{DT_C} = \frac{EO_2}{ET_C}$$

which together with $\angle T_CDO_1 = \angle T_CDX = \angle T_CEX = \angle T_CEO_2$ implies that $\triangle DT_CO_1 \sim \triangle ET_CO_2$, i.e. T_C is the center of spiral similarity carrying $\overline{DE}$ to $\overline{O_1O_2}$, then T_C lies on the circumcircle of $\triangle XO_1O_2$. $\square$

5.3. 2016 USA IMO Team Selection Test, P2

Example 3.

(USA TST 2016, P2) Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

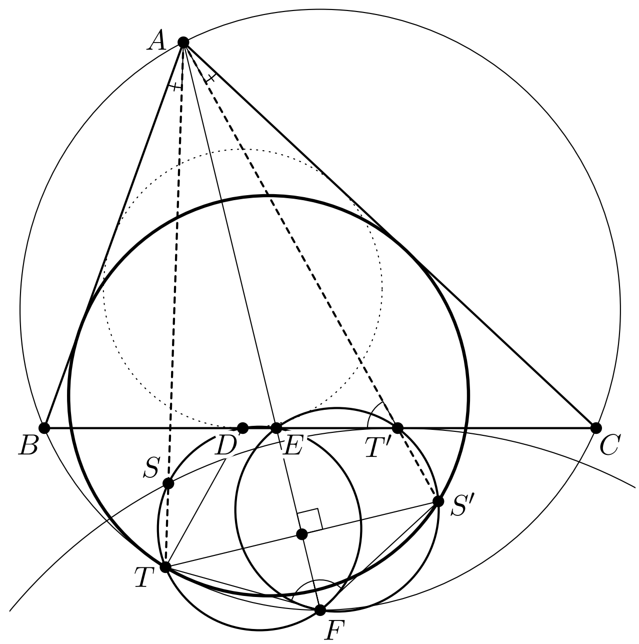

Figure 5.3. An intersection point of the A-excircle and (DEF) lies on AT.

Solution. According to lemma 18, T is the A-mixtilinear intouch point. Define S' to be the reflection of T across AF and T' the tangency point of the A-excircle and BC. By lemma 16, A, T' and S' are collinear. We know that T and T' are images under the composition of inversion centered at A with radius $\sqrt{AB \cdot AC}$ and reflection across AF. Moreover, E and F are sent to each other, and the A-excircle is mapped to the A-mixtilinear incircle; therefore, it suffices to prove that the intersection point of $(ET'F)$ and ω_A lies on AT'. In fact, we show S' is such a point. Indeed, since the circumcenter of the A-mixtilinear incircle is on AF, S' must lie on ω_A.

On the other hand, we have $\triangle BAT \sim \triangle T'AC$ according with lemma 16, so

$$\angle S'FE = \angle EFT = 180^\circ - \angle ABT = 180^\circ - \angle AT'C = \angle AT'E$$

which leads to conclude that $ET'S'F$ is cyclic. Thus, the inverse S of S' lies on AT, (DEF) and the A-excircle, so it coincides with either S_1 or S_2, as desired. □

5.4. Nicaragua Team Selection Test for IMO 2019, P7

Example 4.

(Nicaragua IMO TST 2019, P7) Let ABC be a triangle with $AC \neq AB$, ω its inscribed circle and I the center of ω. Let D and E be the points of tangency of ω with the sides CA and AB, respectively. Lines BI and CI meet DE at K and L, respectively. Points P and Q are located on the side BC so that $\angle LQC = 90° = \angle KPB$. Show that the circumcircles of the triangles PBE and QCD meet on the circumcircle of triangle ABC.

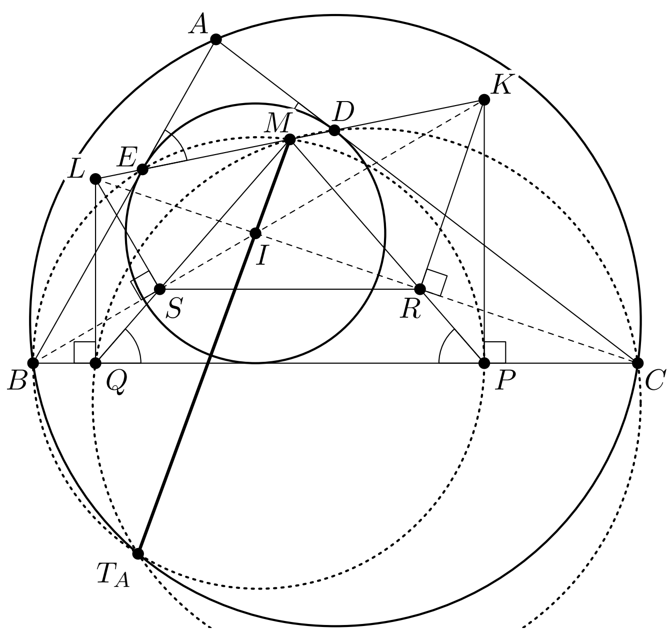

Figure 5.4. Two additional circles passing through the A-mixtilinear intouch point.

Solution. Let T_A, as usual, be the A-mixtilinear intouch point of $\triangle ABC$. We show that this is the required common point. In accordance with lemma 13, we already know that T_A, I and the midpoint of $\overline{LK}$ (say M) lie on a same line. Being $QLKP$ a trapezoid with $\angle LQP = 90° = \angle KPQ$, the perpendicular bisector of $\overline{QP}$ must pass through the midpoint of $\overline{LK}$, i.e. we have $MQ = MP$.

Let R and S be points on CI and BI, respectively, such that $ML = MS = MR = MK$, then $\angle CRK = 90°$ and we infer that $KRPC$ is cyclic. As is known, $\angle BKC = 90°$, hence

$$\angle PRC = \angle PKC = \angle KBC = \angle KLC = \angle MLR = \angle MRL$$

which implies that M, R and P are collinear. Analogously, we can prove that M, S and Q are collinear, thus

$$\angle LMQ = 2\angle MKS = 2\angle LKB = \angle DCQ$$

therefore $MDCQ$ is cyclic. In a similar way we can obtain that $EMPB$ is cyclic as well. Finally, observe that

$$\angle MQC = \angle MPB = \angle AED = \angle IBC + \angle ICB = \angle IT_AB = \angle IT_AC$$

where we have taken into account that T_AI bisects $\widehat{BAC}$. Thus, T_A is the second intersection point of (BEP) and (CDQ), as required. ☐

5.5. 2014 International Mathematical Olympiad Shortlist, G7

Example 5.

(IMO 2014 SL, G7) Let ABC be a triangle with circumcircle Ω and incenter I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X and Y are collinear, then the points I, W and Z are also collinear.

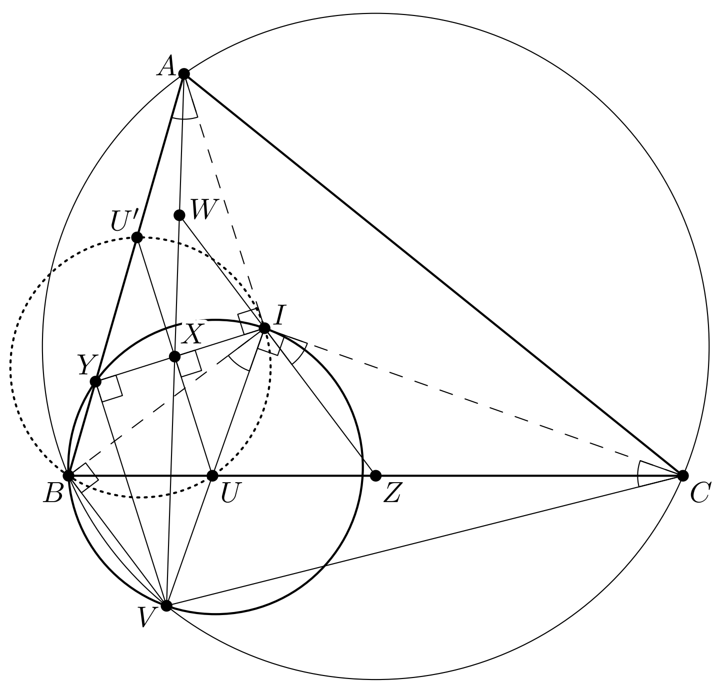

Figure 5.5. Seventh geometry problem of IMO 2014 Shortlist

Solution. It is known that $\angle BIC = 90^\circ + \angle BAI$, so $\angle BAI = \angle BIV$. Construct $U' = \overline{UX} \cap \overline{AB}$. Using the parallelisms, we readily get that $\angle BIV = \angle BAI = \angle BU'U = \angle BYV$, thus, $BYIV$ and $BU'IU$ are cyclic quadrilaterals. Furthermore,

$$\frac{U'X}{XU} = \frac{AX}{AV} \cdot \frac{VA}{VX} \cdot \frac{VY}{AI} = \frac{AX}{VX} \cdot \frac{VY}{AI} = 1$$

hence, X is the midpoint of $\overline{U'U}$. Because I is the midpoint of $\widehat{U'IU}$, we conclude that XY is the perpendicular bisector of $\overline{U'U}$, therefore

$$\angle VBI = \angle VYI = \angle UXI = \angle AIY = 90^\circ$$

which implies that Y is the point where the A-mixtilinear incircle touches AB, and by lemma 7 V must coincide with the tangency point of such a circle and (ABC). Now, recall lemma 11. We immediately recognize that V maps $\overline{BI}$ to $\overline{IC}$, then IV must be a symmedian of $\triangle BIC$, i.e. $\angle BIV = \angle ZIC$, whence $\angle BIZ = 90^\circ$.

On the other hand, $\angle YAI = \angle BAI = \angle BIV = \angle VCI$ and $\angle AIY = 90° = \angle VIC$, so $\triangle YAI \sim \triangle VCI$. Since $\frac{YX}{XI} = \frac{VU}{UI}$, taking into account that W is the circumcenter of $\triangle AIX$ and YI is tangent to (BIC), we obtain that

$$\angle WIA = \angle WAI = \angle XAI = \angle ICU = \angle ICB = \angle YIB$$

thus $\angle WIB = 90° = \angle BIZ$, which indeed implies the required assertion. □

Let us end this chapter solving the following hard and amazing problem.

5.6. 2019 International Mathematical Olympiad, P6

Example 6.

(IMO 2019, P6) Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q. Prove that lines DI and PQ meet on the line through A perpendicular to AI.

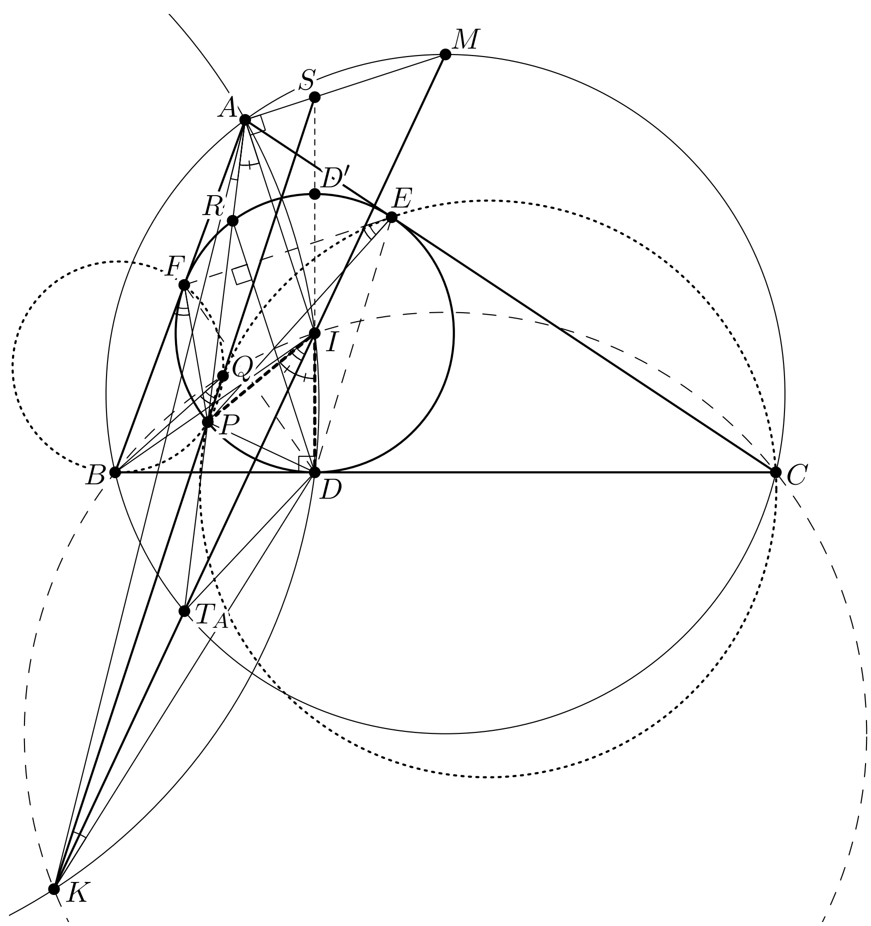

Figure 5.6. Sixth problem of IMO 2019 in all its glory.

Solution. Without loss of generality, assume that $AC > AB$. We start with some preparations. Define T_A to be the A-mixtilinear intouch point. Let P' and R' the intersection points of AT_A with ω, so that $AR' < AP'$. By lemmas 2 and 17, and being I the center of ω, we conclude that P' is the reflection of D across T_AI. Using lemma 14 we get that

$$\angle DR'P' = \angle T_AIP' = \angle DIT_A = \angle IAT_A$$

thus $R'D \parallel AI$ and thus $R'D \perp EF$, hence $R' = R$ and consecutively $P = P'$. The previous angle equalities also imply that T_AI is tangent to the circumcircle of $\triangle AIP$ (1).

Let M be the midpoint of $\widehat{BAC}$. Observe that

$$\angle SAP = 90^\circ + \angle PAI = 90^\circ + \angle PRD = 90^\circ + \angle PDB = 180^\circ - \angle PDS$$

which gives that $ASDP$ is a cyclic quadrilateral (2).

Define K to be the intersection point of MI with (BIC). According to (1) and lemmas 12 and 14 we have that $T_AK^2 = TI^2 = TP \cdot TA$. In concomitance with (2) and lemma 15 we get

$$\angle KPT_A = T_AKA = \angle IKA = \angle IDA = \angle SDA = \angle SPA$$

thus, S, P and K are collinear.

We turn to the solution of the problem. Let $Q' = \overline{SP} \cap (BIC)$, $K \neq Q'$. Observe that

$$\angle BQ'P = \angle BQ'K = \angle BIK = \angle BID - \angle T_AID = \angle FED - \angle PED = \angle FEP = \angle BFP$$

and

$$\angle PQ'C = \angle KQ'C = \angle KIC = \angle T_AID + \angle DIC = \angle PFD + \angle DFE = \angle PFE = \angle PEC$$

hereby $BFQ'P$ and $CEQ'P$ are cyclic quadrilaterals, so $Q' = Q$ and thus Q lies on PS. We are done! $\square$

6. Problems

Exercise 6.1. Prove lemma 4 inverting at M with radius MA.

Exercise 6.2. (Centroamerican MO 2016, P6) Let ABC be a triangle with incenter I and circumcircle Γ. Let $M = \overline{BI} \cap \Gamma$ and $N = \overline{CI} \cap \Gamma$. The line through I parallel to MN intersects AB, AC at P, Q respectively. Show that the circumradii of $\triangle BNP$ and $\triangle CMQ$ are congruent.

Exercise 6.3. (Sharygin 2017, 9^{th} grade, P2) Let I be the incenter of triangle ABC, M the midpoint of $\overline{AC}$ and W the midpoint of $\widehat{AB}$ not containing C. It is known that $\angle AIM = 90°$. Find the ratio $CI : IW$.

Exercise 6.4. Given a scalene triangle ABC, let D, E, F be the intersection points of the lines through I perpendicular to AI, BI, CI and BC, CA, AB, respectively. Show that D, E and F are collinear.

Exercise 6.5. Given a triangle ABC with incenter I and a circle ω which is tangent to AN, AC and also is internally tangent to the circumcircle of ABC. Lines AI, BI, CI meet the circumcircle of ABC at A', B', C', respectively. Let the lines through I and parallel to $A'B'$, $A'C'$ meet $A'C'$, $A'B'$ at B_1, C_1, respectively. Let ω has center J and is tangent to AB, AC at C_2, B_2 and is tangent to the circumcircle of ABC at T. Prove

i. $A'B_1C_1JT$ is cyclic.

ii. C_1 is the circumcenter of TIB_2C and B_1 is the circumcenter of TIC_2B.

Exercise 6.6. (ELMO 2014 SL, G7) Let ABC be a triangle inscribed in circle ω with center O; let ω_A su A-mixtilinear incircle, ω_B its B-mixtilinear incircle, ω_C its C-mixtilinear incircle, and X be the radical center of ω_A, ω_B, ω_C. Let A', B', C' be the points at which ω_A, ω_B, ω_C are tangent to ω. Prove that AA', BB', CC' and OX are concurrent.

Exercise 6.7. (Korea Winter Program Practice Test 2018, P5) Let ABC be triangle with circumcenter O and circumcircle ω. Let S be the center of the circle which is tangent with AB, AC, and ω (in the inside), and let the circle meet ω at point K. Let the circle with diameter $\overline{AS}$ meet ω at T. If M is the midpoint of $\overline{BC}$, show that K, T, M, O are concyclic.

Exercise 6.8. (IMO 2017 SL, G4) In triangle ABC, let ω be the excircle opposite A. Let D, E, and F be the points where ω is tangent to lines BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

Exercise 6.9. (ELMO 2017 SL, G4) Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2 and C_2 are the points opposite to B, C on ω, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

Exercise 6.10. (IMO 2016 SL, G2) Let ABC be a triangle with circumcircle Γ and incenter I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Ω.

Exercise 6.11. (USAMO 2017, P3) Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Exercise 6.12. (Taiwan TST 2015, Round 3, Quiz 1, P2) Let Ω be the circumcircle of the triangle ABC. Two circles ω_1, ω_2 are tangent to each of the circle Ω and the rays $\overrightarrow{AB}$, $\overrightarrow{AC}$, with ω_1 interior to Ω, ω_2 exterior to Ω. The common tangent of Ω, ω_1 and the common tangent of Ω, ω_2 intersect at the point X. Let M be the midpoint of the arc BC (not containing the point A) on the circle Ω, and the segment $\overline{AA'}$ be the diameter of Ω. Prove that X, M and A' are collinear.

Exercise 6.13. (Taiwan TST 2014, Round 3, Day 1, P3) Let M be any point on the circumcircle of $\triangle ABC$. Suppose the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that the circumcircle of $\triangle MX_1X_2$ intersects the circumcircle of $\triangle ABC$ again at the tangency point of the A-mixtilinear incircle.

Exercise 6.14. (ELMO 2014 SL, G8) In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear incircles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Show that ω_A, ω_B, ω_C have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Exercise 6.15. (Mathematical Reflections O451). Let ABC be a triangle, Γ its circumcircle, ω its incircle and I the incenter. Let M be the midpoint of BC. The incircle ω is tangent to AB and AC at F and E; respectively. Suppose EF meets Γ at distinct points P and Q. Let J denote the point on EF such that MJ is perpendicular on EF. Show that IJ and the radical axis of (MPQ) and (AJI) intersect on Γ.

Exercise 6.16. (AoPS Problem Making Contest 2016, P7) Let ABC be a given triangle, ω its A-mixtilinear incircle and I_A its A-excenter. Denote by H the foot of the A-altitude to BC, E the midpoint of $\widehat{BAC}$, M the midpoint of $\overline{BC}$ and N the midpoint of $\overline{AH}$. Suppose that $P = \overline{MN} \cap \overline{AE}$ and ray PI_A meets ω for the first time at S. Show that (BSC) and ω are tangent to each other.

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