Junior problems

J511. Let a, b, c be positive real numbers such that $a+b+c \le 3\sqrt[3]{3abc}$. Prove that

$$9(a+b+c)^3 < \left(a+3b+3c+\frac{2bc}{a}\right)\left(3a+b+3c+\frac{2ca}{b}\right)\left(3a+3b+c+\frac{2ab}{c}\right).$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA

The weighted AM-GM inequality gives us

$$a+3b+3c+\frac{2bc}{a} \;\ge\; 9\,a^{-1/9}\,b^{5/9}\,c^{5/9}.$$

Therefore

$$\prod_{cyclic}\left(a+3b+3c+\frac{2bc}{a}\right) \;\ge\; \prod_{cyclic} 9\,a^{-1/9}\,b^{5/9}\,c^{5/9} \;=\; 9\left(3\sqrt[3]{3abc}\right)^3 \;\ge 9(a+b+c)^3.$$

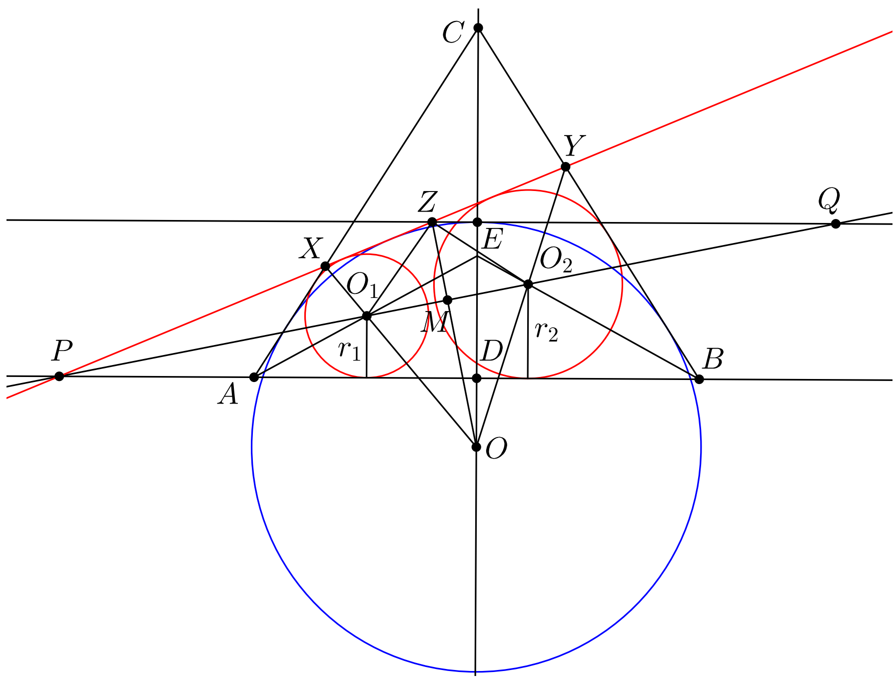
Note: This is actually a strict inequality since equality in the two inequalities used cannot hold simultaneously: The arithmetic and geometric means are equal if and only if $a=b=c$, whereas $a+b+c<3\sqrt[3]{3abc}$ for $a=b=c$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Muhammad Thoriq, Yogyakarta, Indonesia; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania; Polyahedra, Polk State College, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Brian Bradie, Christopher Newport University, VA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Sarah Seales, Northern Arizona University, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India.

J512. Let ABC be a triangle with $AC = BC$ and let a be a positive real number. Consider two variable circles with radii r_1 and r_2 lying inside the triangle such that the first circle is tangent to segments AB, AC, the second one to segments AB, BC, and $r_1 + r_2 = a$. Prove that the common external tangents to these circles, different from line AB, are all tangent to a fixed circle.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Polyahedra, Polk State College, USA



Suppose that the two circles of radii r_1 and r_2 have centers O_1 and O_2, and their other common external tangent intersects AB, AC, and BC at P, X, and Y, respectively. Let Ω be the C-excircle of $\triangle CXY$. Let D be the midpoint of AB. Then XO_1, YO_2, and CD concur at the the center O of Ω. Let E be the intersection of Ω with the ray DC. It suffices to show that $DE = a$. Suppose that the tangent line to Ω at E intersects PX at Z and PO_1 at Q. Then the bisector of $\angle PZQ$ passes through O. Since $PZ = QZ$, OZ is also the perpendicular bisector of PQ. Chasing angles we have

$$\begin{aligned}\angle APX &= \angle BAC - \angle YXC = 90^\circ - \frac{1}{2}\angle ACB - \angle YXC \\ &= \frac{1}{2}(\angle XYC - \angle YXC) = \angle OXY - \angle OYX.\end{aligned}$$

Therefore,

$$\angle OO_1O_2 = \angle OXY - \frac{1}{2}\angle APX = \angle OYX + \frac{1}{2}\angle APX = \angle OO_2O_1,$$

so the midpoint M of PQ is the midpoint of O_1O_2 as well. Thus $DE = r_1 + r_2 = a$, and Ω is the fixed circle claimed.

Also solved by Daniel Lasaosa, Pamplona, Spain.

J513. Let a, b, c be distinct positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sum_{cyc} \frac{(a+b)(a+c) - bc}{(b-c)\left(b^3 - c^3\right)} \geq \left(\sum_{cyc} \frac{a}{|b-c|}\right)^2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

First solution by Polyahedra, Polk State College, USA

By the Cauchy-Schwarz inequality,

$$\sum_{cyc} \frac{(a+b)(a+c) - bc}{(b-c)(b^3 - c^3)} = (a+b+c) \sum_{cyc} \frac{a}{(b-c)^2(b^2+bc+c^2)} \geq \frac{(a+b+c)\left(\sum_{cyc} \frac{a}{|b-c|}\right)^2}{\sum_{cyc} a(b^2+bc+c^2)}$$

$$= \frac{(a+b+c)\left(\sum_{cyc} \frac{a}{|b-c|}\right)^2}{(a+b+c)(ab+bc+ca)} = \left(\sum_{cyc} \frac{a}{|b-c|}\right)^2.$$

Second solution by Daniel Lasaosa, Pamplona, Spain

Consider the following vectors:

$$\vec{u} \equiv \left(\sqrt{\frac{a\left(b^2+bc+c^2\right)}{a+b+c}}, \sqrt{\frac{b\left(c^2+ca+a^2\right)}{a+b+c}}, \sqrt{\frac{c\left(a^2+ab+b^2\right)}{a+b+c}} \right),$$

$$\vec{v} \equiv \left(\sqrt{\frac{a(a+b+c)}{b^2+bc+c^2}} \cdot \frac{1}{|b-c|}, \sqrt{\frac{b(a+b+c)}{c^2+ca+a^2}} \cdot \frac{1}{|c-a|}, \sqrt{\frac{c(a+b+c)}{a^2+ab+b^2}} \cdot \frac{1}{|a-b|} \right).$$

Note that

$$|\vec{u}| = \sqrt{\frac{a\left(b^2+bc+c^2\right)+b\left(c^2+ca+a^2\right)+c\left(a^2+ab+b^2\right)}{a+b+c}} = \sqrt{ab+bc+ca} = 1,$$

$$|\vec{v}| = \sqrt{\sum_{cyc} \frac{(a+b)(a+c)-bc}{(b-c)\left(b^3-c^3\right)}},$$

since $a(a+b+c) = (a+b)(a+c) - bc$ and $(b-c)\left(b^2+bc+c^2\right) = b^3-c^3$. Note further that

$$\vec{u} \cdot \vec{v} = \sum_{cyc} \frac{a}{|b-c|},$$

or the conclusion follows by the scalar product inequality. Note however that equality requires that a real nonzero constant k exists such that $\vec{u} = k\vec{v}$, or equivalently,

$$(a+b+c)k = \left|b^3-c^3\right| = \left|c^3-a^3\right| = \left|a^3-b^3\right|.$$

If this were true, and assuming wlog that $a > b, c$ since a, b, c are distinct and by cyclic symmetry in the problem, we would have $a^3-b^3 = a^3-c^3$, for $b=c$, contradiction. Or, equality cannot hold, and the inequality holds strictly.

Also solved by Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Dumitru Barac, Sibiu, Romania; Kevin Soto, Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India.

J514. Let a, b, c be nonnegative real numbers such that

$$\left(a^2-a+1\right)\left(b^2-b+1\right)\left(c^2-c+1\right)=1.$$

Prove that

$$\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)\leq 27.$$

Proposed by Marius Stănean, Zalău, Romania

First solution by Daniel Lasaosa, Pamplona, Spain

Claim: For all nonnegative reals a, b, the following inequality holds:

$$\left(a^2-a+1\right)\left(b^2-b+1\right)\geq \frac{a^2+ab+b^2}{3},$$

with equality iff $a=b=1$.

Proof: After some algebra, the inequality rewrites as

$$\left(3b^2-3b+2\right)a^2-\left(3b^2-2b+3\right)a+\left(2b^2-3b+3\right)\geq 0$$

Note first that $3b^2-3b+2=\frac{3}{4}(2b-1)^2+\frac{5}{4}>0$. Considering the LHS as a function of a, the respective first and second derivatives are

$$2\left(3b^2-3b+2\right)a-\left(3b^2-2b+3\right), \qquad 2\left(3b^2-3b+2\right)>0.$$

It follows that, when b is fixed and we let a vary, the LHS has a minimum when $2\left(3b^2-3b+2\right)a=\left(3b^2-2b+3\right)$. After some algebra, this minimum for fixed b is

$$15b^4-48b^3+66b^2-48b+15=9(b-1)^4+6\left(b^2+1\right)(b-1)^2,$$

clearly nonnegative, and being zero iff $b=1$. The LHS is therefore nonnegative, being zero iff $b=1$ and $a=\frac{3-2+3}{2(3-3+2)}=1$. The Claim follows.

Returning to the original problem, by the Claim and cyclic permutations of a, b, c, we have

$$\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)\leq$$

$$\leq 3^3\left(a^2-a+1\right)^2\left(b^2-b+1\right)^2\left(c^2-c+1\right)=27,$$

with equality iff $a=b=c=1$. The conclusion follows.

Second solution by Polyahedra, Polk State College, USA
Since $3(a^2 - a + 1) - (a^2 + a + 1) = 2(a - 1)^2 \geq 0$,

$$a^4 + a^2 + 1 = (a^2 + a + 1)(a^2 - a + 1) \leq 3(a^2 - a + 1)^2.$$

Therefore, by the Cauchy-Schwarz inequality,

$$a^2 \cdot 1 + a \cdot b + 1 \cdot b^2 \leq \sqrt{(a^4 + a^2 + 1)(1 + b^2 + b^4)} \leq 3(a^2 - a + 1)(b^2 - b + 1).$$

Multiplying this with the other two analogous inequalities and applying the given condition completes the proof.

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Kevin Soto, Palacios Huarmey, Perú; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India.

J515. Let a, b, c be complex numbers such that $a + b + c = 3$. Prove that any two of the equalities

$$a^3 - 9 = bc(a-9), \qquad b^3 - 9 = ca(b-9), \qquad c^3 - 9 = ab(c-9)$$

imply the third.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Joel Schlosberg, Bayside, NY, USA

If $a + b + c = 3$,

$$\begin{aligned}
&[(a^3 - 9) - bc(a-9)] + [b^3 - 9 - ca(b-9)] + [c^3 - 9 - ab(c-9)] \\
&= a^3 - bc(a-9) + b^3 - ca(b-9) + c^3 - ab(c-9) - 27 \\
&= a^3 - bc(a-9) + b^3 - ca(b-9) + c^3 - ab(c-9) - (a+b+c)^3 \\
&= -3(a+b+c-3)(bc+ca+ab) = 0,
\end{aligned}$$

so if any two of $(a^3 - 9) - bc(a-9)$, $b^3 - 9 - ca(b-9)$, $c^3 - 9 - ab(c-9)$ are zero, the third is also zero.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Muhammad Thoriq, Yogyakarta, Indonesia; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Ioannis D. Sfikas, Athens, Greece.

J516. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3(a+b+c)}{2}}.$$

Proposed by Mircea Becheanu, Montreal, Canada

First solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
\sum_{\text{cyc}} \frac{a}{\sqrt{b+c}} &\geq \frac{(a+b+c)^2}{a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}} \\
&\geq \frac{(a+b+c)^2}{\sqrt{(a+b+c)(a(b+c) + b(c+a) + c(a+b))}} \\
&= \sqrt{\frac{(a+b+c)^3}{2(ab+bc+ca)}} \\
&\geq \sqrt{\frac{3(a+b+c)}{2}}
\end{aligned}$$

as desired.

Second solution by Daniel Lasaosa, Pamplona, Spain

Denote $u = \sqrt{b+c}$, $v = \sqrt{c+a}$ and $w = \sqrt{a+b}$, or $a+b+c = \frac{3Q^2}{2}$ and $a = \frac{v^2+w^2-u^2}{2} = \frac{3Q^2}{2} - u^2$, where Q is the quadratic mean of u, v, w. The proposed inequality then rewrites as

$$Q + 2A \leq Q^2\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) = \frac{3Q^2}{H},$$

where we have denoted by A, H the arithmetic and harmonic means of a, b, c. The proposed inequality is then equivalent to $3Q^2 \geq QH + 2AH$, clearly true since $Q \geq A \geq H$, with equality in either inequality iff $a = b = c$. The conclusion follows, equality holds iff $a = b = c$.

Also solved by Polyahedra, Polk State College, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Brian Bradie, Christopher Newport University, VA, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Henry Ricardo, Westchester Area Math Circle, NY, USA; Adarsh K, IIT Bombay, India; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Jennisha Sunil Agrawal, Disha Delphi Public School, India; Mihaly Bencze, Brașov, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Ioannis D. Sfikas, Athens, Greece.

Senior problems

S511. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\left(\sqrt{a} + \sqrt{b} + \sqrt{c} + 1\right)^2 \le 2(a+b)(b+c)(c+a).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

We have

$$(c+a)(a+b) = a^2 + 1 + 1 + 1$$
$$(a+b)(b+c) = 1 + b^2 + 1 + 1$$
$$(b+c)(c+a) = 1 + 1 + 1 + c^2$$
$$4 = 1 + 1 + 1 + 1.$$

Multiply these equalities and apply Holder's Inequality to get the conclusion.

Second solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan
Since $ab+bc+ca=3$, $(a+b+c)^2 \geq 3(ab+bc+ca)=9$ and $a+b+c \geq 3$. $2(a+b)(b+c)(c+a) = 2(a+b+c)(ab+bc+ca) - 2abc \geq 6(a+b+c) - 2abc$. From Jensen's inequality,

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+1 \leq 4\sqrt{\frac{a+b+c+1}{4}}$$

Therefore it suffices to show

$$\begin{aligned}&4(a+b+c+1) \leq 6(a+b+c) - 2abc\\ &\Leftrightarrow 0 \leq a+b+c-abc-2.\end{aligned}$$

From AM-GM, $abc \leq 1$, we are done.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Kevin Soto, Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India.

S512. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{\sqrt{b^2-bc+c^2}}+\frac{b^3}{\sqrt{c^2-ca+a^2}}+\frac{c^3}{\sqrt{a^2-ab+b^2}} \geq a^2+b^2+c^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

By Cauchy-Schwarz inequality we have

$$\sum_{\text{cyc}} \frac{a^3}{\sqrt{b^2-bc+c^2}} \geq \frac{(a^2+b^2+c^2)^2}{\sum_{\text{cyc}} a\sqrt{b^2-bc+c^2}}.$$

It's enough to show that

$$a^2+b^2+c^2 \geq \sum_{\text{cyc}} a\sqrt{b^2-bc+c^2}.$$

After squaring both sides, the inequality reduces to

$$\sum_{\text{cyc}} a^2(b^2-bc+c^2)+\sum_{\text{cyc}} 2bc\sqrt{(c^2-ca+a^2)(a^2-ab+b^2)} \leq (a^2+b^2+c^2)^2.$$

The AM-GM inequality yields

$$\begin{aligned}\sum_{\text{cyc}} 2bc\sqrt{(c^2-ca+a^2)(a^2-ab+b^2)} &\leq \sum_{\text{cyc}} bc(2a^2+b^2+c^2-ab-ca)\\ &= \sum_{\text{cyc}} bc(b^2+c^2).\end{aligned}$$

Therefor, it suffices to prove

$$\sum_{\text{cyc}} a^2(b^2-bc+c^2)+\sum_{\text{cyc}} bc(b^2+c^2) \leq (a^2+b^2+c^2)^2.$$

This is equivalent to

$$2(a^2b^2+b^2c^2+c^2a^2)-abc(a+b+c)+\sum_{\text{cyc}} bc(b^2+c^2) \leq (a^2+b^2+c^2)^2,$$

$$\sum_{\text{cyc}} bc(b^2+c^2) \leq a^4+b^4+c^4+abc(a+b+c).$$

But this is exactly Schur's inequality so we are done. Equality holds for $a=b=c$ or $a=0, b=c$ and its permutation.

Also solved by Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Kevin Soto, Palacios Huarmey, Perú; Titu Zvonaru, Comănești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Ioannis D. Sfikas, Athens, Greece.

S513. Solve the equation

$$16\{x\}\left(x+2020\{x\}\right)=[x]^2,$$

where $[x]$ and $\{x\}$ are the greatest integer less than or equal to x and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Let $[x]=m$, clearly an integer. The equation then rewrites

$$(m-8\{x\})^2=m^2-16m\{x\}+64\{x\}^2=(16\cdot 2021+64)\{x\}^2=32400\{x\}^2=(180\{x\})^2,$$

yielding either $m=188\{x\}$ when $m\geq 0$ or $m=-172\{x\}$ when $m<0$. Since $0\leq\{x\}<1$, if $m=188\{x\}$ we have $\{x\}=\frac{m}{188}$, for a first family of solutions

$$x=\frac{189m}{188} \qquad \text{where} \qquad m\in\{0,1,2,\dots,187\}.$$

Similarly, if $m=-172\{x\}$, denoting $n=-m$ we have $\{x\}=\frac{n}{172}$,

$$x=-\frac{171n}{172} \qquad \text{where} \qquad n\in\{1,2,3,\dots,171\}.$$

Note that the trivial solution $x=0$ appears in the first, but it could just as well appear in the second family of solutions.

Also solved by Sarah Seales, Northern Arizona University, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Israel Castillo, Nacional Universidad, Perú; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Satvik Dasariraju, Lawrenceville School, NJ, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania; S.Chandrasekhar, Chennai, India; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Ioannis D. Sfikas, Athens, Greece.

S514. Let a, b, c be positive numbers. Prove that

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} \geq \frac{4\left(a^2+b^2+c^2\right)}{ab+bc+ca}+2.$$

Proposed by An Zhenping, Xianyang Normal University, China

First solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

We will use the S.O.S method. Rewite the inequality as

$$\sum_{\text{cyc}}\left(\frac{b}{c}+\frac{c}{b}-2\right) \geq \frac{4(a^2+b^2+c^2-ab-bc-ca)}{ab+bc+ca},$$

$$\sum_{\text{cyc}}\frac{(b-c)^2}{bc} \geq \frac{2(a-b)^2+2(b-c)^2+2(c-a)^2}{ab+bc+ca},$$

$$\sum_{\text{cyc}}\left(\frac{1}{bc}-\frac{2}{ab+bc+ca}\right)(b-c)^2 \geq 0.$$

The standard form of this inequality as

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0$$

where

$$S_a = \frac{1}{bc}-\frac{2}{ab+bc+ca},$$
$$S_b = \frac{1}{ca}-\frac{2}{ab+bc+ca},$$
$$S_c = \frac{1}{ab}-\frac{2}{ab+bc+ca}.$$

Without loss of generality we may assume $a \geq b \geq c > 0$. This implies $S_a \geq S_b \geq S_c$. Now we have

$$S_b+S_c = \frac{1}{ca}+\frac{1}{ab}-\frac{4}{ab+bc+ca} \geq \frac{4}{ab+ac}-\frac{4}{ab+bc+ca} > 0.$$

This show that $S_a \geq S_b > 0$. Form here and note that $(c-a)^2 \geq (a-b)^2$ we obtain

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq (S_b+S_c)(a-b)^2 \geq 0$$

This completes the proof. The equality occurs if and only if $a=b=c$.

Second solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain
After some algebra, the proposed inequality is equivalent to

$$\frac{a^3b^2 - 2a^3bc + a^3c^2 + a^2b^3 + a^2c^3 - 2ab^3c - 2abc^3 + b^3c^2 + b^2c^3}{abc(ab + ac + bc)} \geq 0$$

or, equivalently, to $a^3b^2 + a^3c^2 + a^2b^3 + a^2c^3 + b^3c^2 + b^2c^3 \geq 2a^3bc + 2ab^3c + 2abc^3$, which follows by Muirhead's inequality since $[3,2,0] \succ [3,1,1]$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Kevin Soto, Palacios Huarmey, Perú; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S515. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}\right)^6 \geq 27(a+2)(b+2)(c+2)$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, CA, USA
Replacing $\left(\sqrt[3]{a}, \sqrt[3]{b}, \sqrt[3]{c}\right)$ with (a, b, c) results in an equivalent problem:

$$(a+b+c)^6 \geq 27(a^3+2)(b^3+2)(c^3+2) \text{ if } a, b, c > 0 \text{ and } abc = 1$$

or, after homogenization

$$(a+b+c)^6 \, abc \geq 27(a^3+2abc)(b^3+2abc)(c^3+2abc) \iff$$

$$(a+b+c)^6 \geq 27(a^2+2bc)(b^2+2ca)(c^2+2ab).$$

By AM-GM Inequality we have

$$(a+b+c)^6 = \left((a+b+c)^2\right)^3 = \left((a^2+2bc)+(b^2+2ca)+(c^2+2ab)\right)^3 \geq 27(a^2+2bc)(b^2+2ca)(c^2+2ab)$$

and we are done.

Also solved by Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Kevin Soto, Palacios Huarmey, Perú; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S516. Let a, b, c be nonnegative real numbers such that

$$(a+b)(b+c)(c+a)=2.$$

Prove that

$$\left(a^2+bc\right)\left(b^2+ca\right)\left(c^2+ab\right)+8a^2b^2c^2\leq 1.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that

$$\left(a^2+bc\right)\left(b^2+ca\right)\left(c^2+ab\right)+8a^2b^2c^2=$$

$$=a^3b^3+b^3c^3+c^3a^3+abc\left(a^3+b^3+c^3\right)+10a^2b^2c^2.$$

Squaring both sides in the condition and dividing by 4, we obtain

$$1=a^3b^3+b^3c^3+c^3a^3+abc\left(a^3+b^3+c^3\right)+10a^2b^2c^2+$$

$$+a^2bc(b-c)^2+ab^2c(c-a)^2+abc^2(a-b)^2+\frac{(a-b)^2(b-c)^2(c-a)^2}{4}.$$

The conclusion follows, equality holds iff $a=b=c$. Since in that case we have $2=(a+b)(b+c)(c+a)=8a^3$, equality holds iff

$$a=b=c=\frac{1}{\sqrt[3]{4}} \text{ or } a=b=1, c=0 \text{ and permutations}$$

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

Undergraduate problems

U511. Let A and B be $n \times n$ matrices such that $AB = A + B$. Prove that

$$\operatorname{rank}\left(A^2\right) + \operatorname{rank}\left(B^2\right) \le 2\operatorname{rank}(AB).$$

Proposed by Konstantinos Metaxas, Athens, Greece

First solution by Khakimboy Egamberganov, Paris, France

By the given condition, we get $(A - I)(B - I) = I$. It gives us that $A - I$ and $B - I$ are invertible, and $\operatorname{rank}(A - I) = \operatorname{rank}(B - I) = n$. So by the Sylvester's inequality,

$$\operatorname{rank}(A - I) + \operatorname{rank}(B^2) \le n + \operatorname{rank}(AB^2 - B^2) = n + \operatorname{rank}(AB)$$

and

$$\operatorname{rank}(B^2) \le \operatorname{rank}(AB).$$

Similarly, we can obtain

$$\operatorname{rank}(A^2) + \operatorname{rank}(B - I) \le n + \operatorname{rank}(A^2B - A^2) = n + \operatorname{rank}(AB)$$

and

$$\operatorname{rank}(A^2) \le \operatorname{rank}(AB).$$

The conclusion follows by adding both above inequalities.

Second solution by Li Zhou, Polk State College, USA

From $AB = A + B$ we have $A^2 = (A - I)AB$, $B^2 = AB(B - I)$, and $(A - I)(B - I) = I$. By the Frobenius inequality,

$$\begin{aligned}\operatorname{rank}\left(A^2\right) + \operatorname{rank}\left(B^2\right) &= \operatorname{rank}\left((A - I)AB\right) + \operatorname{rank}\left(AB(B - I)\right)\\ &\le \operatorname{rank}(AB) + \operatorname{rank}\left((A - I)AB(B - I)\right) = 2\operatorname{rank}(AB).\end{aligned}$$

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Muhammad Thoriq, Yogyakarta, Indonesia; Kevin Soto, Palacios Huarmey, Perú; Prajnanaswaroopa S, Bangalore, India.

U512. Consider the polynomial $P(x) = x^6 + 4x^5 + 8x^4 + 12x^3 + 16x^2 + 16x + 8$. Evaluate

$$\int \frac{x^9 + 16x}{P(x)P(-x)} dx$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, VA, USA

Note

$$\begin{aligned} P(x) &= x^6 + 4x^5 + 8x^4 + 12x^3 + 16x^2 + 16x + 8 \\ &= (x^2 + 2x + 2)(x^4 + 2x^3 + 2x^2 + 4x + 4) \\ &= (x^2 + 2x + 2)(x^2(x + 1)^2 + (x + 2)^2). \end{aligned}$$

Moreover,

$$\frac{x^9 + 16x}{P(x)P(-x)} = \frac{f(x)}{P(x)} - \frac{f(-x)}{P(-x)},$$

where

$$\begin{aligned} f(x) &= -\frac{1}{12}(x^4 - 2x^3 - 8x^2 - 4x + 4) \\ &= -\frac{1}{12}(2(x^4 + 2x^3 + 2x^2 + 4x + 4) - (x^4 + 6x^3 + 12x^2 + 12x + 4)) \\ &= -\frac{1}{6}(x^4 + 2x^3 + 2x^2 + 4x + 4) + \frac{1}{12}(x^2 + 2x + 2)(x^2 + 4x + 2). \end{aligned}$$

Thus,

$$\frac{f(x)}{P(x)} = -\frac{1}{6} \cdot \frac{1}{x^2 + 2x + 2} + \frac{1}{12} \cdot \frac{x^2 + 4x + 2}{x^2(x + 1)^2 + (x + 2)^2}.$$

Knowing

$$x^2 + 4x + 2 = (x + 2)\frac{d}{dx}(x(x + 1)) - x(x + 1)\frac{d}{dx}(x + 2),$$

it follows that

$$\int \frac{f(x)}{P(x)} dx = -\frac{1}{6} \tan^{-1}(1 + x) + \frac{1}{12} \tan^{-1} \frac{x(x + 1)}{x + 2} + C,$$

and

$$\int \frac{f(-x)}{P(-x)} dx = \frac{1}{6} \tan^{-1}(1 - x) + \frac{1}{12} \tan^{-1} \frac{x(1 - x)}{2 - x} + C.$$

Thus,

$$\int \frac{x^9 + 16x}{P(x)P(-x)} dx = -\frac{1}{6} \left(\tan^{-1}(1 + x) + \tan^{-1}(1 - x)\right) + \frac{1}{12} \left(\tan^{-1} \frac{x(x + 1)}{x + 2} - \tan^{-1} \frac{x(1 - x)}{2 - x}\right) + C.$$

Now,

$$\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \tan^{-1} \frac{1 + x + 1 - x}{1 - (1 + x)(1 - x)} = \tan^{-1} \frac{2}{x^2}$$

and

$$\tan^{-1}\frac{x(x+1)}{x+2}-\tan^{-1}\frac{x(1-x)}{2-x}=\tan^{-1}\frac{\frac{x(x+1)}{x+2}-\frac{x(1-x)}{2-x}}{1+\frac{x^2(1-x)^2}{4-x^2}}=\tan^{-1}\frac{2x^2}{4-x^2},$$

so

$$\int\frac{x^9+16x}{P(x)P(-x)}\,dx=-\frac{1}{6}\tan^{-1}\frac{2}{x^2}+\frac{1}{12}\tan^{-1}\frac{2x^2}{4-x^2}+C.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Khakimboy Egamberganov, Paris, France; Dumitru Barac, Sibiu, Romania; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, NY, USA; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Kevin Soto, Palacios Huarmey, Perú;; S.Chandrasekhar, Chennai, India; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Ioannis D. Sfikas, Athens, Greece.

U513. Let n be a positive integer. Prove that the polynomial

$$X^n + 2\binom{n}{1}X^{n-1} + \cdots + 2\binom{n}{k}X^{n-k} + \cdots + 2\binom{n}{n-1}X + 1$$

is divisible by $X^2 + 5X + 1$, if and only if $n = 3$.

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Li Zhou, Polk State College, USA

For $n = 1$, 2, and 3, the polynomial is $X + 1$, $X^2 + 4X + 1$, and $(X + 1)(X^2 + 5X + 1)$, respectively. Suppose $n \geq 4$. If $X^2 + 5X + 1$ divides the degree-n polynomial, which can be written as $2(X + 1)^n - X^n - 1$, then the zero $(-5 - \sqrt{21})/2$ of the former is also a zero of the latter, that is,

$$2(-3 - \sqrt{21})^n - (-5 - \sqrt{21})^n - 2^n = 0.$$

However,

$$\begin{aligned} |2(-3 - \sqrt{21})^n - (-5 - \sqrt{21})^n| &= (2 + 3 + \sqrt{21})^n - 2(3 + \sqrt{21})^n \\ &> 2^n + \binom{n}{1}(2)(3 + \sqrt{21})^{n-1} - (3 + \sqrt{21})^n > 2^n, \end{aligned}$$

since $2\binom{n}{1} \geq 8 > 3 + \sqrt{21}$.

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

U514. Let $A, B, C \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be such that

$$x^3 + \frac{1}{6}x^2 - \frac{1}{3}x - \frac{1}{15} = (x - \sin A)(x - \sin B)(x - \sin C).$$

Evaluate $391T^4 - 216T^2 - 32T$, where $T = |\tan A \tan B| + |\tan B \tan C| + |\tan C \tan A|$.

Proposed by Li Zhou, Polk State College, USA

Solution by the author

Denote by $f(x)$ the given cubic polynomial. Let $g(x) = -f(x)f(-x)$. Then

$$\begin{aligned}&(x^2 - \sin^2 A)(x^2 - \sin^2 B)(x^2 - \sin^2 C) = g(x)\\ &= \left(x^3 - \frac{1}{3}x + \frac{1}{6}x^2 - \frac{1}{15}\right)\left(x^3 - \frac{1}{3}x - \frac{1}{6}x^2 + \frac{1}{15}\right) = x^6 - \frac{25}{36}x^4 + \frac{2}{15}x^2 - \frac{1}{225}.\end{aligned}$$

Hence, $\sin^2 A \sin^2 B \sin^2 C = \frac{1}{225}$ and $\cos^2 A \cos^2 B \cos^2 C = g(1) = \frac{391}{900}$. Thus, $\tan^2 A \tan^2 B \tan^2 C = \frac{4}{391}$. Also,

$$\begin{aligned}\sum_{cyc} \tan^2 A \tan^2 B &= \sum_{cyc} \tan^2 A \tan^2 B(\sec^2 C - \tan^2 C)\\ &= \frac{\sum_{cyc} \sin^2 A \sin^2 B}{\cos^2 A \cos^2 B \cos^2 C} - 3\tan^2 A \tan^2 B \tan^2 C = \frac{900}{391} \cdot \frac{2}{15} - \frac{12}{391} = \frac{108}{391}.\end{aligned}$$

Next, $\frac{900}{391} = \sec^2 A \sec^2 B \sec^2 C = (1 + \tan^2 A)(1 + \tan^2 B)(1 + \tan^2 C)$, so

$$\sum_{cyc} \tan^2 A = \frac{900}{391} - 1 - \frac{108}{391} - \frac{4}{391} = \frac{397}{391}.$$

Let $S = \sum_{cyc} |\tan A|$. Then $S^2 = \sum_{cyc} \tan^2 A + 2T = \frac{397}{391} + 2T$ and

$$T^2 = \sum_{cyc} \tan^2 A \tan^2 B + 2|\tan A \tan B \tan C| S = \frac{108}{391} + \frac{4}{\sqrt{391}}S.$$

Therefore,

$$\left(T^2 - \frac{108}{391}\right)^2 = \frac{16}{391}S^2 = \frac{16}{391}\left(\frac{397}{391} + 2T\right),$$

which is equivalent to $391T^4 - 216T^2 - 32T = -\frac{5312}{391}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kevin Soto, Palacios Huarmey, Perú.

U515. Prove that for every positive integer n,

$$\left(1+\frac{1}{1^2}+\frac{1}{2^2}\right)\left(1+\frac{1}{2^2}+\frac{1}{3^2}\right)\cdots\left(1+\frac{1}{n^2}+\frac{1}{(n+1)^2}\right)<\frac{37}{5}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that

$$e<2.72=\sqrt{7.3984}<\sqrt{7.4}=\sqrt{\frac{37}{5}}, \qquad e^2<\frac{37}{5}.$$

It therefore suffices to show that the LHS is at most e^2. Note next that for every positive integer k we have

$$1+\frac{1}{k^2}+\frac{1}{(k+1)^2}=\frac{k^2(k+1)^2+(k+1)^2+k^2}{k^2(k+1)^2}=\left(\frac{k^2+k+1}{k^2+k}\right)^2,$$

or it suffices to prove the stronger bound

$$\left(1+\frac{1}{1\cdot 2}\right)\left(1+\frac{1}{2\cdot 3}\right)\cdots\left(1+\frac{1}{n(n+1)}\right)<e.$$

Now, using the AM-GM, and since no two factors are equal, yields for all $n>1$

$$\prod_{k=1}^{n}\left(1+\frac{1}{k(k+1)}\right)<\left(\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{1}{k(k+1)}\right)\right)^n=$$

$$\left(1+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\right)^n=\left(1+\frac{1}{n}\left(1-\frac{1}{n+1}\right)\right)^n=\left(1+\frac{1}{n+1}\right)^n<$$

$$<\left(1+\frac{1}{n+1}\right)^{n+1}<e.$$

The conclusion follows.

Also solved by Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Brian Bradie, Christopher Newport University, VA, USA; Khakimboy Egamberganov, Paris, France; Prajnanaswaroopa S, Bangalore, India; Li Zhou, Polk State College, USA; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Ioannis D. Sfikas, Athens, Greece.

U516. Evaluate

$$\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\ln n}\right)^{\ln n}$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA

We know that $\lim_{n\to\infty}\left(\sum_{k=1}^{n} 1/k - \ln n\right) = \gamma$, the Euler-Mascheroni constant. More precisely, we have $\sum_{k=1}^{n} 1/k = \ln n + \gamma + O(1/n)$, from which it follows that

$$Q_n = (1+1/2+\cdots+1/n)/\ln n = 1+\gamma/\ln n + O(1/n\ln n) \to 1 \text{ as } n\to\infty.$$

In the well-known inequality $1-1/x \le \ln x \le x-1$ for $x \in (0,\infty)$, replace x by Q_n and multiply through by $\ln n$. Then

$$\ln n\cdot(1-1/Q_n) \;\le\; \ln n\cdot\ln Q_n \;\le\; \ln n\cdot(Q_n-1),$$

implying

$$\lim_{n\to\infty}\ln n\cdot(1-1/Q_n) = \lim_{n\to\infty}\ln n\cdot(Q_n-1)\cdot\lim_{n\to\infty}(1/Q_n) = \lim_{n\to\infty}\ln n\cdot(Q_n-1) = \gamma.$$

The squeeze theorem gives us $\ln n\cdot\ln Q_n \to \gamma$ and exponentiation yields $Q_n^{\ln n} \to e^{\gamma} \approx 1.78107$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Brian Bradie, Christopher Newport University, VA, USA; Khakimboy Egamberganov, Paris, France; Prajnanaswaroopa S, Bangalore, India; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Besfort Shala, University of Primorska, Slovenia; G. C. Greubel, Newport News, VA, USA; Joel Schlosberg, Bayside, NY, USA; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Toyesh Prakash Sharma, St.C.F.Andrews School, Agra, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Alexandru Daniel Pîrvuceanu, National Traian College, Drobeta-Turnu Severin, Romania; Ioannis D. Sfikas, Athens, Greece.

O511. Find all 5-tuples (v, w, x, y, z) of integers satisfying the system of equations

$$\begin{aligned} x^2 + xy - 2yz + 3zx &= 2020, \\ y^2 + yz - 2zx + 3xy &= v, \\ z^2 + zx - 2xy + 3yz &= w, \\ v + w &= 5. \end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Summing the four equations we get $(x+y+z)^2 = 2025$, so $x+y+z = \pm 45$. Substituting $z = \pm 45 - x - y$ into the first equation we get $2y^2 \mp 90y - \left(2x^2 \mp 135x + 2020\right) = 0$. Viewed as a quadratic equation in y, its discriminant $(4x \mp 135)^2 + 6035$ must be a square, say m^2. Therefore,

$$(4x \mp 135 + m)(4x \mp 135 - m) = -6035 = -5 \cdot 17 \cdot 71.$$

It is then easy to see that $8x \mp 270$ must equal $\pm(6035-1)$, $\pm(1207-5)$, $\pm(355-17)$, or $\pm(71-85)$. They yield $x = \pm 788, \pm 184, \pm 76, \pm 32$. It is then routine to obtain the corresponding y, then the corresponding z, and then the corresponding v and w. The 16 5-tuples (v, w, x, y, z) obtained are:

$$\begin{aligned} &(3655037, -3655032, \pm 788, \pm 777, \mp 1520); \\ &(-1169236, 1169241, \pm 788, \mp 732, \mp 11); \\ &(187046, -187041, \pm 184, \pm 174, \mp 313); \\ &(-49597, 49602, \pm 184, \mp 129, \mp 10); \\ &(28793, -28788, \pm 76, \pm 69, \mp 100); \\ &(-3664, 3669, \pm 76, \mp 24, \mp 7); \\ &(6434, -6429, \pm 32, \pm 42, \mp 29); \\ &(-313, 318, \pm 32, \pm 3, \pm 10); \end{aligned}$$

and one can check they all satisfy the system.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Dumitru Barac, Sibiu, Romania; Jennisha Sunil Agrawal, Disha Delphi Public School, India; Konstantina Rasvani, 1st High School, Volos, Greece; Titu Zvonaru, Comănești, Romania.

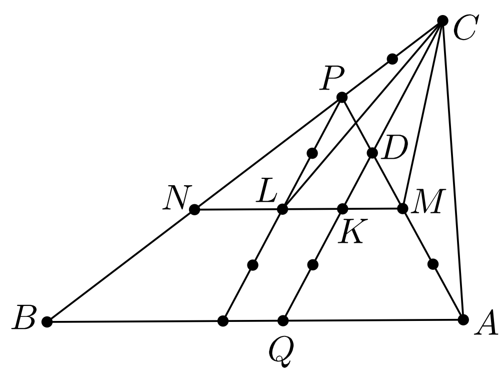
O512. Let ABC be a triangle. Points P and Q lie on sides BC and AB, respectively, such that

$$\angle PAB = \angle CQA = 90^\circ - \frac{1}{2}\angle ACB.$$

Prove that if $AP = 2CP$, then the inradii of triangles ACQ and BCQ are equal.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Li Zhou, Polk State College, USA



Suppose that AP and CQ intersect at D. Then $DA = DQ$ and $\angle CDP = 180^\circ - 2\angle PAB = \angle ACB$, so $\triangle PCD \sim \triangle PAC$. Therefore, $PA = 2PC = 4PD$ and $DC + CA = 3DC$. Let M and N be the midpoints of PA and PB, respectively. Suppose that MN and CQ intersect at K. Locate L on MN such that $PL \parallel CK$. Since $PL = PM = PC$, CL is the bisector of $\angle NCK$. Therefore,

$$\frac{BQ}{BC + CQ} = \frac{NL}{NP + PL} = \frac{NL}{NC} = \frac{KL}{KC} = \frac{KM}{KD + DC} = \frac{QA}{QC + CA}.$$

Denote by $[XYZ]$ the area of $\triangle XYZ$. We then have

$$\frac{[BQC]}{[AQC]} = \frac{QB}{AQ} = \frac{QB + BC + CQ}{AQ + QC + CA},$$

from which the claim follows.

Also solved by Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Konstantina Rasvani, 1st High School, Volos, Greece.

O513. Let x, y, z be positive real numbers such that

$$(x+y+z)^9 = 9^5 x^3 y^3 z^3.$$

Prove that
$$\left(\frac{3(x+y)}{z} + \frac{z^2}{xy} + 2\right)\left(\frac{3(y+z)}{x} + \frac{x^2}{yz} + 2\right)\left(\frac{3(z+x)}{y} + \frac{zx}{y} + 2\right) < 3^7.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that the proposed condition rewrites as

$$\frac{x+y+z}{3} = \sqrt[9]{3} \cdot \sqrt[3]{xyz},$$

or since the AM-GM of x, y, z are not equal, x, y, z cannot be all equal. Note next that the sum of the three factors in the LHS of the proposed inequality rewrites as

$$\sum_{cyc} \frac{z^3 + 3xy(x+y) + 2xyz}{xyz} = \frac{(x+y+z)^3}{xyz}.$$

Therefore, by the AM-GM inequality, we have

$$\left(\frac{3(x+y)}{z} + \frac{z^2}{xy} + 2\right)\left(\frac{3(y+z)}{x} + \frac{x^2}{yz} + 2\right)\left(\frac{3(z+x)}{y} + \frac{zx}{y} + 2\right) \leq$$

$$\leq \left(\frac{(x+y+z)^3}{3xyz}\right)^3 = \frac{(x+y+z)^9}{3^3 x^3 y^3 z^3} = \frac{9^5}{3^3} = 3^7.$$

Equality holds iff $\frac{3(x+y)}{z} + \frac{z^2}{xy} = \frac{3(y+z)}{x} + \frac{x^2}{yz} = \frac{3(z+x)}{y} + \frac{y^2}{zx}$. Now, assuming that $z \geq x, y$, we have $(x-y)(x^2 + xy + y^2 - 3z^2 - 3xz - 3yz) = 0$, where the term in brackets is clearly negative, or $x = y$. Since $z \neq x = y$ because not all of x, y, z are equal, this results in $z^2 - 2xz - 5x^2 = 0$, or since x, z are both positive, $z = x\left(\sqrt{6} + 1\right)$. Substitution shows that the condition given in the problem statement is not met, or the inequality holds strictly. The conclusion follows.

Also solved by Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

O514. Find all pairs of integers (m,n) such that $6(m+1)(n-1)$, $(m-1)(n+1)+6$, and $(m+2)(n-2)$ are simultaneously perfect cubes.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Li Zhou, Polk State College, USA

Clearly, $(m,n)=(-1,2)$ and $(-2,1)$ are solutions. We show they are the only ones. Let a^3, b^3, and c^3 be the three given expressions, respectively. Then $a^3=2b^3+4c^3$. Successively, we must have $a=2u$, $b=2v$, and $c=2w$, thus $u^3=2v^3+4w^3$. This descent implies that one of a,b,c is 0. Therefore, a,b,c must all be 0, since $\sqrt[3]{2}$ is irrational. Now for both a,c to be 0, either $(m,n)=(-1,2)$ or $(-2,1)$.

Also solved by Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Jennisha Sunil Agrawal, Disha Delphi Public School, India; Dumitru Barac, Sibiu, Romania; Anish Ray, Institute of Mathematics and Applications, India.

O515. If a, b are real numbers find the extreme values of the expression

$$\frac{(1-a)(1-b)(1-ab)}{(1+a^2)(1+b^2)}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Let $a=\tan x$, $b=\tan y$, $x,y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. The our expression, which we note with E, becomes

$$E=(\cos x-\sin x)(\cos y-\sin y)(\cos x\cos y-\sin x\sin y),$$

or

$$E=\cos(x+y)\left(\cos(x-y)-\sin(x+y)\right),$$

or

$$E=\cos(x+y)\left(\cos(x-y)-\cos\left(\frac{\pi}{2}-x-y\right)\right),$$

or

$$E=2\cos(x+y)\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right).$$

Let's make the following substitutions $\alpha=\frac{\pi}{4}-x$, $\beta=\frac{\pi}{4}-y$, then

$$E=2\sin(\alpha+\beta)\sin\alpha\sin\beta.$$

By Caucy-Schwarz Inequality and AM-GM Inequality, we have

$$\begin{aligned}E^2&=4\sin^2(\alpha+\beta)\sin^2\alpha\sin^2\beta\\&=4(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2\sin^2\alpha\sin^2\beta\\&\le4(\sin^2\alpha+\sin^2\beta)(\cos^2\beta+\cos^2\alpha)\sin^2\alpha\sin^2\beta\\&=\frac{16}{3}\sin^2\alpha\sin^2\beta\left(\frac{\sin^2\alpha+\sin^2\beta}{2}\right)\left(\frac{3\cos^2\alpha+3\cos^2\beta}{2}\right)\\&\le\frac{16}{3}\left(\frac{\sin^2\alpha+\sin^2\beta+\frac{\sin^2\alpha+\sin^2\beta}{2}+\frac{3\cos^2\alpha+3\cos^2\beta}{2}}{4}\right)^4=\frac{27}{16},\end{aligned}$$

so

$$-\frac{3\sqrt{3}}{4}\le E\le\frac{3\sqrt{3}}{4}.$$

We get the minimum value for $\alpha=\beta=\frac{2\pi}{3}\iff x=y=-\frac{5\pi}{12}\iff a=b=-\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$ and the maximum value for $\alpha=\beta=\frac{\pi}{3}\iff x=y=-\frac{\pi}{12}\iff a=b=-\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.

Also solved by Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

O516. Let ABC be a triangle and let Δ be its area. Prove that

$$a^2 \tan\frac{A}{2} + b^2 \tan\frac{B}{2} + c^2 \tan\frac{C}{2} \geq 2\frac{R}{r}\Delta.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Since $s = \frac{1}{2}(a+b+c)$, $\Delta = rs$, $r^2 s = (s-a)(s-b)(s-c)$ and $\tan\frac{A}{2} = \frac{r}{s-a}$, etc, the stated inequality becomes

$$s\left(a^2+b^2+c^2-2(ab+bc+ca)\right)+5abc \geq 2Rrs. \tag{1}$$

Since $a^2+b^2+c^2 = 2\left(s^2-r^2-4Rr\right)$, $ab+bc+ca = s^2+r^2+4Rr$ and $abc = 4Rrs$, (1) becomes

$$2(R-r) \geq R. \tag{2}$$

But (2) is equivalent to the familiar inequality

$$R \geq 2r.$$

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