Polar duality in Olympiad geometry

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Abstract

In this article we present a way to use polar duality in olympiad geometry. Polar duality is a geometric "transformation" that swaps points and lines. The idea of using it in problem solving is to use this transformation to get equivalent geometry problems that are easier to solve than the original ones. With help of some new notation, we can easily recalculate angle conditions in the transformed problems. This article contains some examples on how to use this method and some problems for you to try it on. For each problem there is a hint at the end of the article. As most of the presented theory is not hard to prove, we often just present observations and let the reader convince themselfs that they really hold.

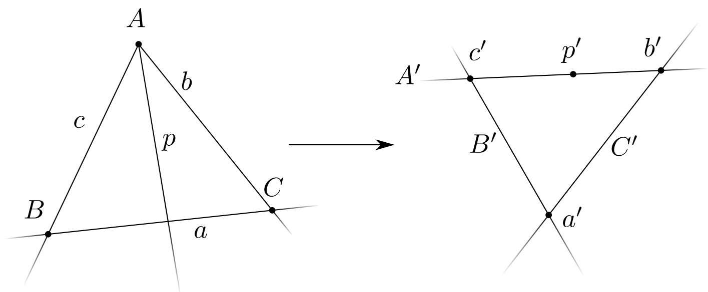
1 General point-line duality

Imagine that you have a set of lines $\mathcal{L}$ and a set of points $\mathcal{P}$. Given a point and a line from these sets, we say those two form a *good pair* if the point lies on a line. Formally, we define a relation called incidence on $\mathcal{P} \times \mathcal{L}$ such that a pair (P, L) is in the relation if and only if the point P lies on the line L. We want duality to be a transformation that swaps points and lines and preserves this incidence relation and swaps points and lines. So for example if we have 3 points on one line, we want to get 3 lines passing through one point.

Example.

Draw points A, B, C, lines $a = BC$, $b = CA$, $c = AB$ and one more line p such that $A \in p$. What will the dual look like?

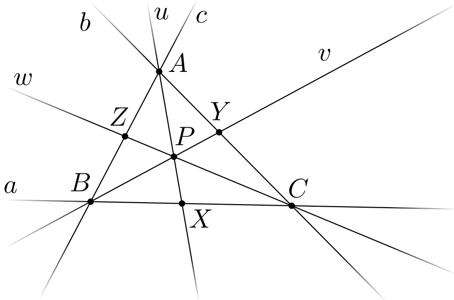
Solution. We will get lines A', B', C'. Denote the intersections of $c' = A' \cap B'$, $b' = A' \cap C'$, $a' = B' \cap C'$. Now we may choose p' to be any point on the line A'.



Exercise.

As you can see, we really just swapped *an intersection of two lines* with *a line through two points*. Try it for yourself with this diagram.

I'd like to thank Matěj Doležálek and Josef Tkadlec for proofreading this article.



2 Standard polar duality

Let O be the origin of our plane.

Definition.

Dot product: Take two points X, Y different from O. Then we define their dot product as $X \cdot Y = |XO| \cdot |YO| \cos(\angle XOY)$. You may notice that if we denote X' the orthogonal projection of X onto line YO, then dot product is $|OX'| \cdot |OY|$. It is important that the dot product is commutative ($X \cdot Y = Y \cdot X$).

Now we are ready to define a standard polar duality with coefficient $k \in \mathbb{R} \setminus \{0\}$.

Definition.

Polar: Take a point P different from O. Then we define its polar to be the set of points X such that $P \cdot X = k$, where k is the *coefficient* of our duality. These points form a line p perpendicular to PO. We call p the *polar* of P and P the *pole* of p.

Observation. For every point different from origin, there is exactly one polar, and for every line not passing through origin, there is exactly one pole. Further, this duality is self-inversive, since taking the pole of a polar yields the original point.

Observation. This point-line duality satisfies what we want from a general point-line duality. Namely, for every point P and line ℓ, the point P lies on ℓ if and only if the pole of ℓ lies on the polar of P. [1] Indeed, denote Q the pole of ℓ, then both statements are equivalent with $P \cdot Q = k$.

Observation. When we have points P, P^* that are inverses in the inversion with center O and coefficient k, then the polar of P is the line passing through P^* perpendicular to OP.

3 Angle chasing with duality

We will define some custom angle notation to make angle chasing simpler. We will refer to this notation as the polar format of angles. We define angles with respect to the center of duality O:

Definition.

(i) $\overset{O}{\angle} AB$: The angle between two points is the directed angle from line OA to line AB.
(ii) $\overset{O}{\angle} pq$: The angle between two lines is the directed angle from p to q. (We don't really use O here, but it will be useful in our notation)
(iii) $\overset{O}{\angle} Ap$: The angle between a point A and a line p is the directed angle between OA and p. Analogously, $\overset{O}{\angle} pA$ is the directed angle between p and OA.

The reason why this angle format is really great for dealing with polars is the following lemma.

[1] This property is often called La-Hire's theorem, but it is really just the fundamental property of how we want polars to work.

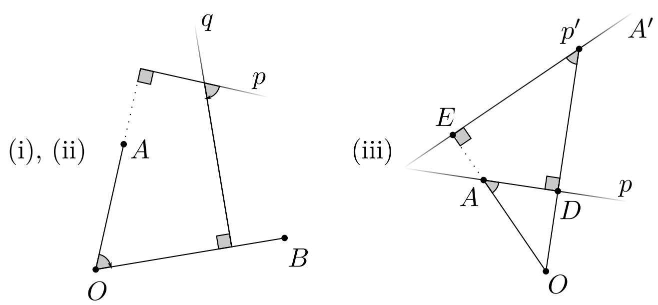
Lemma. (Duality Angle Chasing)

Denote with apostrophe the duals of points and lines in some duality with center O. Then

(i) ${}^{O}\!\angle AB = {}^{O}\!\angle A'B'$,
(ii) ${}^{O}\!\angle pq \;\; = {}^{O}\!\angle p'q'$,
(iii) ${}^{O}\!\angle Ap \;\; = {}^{O}\!\angle A'p'$.

Proof. The proof is straighforward, using only the fact that OA is perpendicular to A'.

- (i),(ii) Denote p the polar of A and q the polar of B. Then there exists a rotation with angle AOB that maps OA to OB. Since $p \perp OA$ and $q \perp OB$, this rotation maps p, q to parallel lines, hence the angle between them is the same as the angle between OA and OB.
- (iii) Denote D and E the feet of the altitudes from O onto p and A', respectively. Then $DAEp'$ (right angles) is cyclic, and thus ${}^{O}\!\angle Ap = {}^{O}\!\angle A'p'$. □

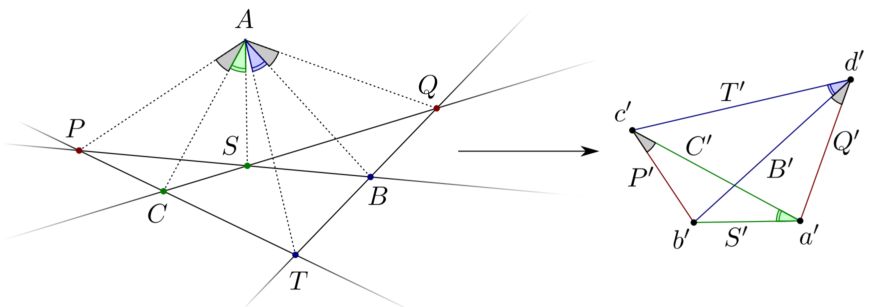


The idea of what the first two conditions do to angles is that they move angles that are at the center of our duality “away” and move angles angles that are “far away” back to the duality center. The idea is better illustrated with an example.

Example. (Isogonal Line Lemma)

Let AS and AT be isogonals in $\angle BAC$. Denote $P = CT \cap BS$ and $Q = CS \cap BT$. Prove that AP and AQ are isogonal in $\angle CAB$.

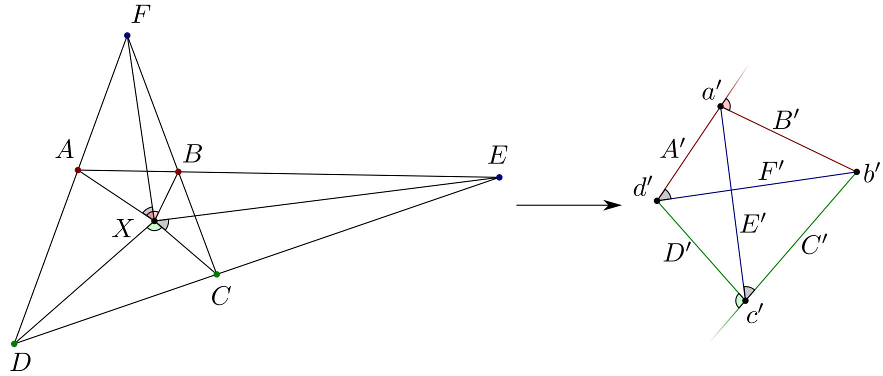
Solution. All the angle conditions here are crammed around point A, so if we apply polar duality with center A, we might get these angles to more usable places. So let’s do it. Let us first transform the angle conditions into polar format. From isogonality, we know that ${}^{A}\!\angle CS = {}^{A}\!\angle TB$ and we want to prove that ${}^{A}\!\angle CP = {}^{A}\!\angle QB$. After applying the duality, we get that ${}^{A}\!\angle C'S' = {}^{A}\!\angle T'B'$ and we want to prove ${}^{A}\!\angle C'P' = {}^{A}\!\angle Q'B'$. So we draw lines C', B', S', T' so that ${}^{A}\!\angle C'S' = {}^{A}\!\angle T'B'$. Denote $a' = C' \cap S'$, $b' = S' \cap B'$, $c' = C' \cap T'$ and $d' = T' \cap Q'$. Then the angle condition gives us that $a'b'c'd'$ is concyclic. Now $P' = b'c'$ and $Q' = a'd'$ and we want to prove that ${}^{A}\!\angle C'P' = {}^{A}\!\angle Q'B'$. But that follows from concyclicity. □



Example.

Let $ABCD$ be a convex quadrilateral. Let lines AB and CD meet at E, AD and BC meet at F. Suppose X is a point inside the quadrilateral such that $\angle AXF = \angle EXC$. Prove that $\angle AXB + \angle CXD = 180°$.

Solution. We consider a duality with center X, seeing that all the angles in the figure are concetrated here. We rewrite angles into polar format. We know that ${}^{X}\angle AF = {}^{X}\angle EC$ and we want to prove that ${}^{X}\angle AB + {}^{X}\angle CD = 180°$. Denote $a = AB$, $b = BC$, $c = CD$, $d = DA$, so that we have reasonable notation for the poles. Now apply the duality. We have points a', b', c', d'. We know, that $E' = a'c'$ and $F' = b'd'$. And from the angle condition, we know that ${}^{X}\angle A'F' = {}^{X}\angle E'C'$. From this, we get that $a'b'c'd'$ is concyclic, and thus ${}^{X}\angle A'B' + {}^{X}\angle C'D' = 180°$, which dualized is exactly what we wanted to prove. □



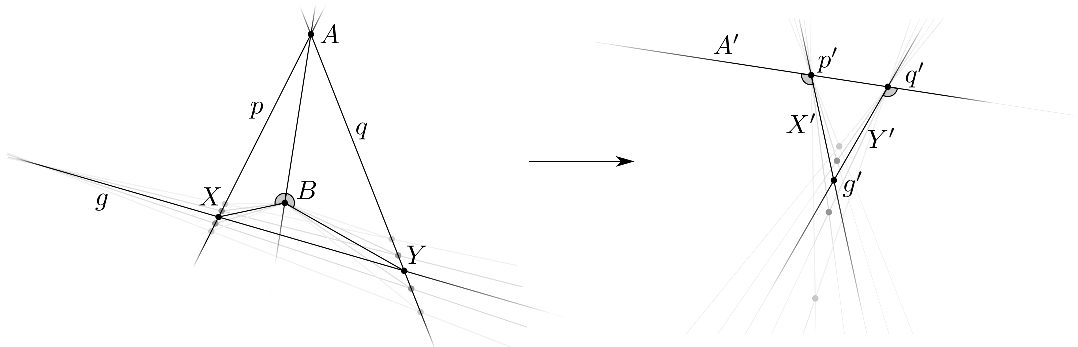
Do you see the magic? Both of these proofs were just duals of the inscribed angle theorem. At first glance, they really don't look the same, but with duality goggles, these problems are dual to the same simple theorem. And there are a few more duals of this trivial fact. Here we just took dual with respect to an arbitrary point X and we looked at different inscribed angles. We could take the dual with respect to some special point in the diagram to get completely different results.

Here is one more example for you, that is not dual of inscribed angle theorem.

Example.

Let p, q be two non-parallel lines. Denote $A = p \cap q$. Choose an arbitrary point B that does not lie on p, q. Then consider a variable point X on the the line p, let Y be a point on q such that $|\angle XBA| = |\angle ABY|$. Prove that lines XY all pass through a fixed point.

Solution. Consider a duality with center B. Then how will the dual look like? We have points p', q' on the line A' and a fixed point B. Then X' and Y' are moving lines, where X' passes through p' and Y' passes through q'. Let us transform the angle condition into polar format. We get ${}^{B}\angle XA = {}^{B}\angle AY$. After applying the duality, we get ${}^{B}\angle X'A' = {}^{B}\angle A'Y'$. Denote $g' = X' \cap Y'$. Then from this angle condition we get, that $p'q'g'$ is isosceles. Hence g' lies on the perpendicular bisector of $p'q'$ and that is a fixed line. Hence $g = XY$ had to pass through a fixed point, the pole of this line. □



We have seen what the first two conditions of *Duality Angle Chasing* are capable of. It is time to use the third one. Angles from the third condition "mirror" and "stay in the same distance". Try it on this exercise.

Exercise.

Let BAC be an angle and AD its bisector. What will the dual be with respect to a point on AB, with respect to a point on AD and with respect to an arbitrary point?

Here are some problems for you to try this angle chasing on. Most of these should be easy, or even trivial after the right duality is applied.

Problem 1. (Blanchet) Let ABC be a triangle and D the foot of A-altitude. Let E, F be points on sides AC, AB such that lines BE, CF and AD are concurrent. Prove that $|\angle EDA| = |\angle FDA|$.

Problem 2. In parallelogram $ABCD$ denote F point such that $|\angle CDF| = |\angle CBF|$. Prove that $|\angle FCB| = |\angle FAB|$.

Problem 3. Let ABC be a triangle and I its incenter. Let P be a point on BI and Q point on CI, such that $|\angle PAC| = |\angle QAB|$. Prove that lines PB, QC and AI are concurrent.

Problem 4. Let ABC be a triangle such that $|\angle BAC| = 120°$. Denote AD, BE, CF the bisectors of the interior angles of ABC. Points D, E, F lie on sidelines of ABC. Prove that $|\angle EDF| = 90°$.

Problem 5. Let $ABCD$ be a convex quadrilateral. Suppose that circles with diameters AB and CD intersect at points X, Y. Denote $P = AC \cap BD$ and $Q = AD \cap BC$. Prove that points P, Q, X, Y are concyclic.

Problem 6. (Jacobi) Let ABC be a triangle. Let P, Q, R be points not on its sides, such that $|\angle PAB| = |\angle CAQ|$, $|\angle ABP| = |\angle RBC|$ and $|\angle BCR| = |\angle QCA|$. Prove that AR, CP and BQ are collinear.

Problem 7. (2012 USAMO Day 2 #5) Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

Problem 8. Let ABC be an acute triangle and P a point inside. Denote A_0 point on ray CB, such that $|\angle PCB| = |\angle A_0PB|$. Denote B_0 point on ray AC, such that $|\angle PAC| = |\angle B_0PC|$. Denote C_0 point on ray AB such that $|\angle BAP| = |\angle BPC_0|$. Prove that A_0, B_0, C_0 are collinear.

There is a small problem with duality, namely that we can't do much about circles other than to transform them into angle conditions. The following lemma can help if there is just one important circle.

Lemma. (Dual Circle)

Let $ABPC$ be a cyclic quadrilateral. Denote $a = BC$, $b = CA$, $c = AB$. Then if we take any duality centered at P, the quadrilaterals $ABPC$ and $a'b'Pc'$ are similar.

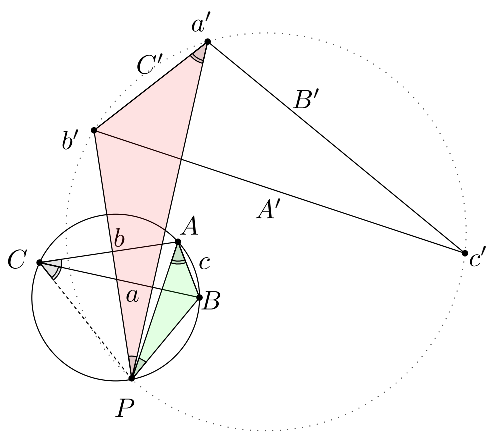
Proof. It sufficies to prove that triangles ABP and $a'b'P$ are similar. From concyclicity, $|\angle BAP| = |\angle BCP|$. Let us transform the angle $|\angle BCP|$ into polar format:

$$ {}^{P}\!\angle \overleftrightarrow{BC}\, C = {}^{P}\!\angle aC = {}^{P}\!\angle a'C'. $$

Hence $|\angle b'a'P| = |\angle BAP|$. Analogously $|\angle APB| = |\angle ACB|$. The angle $|\angle ACB|$ in polar format:

$$ {}^{P}\!\angle \overleftrightarrow{AC}\,\overleftrightarrow{CB} = {}^{P}\!\angle ba = {}^{P}\!\angle b'a'. $$

Hence $|\angle a'Pb'| = |\angle APB|$. From these two angle equalities we deduce, that $\triangle ABP \sim \triangle a'b'P$. $\square$



Corollary.

When you have a circle ω circumscribed to a triangle formed by lines a, b, c, then after duality with center $P \in \omega$, the points a', b', c', P are concyclic.

Example. (Miquel point)

Consider four lines a, b, c, d. They form four triangles. Then the circumcircles of those triangles pass through a single point.

Solution. Denote M the intersection of circumcircles of triangle generated by lines (abc), (abd) different from $a \cap b$. Then from two applications of our corollary, we get that after duality with center M, the points a', b', c', M are concyclic and points a', b', d', M are concyclic. But from that we get that $a'b'c'd'M$ is cyclic, hence again from our corollary (applied in the other way) M lies on circumcircle of lines (acd) and (bcd). $\square$

You may view the transformation as "merging" those four circles into one. This gives us new insight into what the Miquel point is. Try to see the following Miquel point properties using duality.

Problem 9. Let $ABCD$ be a quadrilateral and M the Miquel point of lines[1] AB, BC, CD, DA. Denote $P = AB \cap CD$ and $Q = CB \cap DA$. Then

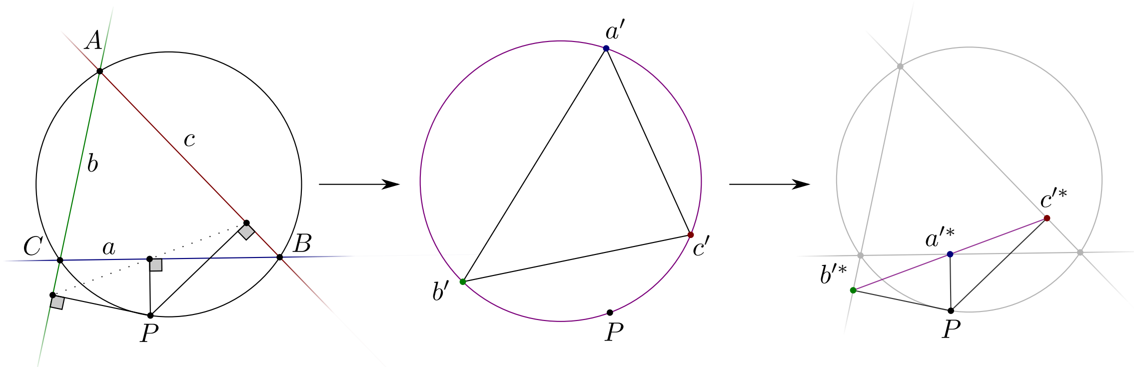
- M lying on PQ is equivalent to $ABCD$ being concyclic.
- $|\angle AMB| = |\angle CMD|$.

We conclude this section with one more example that features a harmonic quadrilateral.

Example. (Simson line)

Let ABC be a triangle and P a point on its circumcircle on the arc BC not containing A. Denote a, b, c the sides of triangle. Then the feet of perpendiculars from P to a, b, c are collinear. Denote the feet X, Y, Z respectively. Then X is the midpoint of YZ if and only if the quadrilateral $ABPC$ is harmonic.

Solution. We'll use apostrophe for duals and asterisk for inverses. We will use the fact that when you apply a polar duality with center P to the line ℓ followed by an inversion with center P with the same coefficient, then ℓ'^* is the foot of the altitude from P to ℓ. First we apply the polar duality centered at P. From the *Dual Circle* lemma, we get that quadrilaterals $ABPC$ and $a'b'Pc'$ are similar, hence $a'b'Pc'$ is cyclic and has the same cross ratio as $ABPC$. Now we apply the inversion centered at P with the same coefficient. Because $a'b'Pc'$ is cyclic, we get that a'^*, b'^*, c'^* are collinear. But these are exactly the points X, Y, Z. Additionally, since inversion preserves cross ratio, $(a', b', P, c') = (a'^*, b'^*, P^*, c'^*)$. But because P^* is the point at infinity, X being the midpoint of YZ is really equivalent to $(A, B, P, C) = -1$. $\square$



Here are few more problems using this inversion-duality combinations.

Problem 10. Let ABC be a triangle and P point on its circumcircle. Denote C_1 point on AC such that $C_1P \perp CP$. Analogously B_1 point on AB such that $B_1P \perp BP$. Line B_1C_1 intersects BC at point X. Prove that midpoint of PX lies on the Simson line of P with respect to ABC.

[1] Often denoted just as the Miquel point of $ABCD$.

Problem 11. Let $ABCD$ be a cyclic quadrilateral. Denote M its Miquel point and P the intersection of AC and BD. Reflect M to M_1, M_2, M_3, M_4 across lines AB, BC, CD, DA. Prove that M_1, M_2, M_3, M_4 and P are collinear.

Problem 12. Let $ABCD$ be a cyclic quadrilateral. Denote P point on its circumcircle. Denote ℓ_1, ℓ_2, ℓ_3, ℓ_4 denote Simson lines of P with respect to ABC, BCD, CDA, DAB respectively. Reflect P over ℓ_i to get P_i. Prove that all four points P_i lie on one line.

4 Duality with respect to a circle

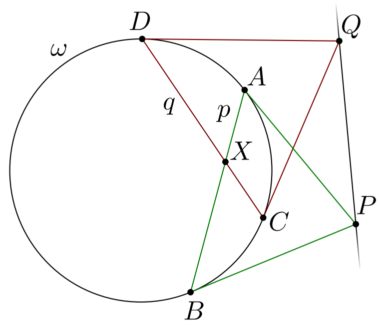
By now, we have used polars and duality just with its center. But if you have met polars somewhere else, they were probably defined with respect to a circle. So for a polar duality with coefficient k^2 and center P we draw a circle with center P and radius k. Then this circle is special for this duality, since every point X on this circle gets mapped to the tangent to this circle through X. This is the only observation we will need to get standard constructions of polars with respect to circle.

Definition.

By duality with respect to a circle with center P and radius r, we mean the duality with center P and coeficient r^2.

Observation. (Polar construction 1) For a point X lying outside of the circle ω, denote Y, Z the points of tangency of the tangents from X to ω. Then YZ is the polar of X under the duality with respect to ω.

Observation. (Polar construction 2) For point X lying inside of ω, denote AB and CD two different chords of ω passing through X. Let the tangents to ω at A and B meet at P. Tangents at C and D to ω meet at Q. Then PQ is the polar of X with respect to ω.



As all dualities with the same center are different only by homothethy, taking a dual with respect to a circle is mostly useful to find pole-polar relations hidden in the diagram or to make dualization process easier. Try it on the following problems.

Problem 13. Let ABC be a triangle with ω being its incircle and I its incenter. Denote p line tangent to ω different from AB, BC, CA. On p denote A_0, B_0, C_0 points, such that $|\angle AIA_0| = |\angle BIB_0| = |\angle CIC_0| = 90°$. Prove that lines AA_0, BB_0, CC_0 are concurrent.

Problem 14. Let ABC be a triangle with incenter I. Denote p line AC reflected around BI. Analogously q is line AB reflected around CI. Denote $G = p \cap q$. Prove that $GI \perp BC$.

Problem 15. Let $ABCD$ be cyclic circumscribed quadrilateral, with incenter I and circumcenter O. Denote $P = AC \cap BD$. Prove that P, I, O are collinear.

5 Polars in the projective world

As you probably know, there are no poles of lines passing through origin and there is no polar of origin. So let's augment our plane a bit. We will work in $\mathbb{RP}^2$. That means we have the plane $\mathbb{R}^2$ and for every line direction we introduce one *point at infinity* and all of these points lie on one *line at infinity*. And we define the polar of origin to be the *line at infinity* and for a line passing through origin, we define its pole to be the point at infinity in the direction perpendicular to the given line. Convince yourself that this still satisfies what we want a general point-line duality to do.

Observation. See that our two polar constructions from the last section are projectively invariant, that is, observe that they use just tangents and lines. That means that every projective transformation preserving a circle also preserves the pole/polar relation with respect to such circle.

Observation. Duality preserves cross ratios of points on a line and of pencils. Because if P is the center of the duality and points A, B, C, D lie on a line q, then $(A, B, C, D) = (PA, PB, PC, PD)$ and polars of A, B, C, D are perpendicular to these lines and are concurrent in q'. And if q passes through P, then duality is just an inversion on this line.

Let's see the power of polars with projective transformations on the following example.

Example.

Let $ABCD$ be a cyclic quadrilateral with circumcircle ω. Denote $P = AC \cap BD$, $Q = BC \cap DA$ and $R = AB \cap CD$. Then the triangle PQR is self-polar with respect to ω. In other words, $\overleftrightarrow{PQ} = R'$, $\overleftrightarrow{QR} = P'$ and $\overleftrightarrow{RP} = Q'$ under the duality with respect to ω.

Solution. Apply a projective transformation that sends PQ to infinity and preserves ω. Then $ABCD$ becomes a rectangle and PQR becomes trivially self-polar. Hence it was self-polar before our transformation. □

The example statement itself was in the language of polars, but we could have stated it for example as follows: Prove that center of ω is the orthocenter of $R'P'Q'$. We can deduce this easily using those pole-polar relations we got.

Try combining duality with projective transformations and cross ratios on the following problems.

Problem 16. Let ABC be a triangle and I its incenter. Denote D, E, F the tangency points of incircle with lines BC, AC, AB respectively. Denote $S = EF \cap BC$. Prove that $SI \perp AD$.

Problem 17. Let $ABCD$ be circumscribed quadrilateral with incenter I. Denote $P = BC \cap AD$ and $Q = AB \cap CD$. Denote M the perpendicular projection of I onto PQ. Prove that $|\angle DMI| = |\angle BMI|$

Observation. Note that applying two polar dualities with different centers gives projective transformation, as it preserves incidencies and maps lines to lines.

To close this section we mention a fun statement and leave the proof to the reader. Suppose that ω is some conic. Consider any duality. Then the set of all polars of points $X \in \omega$ is a set of tangents to some fixed conic.

6 General Triangle Self-Duality

In example from the last section, we got to see that for some triangles, there exists a duality that we called self-duality that just swaps vertices and sidelines. For clarity, here is a definition.

Definition.

We call a duality the *self-duality* with respect to a triangle ABC, when image of its verticies under this duality are its sidelines. So if we denote sidelines the standard way, then $A' = a$, $B' = b$ and $C' = c$.

Now let's take a look at when such a duality exists. Let ABC be a triangle. Where must the center of such duality lie in order for $a = A'$? It has to lie on the A-altitude, since the line connecting the point A and the center of duality is perpendicular to the polar of A. Analogously, the center must lie on the other two altitudes. So it has to be the orthocenter of ABC. But will it always exist? The following lemma answers this question.

Lemma. (Self-duality)

For a not-right-angled triangle ABC with orthocenter H there exists a self-duality for this triangle. And this duality has center H.

Proof. Consider a duality with center H. Choose the coefficient of duality in such a way that $A' = a$. Then what is the image of B under this duality? B lies on A', hence the polar of B passes through A. It also has to be perpendicular to BH. The only line satisfying these criteria is b. Hence $B' = b$. Analogously, $C' = c$. □

Exercise.

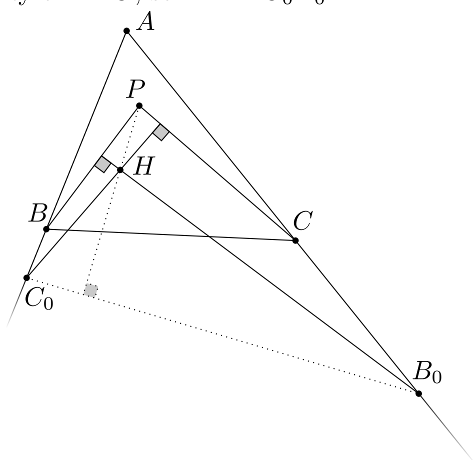
Convince yourself that we really need that ABC isn't right-angled.

As with the duality with respect to a circle, setting exact coefficient is just change up to homothethy. So the information we want to get is some pole-polar relationwith respect to this duality that is hidden in the diagram. See the following example.

Example.

Let ABC be an acute triangle with orthocenter H. Inside of ABC there is a point P. Denote B_0 the point on AC such that $B_0H \perp BP$. Analogously, denote C_0 on AB such that $C_0H \perp CP$. Prove that $PH \perp C_0B_0$.

Solution. We will find the poles of lines PB and PC in the self-duality of ABC. Line PB passes through B, hence its pole lies on b. Denote $Q = \overleftrightarrow{PB}'$. Because H is the center of duality, $HQ \perp PB$. From this, we get $Q = B_0$. Analogously, C_0 is the pole of PC. Hence the line C_0B_0 is the polar of $P = PB \cap PC$ under self-duality of ABC, so $PH \perp C_0B_0$. □



Now it's time for you to try this on the following set of problems.

Problem 18. Let ABC be a triangle and H its orthocenter. Then let D, E, F be points on lines BC, AC, AB, such that AD, BE, CF are concurrent. A perpendicular to DH passing through A intersects BC at A_0. Analogously, a perpendicular to EF through B intersects AC at B_0 and a perpendicular to FH through C intersects AB at C_0. Prove that A_0, B_0, C_0 are collinear.

Problem 19. Let ABC be a triangle with orthocenter H. Denote M_a the midpoint of BC, M_b the midpoint of AC and M_c the midpoint of AB. Denote X_a the intersection of ray M_aH with circumcircle of ABC. Denote Y_a the intersection of AX_a with BC. Analogously construct point X_b, X_c and Y_b, Y_c. Prove that Y_a, Y_b, Y_c are collinear.

Problem 20. Let ABC be a triangle with orthocenter H. On side BC choose point D. Line through H perpendicular to DH intersects lines AB and AC at points K, L. Prove that $\frac{|KH|}{|HL|} = \frac{|BD|}{|DC|}$.

Problem 21. Let ABC be a triangle with orthocenter H. On AC choose point B', such that $|\angle AHB'| = |\angle ABC|$. Analogously on AB point C', such that $|\angle AHC'| = |\angle ACB|$. Prove that $B'C' \parallel BC$.

Problem 22. Denote D, E, F the feet of altitudes in ABC. Denote M the midpoint of BC. Line EF intersects BC at X. Prove that line perpendicular to XA through M passes through the orthocenter of ABC.

Problem 23. (Droz-Farny) Let ABC be a triangle with orthocenter H. Lines p, q are perpendicular to each other and both pass through H. line p intersects BC at A_0, AC at B_0 and AB at C_0. Analogously line q intersects BC at A_1, AC at B_1 and AB at C_1. Prove that midpoints of segments A_0A_1, B_0B_1 and C_0C_1 are collinear.

Problem 24. Let ABC be a triangle, H its orthocenter and M the midpoint of BC. Let K be the intersection of ray MH with circumcircle of ABC. Line parallel to BC passing through H intersects AK att X. External angle bisector of BHC intersects BC at Y. Denote S the midpoint of arc BC of circumcircle of ABC not containing A. Prove that $SH \perp XY$.

Problem 25. Let H be the orthocenter in a triangle ABC. External angle bisector of BHC intersects BC at X. Internal angle bisector of AB intersects AB at Y and internal angle bisector of AHC intersects AC at Z. Prove that X, Y, Z are collinear.

7 Hints

1. Duality with center D.
2. Dual with center F. You will get inscribed angel theorem once again.
3. Duality with center A.
4. Duality with center D gives you buch isosceles and one equilateral triangle.
5. Prove that $|\angle PXQ| = 90°$.
6. Dual with center A. Find similar triangles and a circle. It's also nice to see what the duality with center at incenter does.
7. Dual with respect to P. Angle chase with circumcircle of dual triangle.
8. Duality with center P.
9. Transform concyclicity into equivalent angle condition.
10. Draw bisector of PX. Find its pole and prove that it lies on the circumcircle of dual triangle.
11. Use homothethy to work with foot from M instead of reflection. Define center of MP as foot to some line and find its pole.
12. Dual the quadrialteral and invert the simson lines. Both with center P.
13. Duality with respect to ω.
14. Duality with respect to incircle.
15. Find polar of P with respect to incenter and with respect to circumcenter.
16. prove that AD is the polar of SI with respect to incircle.
17. Prove that $(MI, MB, PQ, MD) = -1$ using duality with respect to incircle.
18. Find polar of $AD \cap BE$ in self-duality.
19. Notice that $AX_a \perp M_aH$.
20. Calculate $\frac{|KH|}{|HL|}$ as a cross ratio with point at infinity.
21. Hind polars of C' and B' in self-duality with respect to ABC.
22. Reflect A through M to A_0.
23. Use harmonic cross-ratio to get all information in dual diagram. Then it's just angle chasing.
24. Find the pole of angle bisector of BAC in self-duality.
25. Self-duality with respect to ABC. Get angle bisectors "out".

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