Junior problems

J517. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $a_1 = 1$, $a_2 = 2$ and

$$\frac{a_{n+1}^3 + a_{n-1}^3}{9a_n} + a_{n+1}a_{n-1} = 3a_n^2.$$

Find a_n in a closed form.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, FL, USA

Let f_n be the nth Fibonacci number defined by $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. We show by induction that $a_n = f_{2n-1}$ for all $n \geq 1$. The claim is true for $n = 1$ and $n = 2$. Assume that it is true up to some $n \geq 2$. Multiplying the given equation by $9a_n$ we get

$$\begin{aligned} 0 &= a_{n+1}^3 + a_{n-1}^3 + 9a_{n+1}a_n a_{n-1} - 27a_n^3 \\ &= (a_{n+1} + a_{n-1} - 3a_n)\left[(a_{n+1} + a_{n-1})^2 + 3a_n(a_{n+1} + a_{n-1}) + 9a_n^2 - 3a_{n+1}a_{n-1}\right]. \end{aligned}$$

Since $(a_{n+1} + a_{n-1})^2 - 3a_{n+1}a_{n-1} = (a_{n+1} - a_{n-1})^2 + a_{n+1}a_{n-1} > 0$, we must have

$$a_{n+1} = 3a_n - a_{n-1} = 3f_{2n-1} - f_{2n-3} = 2f_{2n-1} + f_{2n-2} = f_{2n-1} + f_{2n} = f_{2n+1},$$

completing the induction.

Also solved by Ioannis D. Sfikas, Athens, Greece; Andrew Yang Hotchkiss School, Lakeville, CT, USA; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlosberg, Bayside, NY, USA; Daniel Lasaosa, Pamplona, Spain; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; SQ Problem Solving Group, Yogyakarta, Indonesia; Taes Padhihary, Disha Delphi Public School, India; Brian Bradie, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam.

J518. Find all real numbers x, y, z such that

$$x(y+z)^2 = y(z+x)^2 = z(x+y)^2 = 108.$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

If any two of x, y, z are real, without loss of generality by symmetry we have $y = z$, or

$$4xy^2 = y^3 + 2xy^2 + x^2y = 108, \qquad y^3 - 2xy^2 + x^2y = y(x-y)^2 = 0,$$

and since clearly $y \neq 0$, we have $x = y = z$. If $y \neq z$, then $z(x+y)^2 = y(z+x)^2$ rewrites as $0 = (y-z)\left(x^2 - yz\right)$, yielding $x^2 = yz$, and analogously $y^2 = zx$ if $z \neq x$ and $z^2 = xy$ if $x \neq y$. Or, if x, y, z are all distinct, we have $xyz = x^3 = y^3 = z^3$, contradiction. It follows that the only solutions may occur when $x = y = z$, yielding $4x^3 = 108 = 4 \cdot 3^3$. We conclude that the only solution is $x = y = z = 3$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Michail Prousalidis, Evangeliki Model Junior High School of Smyrna Athens, Greece; Andrew Yang Hotchkiss School, Lakeville, CT, USA; Polyahedra, Polk State College, FL, USA; SQ Problem Solving Group, Yogyakarta, Indonesia; Taes Padhihary, Disha Delphi Public School, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlosberg, Bayside, NY, USA; Nakis Konstantinos Pantelis, Ionidios Model High School, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; G. C. Greubel, Newport News, VA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Dhyey Dharmendrakumar Mavani, P. P. Savani Cambridge International School, India.

J519. Let x, y, z be positive numbers such that $xyz(x+y+z) = 3$. Prove that

$$\left(2x^2 - xy + 2y^2\right)\left(2y^2 - yz + 2z^2\right)\left(2z^2 - zx + 2x^2\right) \geq 27.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

We have

$$2x^2 - xy + 2y^2 = \frac{3}{4}(x+y)^2 + \frac{5}{4}(x-y)^2 \geq \frac{3}{4}(x+y)^2.$$

Writing two similar inequalities and multiplying them up we get

$$\begin{aligned}(2x^2 - xy + 2y^2)(2y^2 - yz + 2z^2)(2z^2 - zx + 2x^2) &\geq \frac{27}{64}(x+y)^2(y+z)^2(z+x)^2 \\ &\geq \frac{27}{64}\left[\frac{8}{9}(x+y+z)(xy+yz+zx)\right]^2 \\ &= \frac{1}{3}(x+y+z)^2(xy+yz+zx)^2 \\ &\geq (xy+yz+zx)^3.\end{aligned}$$

On the other hand we have

$$(xy+yz+zx)^2 \geq 3xyz(x+y+z) = 9.$$

It follows that

$$xy + yz + zx \geq 3.$$

From which the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, California, USA.

J520. Find all positive integers n for which $2^{3n-1}5^{n+1}+96$ is a perfect square.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Joel Schlosberg, Bayside, NY, USA

If $n=1$, $2^{3n-1}5^{n+1}+96=196=14^2$.
If $n=2$, $2^{3n-1}5^{n+1}+96=4096=64^2$.
If $n\geq 3$,

$$\frac{2^{3n-1}5^{n+1}+96}{32}=2^{3n-6}5^{n+1}+3$$

is an odd integer. But if $2^{3n-1}5^{n+1}+96=s^2$ for an integer s, $32\mid s^2$ implies that $8\mid s$ and thus $64\mid s^2=2^{3n-1}5^{n+1}+96$, making $\frac{2^{3n-1}5^{n+1}+96}{32}$ even. Therefore, $n=1,2$ are the only values for which $2^{3n-1}5^{n+1}+96$ is a perfect square.

Also solved by Daniel Lasaosa, Pamplona, Spain; Andrew Yang Hotchkiss School, Lakeville, CT, USA; Polyahedra, Polk State College, FL, USA; SQ Problem Solving Group, Yogyakarta, Indonesia; Taes Padhihary, Disha Delphi Public School, India; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Piteşti, Romania; Dumitru Barac, Sibiu, Romania; Kaitlynn Lane, SUNY Brockport, USA; Satvik Dasariraju, Lawrenceville School, NJ, USA; Titu Zvonaru, Comănești, Romania; G. C. Greubel, Newport News, VA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam.

J521. Is it possible to write the integers $1, 2, \ldots, 2020$ in a row so that the sum of any eleven neighboring numbers is divisible by 5?

Proposed by Li Zhou, Polk State College, Florida, USA

Solution by Polyahedra, Polk State College, USA

The answer is no. For contradiction, suppose that $a_1, a_2, \ldots, a_{2020}$ is such a writing. Then for any $1 \le i \le 2009$, $a_i + a_{i+1} + \cdots + a_{i+10} \equiv 0$ and $a_{i+1} + a_{i+2} + \cdots + a_{i+11} \equiv 0 \pmod 5$.

By subtraction we get $a_i \equiv a_{i+11} \pmod 5$. Therefore, for $1 \le k \le 11$, all elements in the set $A_k = \{a_i : i \equiv k \pmod{11}\}$ have the same residue modulo 5. Notice also that $|A_k| = 184$ for $1 \le k \le 7$ and $|A_k| = 183$ for $8 \le k \le 11$. On the other hand, the set $\{1, 2, \ldots, 2020\}$ has 404 elements for each of the residues $0, \pm1, \pm2$ modulo 5.

By the pigeonhole principle, there are three sets A_l, A_m, A_n whose elements fall into the same residue class modulo 5. But $3 \times 183 > 404$, a desired contradiction.

Also solved by Andrew Yang Hotchkiss School, Lakeville, CT, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Daniel Lasaosa, Pamplona, Spain; Michail Prousalidis, Evangeliki Model Junior High School of Smyrna Athens, Greece; Titu Zvonaru, Comănești, Romania; Dhyey Dharmendrakumar Mavani, P. P. Savani Cambridge International School, India.

J522. Let a, b, c be nonnegative real numbers. Prove that

$$\left(a^2 + 4b^2\right)\left(b^2 + 4c^2\right)\left(c^2 + 4a^2\right) \geq 64abc(2a - b)(2b - c)(2c - a)$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

After some algebra, the inequality rewrites as

$$4\left(a^2b + b^2c + c^2a\right)^2 + \left(4ab^2 + 4bc^2 + 4ca^2 - 17abc\right)^2 + 224abc\left(a^2b + b^2c + c^2a - 3abc\right) \geq 0.$$

The first two terms are clearly non-negative since they are squares, and the last one is nonnegative because of the AM-GM inequality applied to a^2b, b^2c, c^2a. Since a, b, c are nonnegative, the first term can be zero iff $a^2b = b^2c = c^2a = 0$, ie iff at least two out of a, b, c are zero. The conclusion follows, equality holds iff (a, b, c) is a permutation of $(r, 0, 0)$, where r is any nonnegative (possibly zero) real number.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

Senior problems

S517. Let a, b, c be real numbers such that

$$a^3 + b^3 + c^3 - 1 = 3(a-1)(b-1)(c-1).$$

Prove that $a + b + c \le 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

We will use compact notations, namely let $s := a + b + c, p := ab + bc + ca, q := abc$.
Without loss of generality we can assume that $s > 1$. Since $a^3 + b^3 + c^3 = s^3 + 3q - 3sp$ then

$$a^3 + b^3 + c^3 - 1 = 3(a-1)(b-1)(c-1)$$

becomes

$$s^3 + 3q - 3sp = 3(q - p + s - 1) \iff s^3 - 3s + 2 = 3p(s-1) \iff$$
$$(s+2)(s-1)^2 = 3p(s-1) \iff$$
$$(s+2)(s-1) = 3p.$$

Noting that $3p = 3(ab + bc + ca) \le (a + b + c)^2 = s^2$ we obtain that

$$(s+2)(s-1) \le s^2 \iff s \le 2.$$

Also solved by Israel Castillo Pilco, Huaral, Peru; Daniel Lasaosa, Pamplona, Spain; Andrew Yang Hotchkiss School, Lakeville, CT, USA; SQ Problem Solving Group, Yogyakarta, Indonesia; Taes Padhihary, Disha Delphi Public School, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Dumitru Barac, Sibiu, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

S518. Let ABC be a triangle with $BC = a$, $AB = AC = b$ and $a^3 - b^3 = 3ab^2$. Calculate $\angle BAC$.

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain
From isosceles triangle ABC, we have $\frac{a}{b} = 2\sin\frac{A}{2}$. Hence the given equality may be writte as

$$8\sin^3\frac{A}{2} - 6\sin\frac{A}{2} - 1 = 0. \tag{1}$$

Now we apply the identity

$$\sin 3x = 3\sin x - 4\sin^3 x$$

with $x = \frac{A}{2}$, obtaining

$$\sin\frac{3A}{2} = 3\sin\frac{A}{2} - 4\sin^3\frac{A}{2}.$$

When this is substituted into (1), we get

$$2\sin\frac{3A}{2} + 1 = 0$$

i.e.

$$A = 120^\circ k + (-1)^k \cdot 140^\circ, \qquad k \in \mathbb{Z}$$

whose only admissible solution is $\angle BAC = 140^\circ$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Andrew Yang Hotchkiss School, Lakeville, CT, USA; SQ Problem Solving Group, Yogyakarta, Indonesia; Taes Padhihary, Disha Delphi Public School, India; Brian Bradie, Newport News, VA, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Toyesh Prakash Sharma, St. C.F Andrews School, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, California, USA; Dhyey Dharmendrakumar Mavani, P. P. Savani Cambridge International School, India.

S519. Prove that in any triangle ABC

$$2\sqrt{3} \le \operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C \le \frac{2\sqrt{3}}{9}\left(1+\frac{R}{r}\right)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note that

$$\sin A + \sin B + \sin C = 2\sin\frac{A}{2}\cos\frac{A}{2} + 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} \le 2\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right),$$

with equality iff $B = C$, or denoting $x = \sin\frac{A}{2}$,

$$\frac{27}{4} - (\sin A + \sin B + \sin C)^2 \ge \frac{11 - 32x + 32x^3 + 16x^4}{4} = \frac{(1-2x)^2\left(11+12x+4x^2\right)}{4} \ge 0,$$

with equality iff $x = \frac{1}{2}$. Or, $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$, and equivalently using the Sine Law $a+b+c \le 3\sqrt{3}R$, with equality iff ABC is equilateral.

Using the AM-HM inequality between $\sin A, \sin B, \sin C$, for the left inequality it suffices to prove that $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$, clearly true. The left inequality follows, equality holds iff ABC is equilateral.

Note that $\left(1+\frac{R}{r}\right)^2 \ge \frac{9R}{2r}$ is equivalent to $0 \le 2R^2 - 5Rr + 2r^2 = (R-2r)(2R-r)$, clearly true since $R \ge 2r$ with equality iff ABC is equilateral. After using the Sine Law and well-known relations for the area S of ABC, for the right inequality it suffices to show that

$$\frac{ab+bc+ca}{2S} \le \frac{\sqrt{3}R}{r}, \qquad \frac{3(ab+bc+ca)}{(a+b+c)^2}\cdot\frac{a+b+c}{3\sqrt{3}R} \le 1.$$

Now the second factor in the LHS is not larger than 1 with equality iff ABC is equilateral, whereas the first factor is also not larger than 1 because of the scalar product inequality, with equality iff $a = b = c$. The right inequality follows, equality holds iff ABC is equilateral.

Also solved by Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Dumitru Barac, Sibiu, Romania; Daniel Văcaru, Pitești, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, California, USA; Dhyey Dharmendrakumar Mavani, P. P. Savani Cambridge International School, India.

S520. Let a, b, c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that

$$\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}\geq\frac{2r}{R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania
Using Cauchy-Schwarz Inequality we get

$$\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}\geq\frac{(\sum a)^2}{2\sum a^2+\sum ab}=\frac{4s^2}{4(s^2-r^2-4Rr)+s^2+r^2+4Rr}=\frac{4s^2}{5s^2-3r^2-12Rr}$$

Therefore, it suffices to show that

$$\frac{4s^2}{5s^2-3r^2-12Rr}\geq\frac{2r}{R}\Leftrightarrow s^2(4R-10r)+6r^2(r+4R)\geq 0$$

Using the inequality $s^2\geq 3r(r+4R)$ we have

$$s^2(4R-10r)+6r^2(r+4R)\geq 0\Leftrightarrow 3r(r+4R)(4R-10r)+6r^2(r+4R)\geq 0\Leftrightarrow$$

$$12r(r+4R)(R-2r)\geq 0,$$

which is true.

Also solved by Daniel Lasaosa, Pamplona, Spain; Daniel Văcaru, Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, California, USA; Dhyey Dharmendrakumar Mavani, P. P. Savani Cambridge International School, India.

S521. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{8a^3}{(b+c)^3}+\frac{b+c}{a}\right)\left(\frac{8b^3}{(c+a)^3}+\frac{c+a}{b}\right)\left(\frac{8c^3}{(a+b)^3}+\frac{a+b}{c}\right)\geq$$

$$\frac{143(a+b)(b+c)(c+a)}{8abc}-116.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Denote $x=\frac{2a}{b+c}$, $y=\frac{2b}{c+a}$, $z=\frac{2c}{a+b}$ then $xy+yz+zx+xyz=4$ and the inequality becomes

$$(x^4+2)(y^4+2)(z^4+2)\geq 143-116xyz,$$

or

$$x^4y^4z^4+2(x^4y^4+y^4z^4+z^4x^4)+4(x^4+y^4+z^4)+8\geq 143-116xyz,$$

or

$$\left[2(x^2+y^2+z^2)-x^2y^2z^2\right]^2+2(x^2y^2+y^2z^2+z^2x^2-2)^2\geq 143-116xyz,$$

or

$$\left[2(x+y+z)^2-4(xy+yz+zx)-x^2y^2z^2\right]^2+2\left[(xy+yz+zx)^2-2xyz(x+y+z)-2\right]^2$$
$$\geq 116(xy+yz+zx)-321,$$

By Schur's Iequality (see point 9 of Note 2 from "116 Algebraic Inequalities"), we have

$$x+y+z\geq xy+yz+zx,$$

and from Popoviciu's Inequality (see Example 95 from "116 Algebraic Inequalities"), we have

$$x+y+z\leq\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\iff xy+yz+zx\geq xyz(x+y+z).$$

Therefore, if we denote $t=xy+yz+zx\in[3,4]$, we deduce that

$$\begin{aligned}2(x+y+z)^2-4(xy+yz+zx)-x^2y^2z^2 &\geq 2t^2-4t-(4-t)^2\\ &=t^2+4t-16\geq 0\end{aligned}$$

and

$$(xy+yz+zx)^2-2xyz(x+y+z)-2\geq t^2-2t-2\geq 0.$$

These being established, it only remains to show that

$$(t^2+4t-16)^2+2(t^2-2t-2)^2-116t+321\geq 0,$$

or

$$(t-3)^3(3t^2+18t+65)\geq 0$$

obviously true.

Equality holds when $t=3$ which implies $a=b=c$.

S522. Let $a_1, \ldots, a_n$ and $x_1, \ldots, x_n$, $(n \geq 2)$, be positive real numbers such that

$$\prod_{i=1}^{n} a_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} x_i = n.$$

Prove that

$$\sum_{i=1}^{n} \frac{1}{(n-1)a_i x_i + 1} \geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

First solution by Henry Ricardo, Westchester Area Math Circle

Let $k = 1/n - 1$. For $i \in \{1, 2, \ldots, n\}$, the AGM inequality gives us

$$\begin{aligned}
\sum_{\substack{j=1 \\ j \neq i}}^{n} (a_j x_j)^k &\geq (n-1) \sqrt[n-1]{(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)^k (x_1 \cdots x_{i-1} x_{i+1} \cdots x_n)^k} \\
&= (n-1) \frac{\sqrt[n-1]{(a_1 a_2 \cdots a_n)^k}}{\sqrt[n-1]{a_i^k}} \cdot \frac{\sqrt[n-1]{(x_1 x_2 \cdots x_n)^k}}{\sqrt[n-1]{x_i^k}} \\
&= \frac{(n-1)(a_i x_i)^{-k/(n-1)}}{\sqrt[n]{x_1 x_2 \cdots x_n}} \geq (n-1)(a_i x_i)^{1/n} = (n-1)(a_i x_i)^{k+1}
\end{aligned}$$

since the AGM implies $\sqrt[n]{x_1 x_2 \cdots x_n} \leq 1$. This inequality, in turn, can be written as

$$\sum_{j=1}^{n} (a_j x_j)^k \geq (a_i x_i)^k + (n-1)(a_i x_i)^{k+1} = (a_i x_i)^k \left(1 + (n-1) a_i x_i\right),$$

or

$$\frac{1}{(n-1)a_i x_i + 1} \geq \frac{(a_i x_i)^k}{\sum_{j=1}^{n} (a_j x_j)^k}.$$

Summing this last inequality over i yields the desired inequality. Equality holds if and only if $a_1 = \cdots = a_n = x_1 = \cdots = x_n = 1$.

Second solution by Daniel Lasaosa, Pamplona, Spain

Denote by s_k the sum

$$s_k = (n-1)^k \sum \prod a_i x_i,$$

where each product contains exactly k distinct factors $a_i x_i$ for k distinct indices $i \in \{1, 2, \ldots, n\}$, and the sum is taken over all $\binom{n}{k}$ possible distinct k-tuples. Multiplying both sides of the proposed inequality by the (clearly positive) product of denominators, the inequality rewrites as

$$n + (n-1)s_1 + (n-2)s_2 + \cdots + 2s_{n-2} + s_{n-1} \geq 1 + s_1 + s_2 + \cdots + s_{n-1} + s_n,$$

$$(n-1) + (n-2)s_1 + (n-3)s_2 + \cdots + s_{n-2} \geq s_n = (n-1)^n P,$$

where $P = x_1 x_2 \cdots x_n$. This is so because when we make common denominator s_n in the fractions in the LHS, a certain k-tuple of indices appears in exactly $n-k$ numerators, those whose original denominators do not contain any of those k indices, and moreover 1 appears in each one of the n numerators. Note that by the AM-GM inequality,

$$P \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n = 1,$$

with equality iff $x_1 = x_2 = \cdots = x_n = 1$. Now, using the AM-GM inequality and $a_1 a_2 \cdots a_n = 1$, we have

$$s_k \geq (n-1)^k \binom{n}{k} \left(P^{\binom{n-1}{k-1}}\right)^{\frac{1}{\binom{n}{k}}} = \binom{n}{k} \left((n-1)\sqrt[n]{P}\right)^k,$$

since each one of the factors $a_i x_i$ appears in exactly $\binom{n-1}{k-1}$ k-tuples, exactly as many as groups of $k-1$ other indices may be picked out from the remaining $n-1$ to complete the k-tuple in which $a_i x_i$ appears. Since $P^k \geq P^n$ for each $k \in \{0, 1, \ldots, n-1\}$ because $P \leq 1$, with equality iff $P = 1$, it suffices to show that

$$(n-1) + (n-2)\binom{n}{1}(n-1) + (n-3)\binom{n}{2}(n-1)^2 + \cdots + \binom{n}{n-1}(n-1)^{n-1} \geq (n-1)^n.$$

Now, after trivial induction, the first k terms in the LHS add up to $\binom{n-1}{k-1}(n-1)^k$. This is true for the first term since for $k=1$ this expression becomes $(n-1)$, and if it is true for the first k terms, then the sum of the first $k+1$ terms is

$$\binom{n-1}{k-1}(n-1)^k + (n-k-1)\binom{n}{k}(n-1)^k = \frac{(n-1)!}{k!(n-k)!}(n-1)^k \left(k + n(n-k-1)\right) =$$

$$= \frac{(n-1)!(n-k)}{k!(n-k)!}(n-1)^{k+1} = \binom{n-1}{k}(n-1)^{k+1}.$$

Or, the LHS equals $(n-1)^n$, ie it equals the RHS. The conclusion follows, equality holds iff $P=1$, ie $x_1 = x_2 = \cdots = x_n$, and simultaneously all products $a_i x_i$ are equal. Thus, equality holds iff $a_i = x_i = 1$ for $i = 1, 2, \ldots, n$.

Also solved by Taes Padhihary, Disha Delphi Public School, India.

Undergraduate problems

U517. We say that the polynomial $a_n x^n + \cdots + a_1 x + a_0$ with real coefficients is powerful if

$$|a_n| + \cdots + |a_1| = |a_0|$$

Prove that if $P(x)$ is a polynomial with nonzero real coefficients of degree d, such that $P(x)(x-1)^s(x+1)^t$ is powerfull for some nonnegative integers s and t, then either $P(x)$ or $(-1)^d P(-x)$ has non-increasing coefficients.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Without loss of generality assume that $a_d > 0$. First of all, we show that $(s,t) = (0,1),(1,0)$. Assume by contradiction that $t \geq 2$, let

$$R(x) = P(x)(x-1)^t(x+1)^s = b_m x^m + \cdots + b_0.$$

It is known that $b_0 = \pm a_0, b_m = a_d > 0$. From the problem assumption, we find that:

$$|b_0| = |b_1| + \cdots + |b_m|.$$

Since $R(1) = 0$, we find that $b_m + b_{m-1} + \cdots + b_0 = 0$. That is,

$$|b_0| = |b_1 + \cdots + b_m| = |b_1| + \cdots + |b_m|.$$

Therefore, $b_1,...,b_m$ have the same sign. Since $b_m > 0$, we find that $b_1,...,b_m > 0$. On the other hand, since $R(x)$ is divisible by $(x-1)^2$ we find that $mb_m + (m-1)b_{m-1} + \cdots + b_1 = 0$, absurd. By the same argument, we can find that $s \leq 1$.

Assume that $s = t = 1$. Then, $P(x)(x^2-1) = b_m x^m + \cdots + b_0$. Therefore, $\sum_{i=0}^{m} b_i = \sum_{i=0}^{m}(-1)^i b_i = 0$. That is, $\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} b_{2i} = 0$. Hence, $|b_0| \leq \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} |b_{2i}|$. Thus, we find that $b_1 = b_3 = ... = 0$. Hence, $R(x) = S(x^2)$ for some polynomial $S(x)$ with integer coefficients. Thus, $P(x) = \dfrac{S(x^2)}{x^2-1}$ would be a polynomial in x^2 with real coefficients. Therefore, odd monomials would be zero. This contradicts to the fact $P(x)$ has non-zero coefficients. Therefore, one of s,t would be zero. This yields to $(x-1)^t(x+1)^s \in \{x-1, x+1\}$. Now we prove that following lemma.

Lemma: Let $a_d > 0, P(x) = a_d x^d + \cdots + a_0$ the the polynomial $(x-1)P(x)$ is powerful if $P(x)$ has non-increasing coefficients.

Proof: $(x-1)P(x) = -a_0 + (a_0 - a_1)x + \cdots + (a_{d-1} - a_d)x^d + a_d x^{d+1}$. Thus,

$$|a_0| = |a_0 - a_1| + \cdots + |a_{d-1} - a_d| + |a_d|,$$

If $a_0 \geq a_1 \geq \cdots \geq a_d$. This completes the proof.

Back to out problem, assume that $(x-1)^t(x+1)^s = x-1$, then by lemma we find that the coefficients of $P(x)$ are non-increasing. Now, if $(x-1)^t(x+1)^s = x+1$, we see that $(-1)^d P(-x)$ is a polynomial with positive leading coefficient and the sum of absolute values of the coefficients of $P(x)(x+1)$ is equal to the sum of absolute values of the coefficients $(-1)^d P(-x)(x-1)$. The conclusion follows.

U518. Evaluate

$$\lim_{n\to\infty}\left(\frac{1^2}{n^3+n^2+1}+\frac{2^2}{n^3+n^2+2}+\cdots+\frac{n^2}{n^3+n^2+n}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Let L be the proposed limit,

$$L_1=\lim_{n\to\infty}\left(\frac{1^2}{n^3+n^2+1}+\frac{2^2}{n^3+n^2+1}+\cdots+\frac{n^2}{n^3+n^2+1}\right), \text{ and}$$

$$L_n=\lim_{n\to\infty}\left(\frac{1^2}{n^3+n^2+n}+\frac{2^2}{n^3+n^2+n}+\cdots+\frac{n^2}{n^3+n^2+n}\right)$$

It is clear that $L_n\le L\le L_1$. Since $\sum_{k=1}^{n}k^2=\frac{n(n-1)(2n+1)}{6}$, then $L_n=L_1=\frac{1}{3}=L$.

Also solved by Khakimboy Egamberganov, Paris, France; Davrbek Oltiboev, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Andrew Yang Hotchkiss School, Lakeville, CT, USA; Brian Bradie, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Joel Schlosberg, Bayside, NY, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Toyesh Prakash Sharma, St. C.F Andrews School, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, California, USA; Ioannis D. Sfikas, Athens, Greece; Dhyey Dharmendrakumar Mavani, P. P. Savani Cambridge International School, India.

U519. Let k be a fixed integer. Evaluate

$$\sum_{n=k+1}^{\infty} \frac{1}{n\left(n^2-1^2\right)^2\left(n^2-2^2\right)^2\cdots\left(n^2-k^2\right)^2}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Newport News, VA, USA

Note

$$\frac{1}{n\prod_{j=1}^{k}(n^2-j^2)^2} = \frac{n}{\prod_{j=-k}^{k}(n+j)^2} = \frac{1}{4k}\left(\frac{1}{\prod_{j=-k}^{k-1}(n+j)^2} - \frac{1}{\prod_{j=-k+1}^{k}(n+j)^2}\right),$$

so

$$\sum_{n=k+1}^{\infty} \frac{1}{n(n^2-1^2)^2(n^2-2^2)^2\cdots(n^2-k^2)^2} = \frac{1}{4k}\cdot\frac{1}{\prod_{j=-k}^{k-1}(k+1+j)^2} = \frac{1}{4k((2k)!)^2}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; G. C. Greubel, Newport News, VA, USA; Khakimboy Egamberganov, Paris, France; Davrbek Oltiboev, National University of Uzbekistan, Tashkent, Uzbekistan.

U520. Evaluate

$$\lim_{x\to 0}\frac{(1-\cos x)(1-\cos 2x)\cdots(1-\cos nx)}{\sin^{2n}x}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Daniel Lasaosa, Pamplona, Spain

Using Landau's little-o notation, we have

$$\cos kx = 1-\frac{k^2x^2}{2}+o\left(x^2\right), \qquad \sin^{2n}x = x^{2n}+o\left(x^{2n}\right),$$

or

$$(1-\cos x)(1-\cos 2x)\cdots(1-\cos nx) = \frac{(n!)^2\,x^{2n}}{2^n}+o\left(x^{2n}\right),$$

and finally

$$\frac{(1-\cos x)(1-\cos 2x)\cdots(1-\cos nx)}{\sin^{2n}x} = \frac{(n!)^2}{2^n}\cdot\frac{1+o(1)}{1+o(1)}.$$

Since $\lim_{x\to 0}o(1)=0$, we have

$$\lim_{x\to 0}\frac{(1-\cos x)(1-\cos 2x)\cdots(1-\cos nx)}{\sin^{2n}x} = \frac{(n!)^2}{2^n}.$$

Second solution by Daniel Lasaosa, Pamplona, Spain

Using the De Moivre relations, we have

$$\sin kx = k\sin x\cos^{k-1}x+\sin^3 xP\left(\cos x,\sin x\right),$$

where $P((\cos x,\sin x)$ is a homogeneous polynomial of degree $k-3$ in $\cos x,\sin x$ with integral coefficients. It follows that

$$\lim_{x\to 0}\frac{\sin kx}{\sin x}=k.$$

Therefore,

$$\lim_{x\to 0}\frac{1-\cos kx}{\sin^2 x}=\lim_{x\to 0}\frac{1}{1+\cos kx}\cdot\frac{1-\cos^2 kx}{\sin^2 x}=\lim_{x\to 0}\frac{1}{1+\cos kx}\cdot\left(\frac{\sin kx}{\sin x}\right)^2=\frac{k^2}{2}.$$

Then,

$$\lim_{x\to 0}\frac{(1-\cos x)(1-\cos 2x)\cdots(1-\cos nx)}{\sin^{2n}x}=\lim_{x\to 0}\prod_{k=1}^{n}\frac{1-\cos kx}{\sin^2 x}=\prod_{k=1}^{n}\frac{k^2}{2}=\frac{(k!)^2}{2^n}.$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Jamal-Dine Chergui, Ecole Normale Supérieure de Tétouan, Marocco; Andrew Yang Hotchkiss School, Lakeville, CT, USA; Taes Padhihary, Disha Delphi Public School, India; Brian Bradie, Newport News, VA, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; G. C. Greubel, Newport News, VA, USA; Davrbek Oltiboev, National University of Uzbekistan, Tashkent, Uzbekistan; Toyesh Prakash Sharma, St. C.F Andrews School, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Atila Araujo Lobo, Universidade de Brasilia, Brazil; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Dumitru Barac, Sibiu, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Joel Schlosberg, Bayside, NY, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania ; Arkady Alt, San Jose, California, USA; Dhyey Dharmendrakumar Mavani, P. P. Savani Cambridge International School, India.

U521. Find all automorphisms of the group $S_3 \times \mathbb{Z}_3$.

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

We may define S_3 by $\{e, u, u^2, v, uv, u^2v\}$ with the unit e, associative property, and multiplication rules $u^3 = v^2 = e$ and $vu = u^2v$. We may define $\mathbb{Z}_3$ by $\{0, 1, 2\}$ with unit 0, associative property, and addition table $1 + 1 = 2$ and $2 + 2 = 1$.

In S_3, u, u^2 have order 3 and v, uv, u^2v have order 2. Or, $f(u) \in \{u, u^2\}$ and $f(v) \in \{v, uv, u^2v\}$. Each of the possible $2 \cdot 3 = 6$ possible combinations of values yield an automorphism f in S_3. In $\mathbb{Z}_3$, both $1, 2$ have order 3, or there are two possible automorphisms $g : \mathbb{Z}_3 \to \mathbb{Z}_3$, defined by $g(1) = 1$ and $g(1) = 2$ respectively. Note that any automorphism of $S_3 \times \mathbb{Z}_3$ must have a restriction to S_3 and to $\mathbb{Z}_3$ which are themselves automorphisms in S_3 and $\mathbb{Z}_3$, respectively.

Let $[x, y] \in S_3 \times \mathbb{Z}_3$ such that $x \in S_3$, $y \in \mathbb{Z}_3$. Note that the order of $[x, y]$ is the least common multiple of the orders of x, y, ie the order of the elements in $S_3 \times \mathbb{Z}_3$ is:

- 1 for the unit $[e, 0]$.
- 2 for $[v, 0]$, $[uv, 0]$ and $[u^2v, 0]$.
- 3 for $[e, 1]$, $[e, 2]$, $[u, 0]$, $[u^2, 0]$, $[u, 1]$, $[u^2, 1]$, $[u, 2]$ and $[u^2, 2]$.
- 6 for $[v, 1]$, $[uv, 1]$, $[u^2v, 1]$, $[v, 2]$, $[uv, 2]$ and $[u^2v, 2]$.

Note next that $S_3 \times \mathbb{Z}_3$ may clearly be generated from $[u, 0]$, $[v, 0]$ and $[e, 1]$. Note further that $[v, 1]^3 = [v, 0]$ and $[v, 1]^4 = [e, 1]$, or $h([v, 1])$ and $h([u, 0])$ fully determine the automorphism. Now, $h([v, 1])$ may in principle be any one of the 6 elements of order 6. For any one of these 6 choices, it is easily found that $h([e, 1]) \in \{[e, 1], [e, 2]\}$ and consequently $h([e, 2]) \in \{[e, 2], [e, 1]\}$ in the same order, and that $h([v, 0]) \in \{[v, 0], [uv, 0], [u^2v, 0]\}$. Or, $h([u, 0])$ must be an element of order 3, yet different from $[e, 1]$ and $[e, 2]$ because h is bijective, yielding $h([u, 0]) \in \{[u, 0], [u^2, 0], [u, 1], [u^2, 1], [u, 2], [u^2, 2]\}$. After some tedious checking these 6 values of $h([u, 0])$ indeed yield automorphisms for each one of the 6 possible values of $h([v, 1])$, or there are exactly $6 \cdot 6 = 36$ automorphisms of group $S_3 \times \mathbb{Z}_3$.

U522. Let $f:(0,\infty)\to(0,\infty)$ be a continuous function. For any positive integer n we denote $t_n = n\sqrt[n]{n}$. Evaluate

$$\lim_{n\to\infty}\int_{t_n}^{t_{n+1}} f\left(\frac{x}{n}\right)dx.$$

Proposed by Florin Rotaru, Focşani,Romania

Solution by Brian Bradie, Newport News, VA, USA

Let $u = x/n$. Then

$$\int_{t_n}^{t_{n+1}} f\left(\frac{x}{n}\right)dx = n\int_{t_n/n}^{t_{n+1}/n} f(u)\,du.$$

Because f is continuous on $[t_n/n, t_{n+1}/n]$, there exist $c_n, d_n \in [t_n/n, t_{n+1}/n]$ such that

$$f(c_n) \le f(x) \le f(d_n)$$

for all $x \in [t_n/n, t_{n+1}/n]$. Thus,

$$n\left(\frac{t_{n+1}}{n} - \frac{t_n}{n}\right) f(c_n) \le \int_{t_n}^{t_{n+1}} f\left(\frac{x}{n}\right)dx \le n\left(\frac{t_{n+1}}{n} - \frac{t_n}{n}\right) f(d_n).$$

Now

$$\lim_{n\to\infty}(t_{n+1} - t_n) = 1,$$

and

$$\lim_{n\to\infty}\frac{t_n}{n} = \lim_{n\to\infty}\frac{t_{n+1}}{n} = 1,$$

so

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 1.$$

It then follows that

$$\lim_{n\to\infty} n\left(\frac{t_{n+1}}{n} - \frac{t_n}{n}\right) f(c_n) = \lim_{n\to\infty} n\left(\frac{t_{n+1}}{n} - \frac{t_n}{n}\right) f(d_n) = f(1),$$

and

$$\lim_{n\to\infty}\int_{t_n}^{t_{n+1}} f\left(\frac{x}{n}\right)dx = f(1)$$

by the squeeze theorem.

Also solved by Besfort Shala, University of Primorska, Slovenia; Daniel Lasaosa, Pamplona, Spain; Khakimboy Egamberganov, Paris, France; Anish Ray, Institute of Mathematics and Applications, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlosberg, Bayside, NY, USA; Toyesh Prakash Sharma, St. C.F Andrews School, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, California, USA.

Olympiad problems

O517. Prove that for any positive real numbers a, b, c

$$\sqrt{\frac{2ab}{a^2+b^2}}+\sqrt{\frac{2bc}{b^2+c^2}}+\sqrt{\frac{2ca}{c^2+a^2}}+\frac{3(a^2+b^2+c^2)}{ab+bc+ca}\geq 6.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

By the AM-GM inequality we have

$$\sqrt{\frac{2bc}{b^2+c^2}}=\frac{2bc}{\sqrt{(2bc)(b^2+c^2)}}\geq\frac{4bc}{2bc+b^2+c^2}=\frac{4bc}{(b+c)^2}.$$

The AM-GM inequality also gives us

$$\begin{aligned}\sum_{\text{cyc}}\sqrt{\frac{2bc}{b^2+c^2}}&\geq\sum_{\text{cyc}}\frac{4bc}{(b+c)^2}\\ &\geq 3\sqrt[3]{\frac{4bc}{(b+c)^2}\cdot\frac{4ca}{(c+a)^2}\cdot\frac{4ab}{(a+b)^2}}\\ &=3\sqrt[3]{\left(\frac{8abc}{(a+b)(b+c)(c+a)}\right)^2}\\ &\geq\frac{24abc}{(a+b)(b+c)(c+a)}.\end{aligned}$$

The last inequality is true because

$$0\leq\frac{8abc}{(a+b)(b+c)(c+a)}\leq 1$$

Hence, it suffices to show that

$$\frac{8abc}{(a+b)(b+c)(c+a)}+\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 2.$$

Without loss of generality we may assume $a=\min\{a,b,c\}$. Now we note that: if $x\geq y>0$ then for any $z\geq 0$,

$$\frac{x}{y}\geq\frac{x+z}{y+z}$$

Equality holds for $x=y$ or $z=0$. Applying this result we obtain

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\frac{2a^2+b^2+c^2}{a^2+ab+bc+ca}=\frac{2a^2+b^2+c^2}{(a+b)(a+c)}.$$

Next, we need to prove

$$\frac{8abc}{(a+b)(b+c)(c+a)}+\frac{2a^2+b^2+c^2}{(a+b)(a+c)}\geq 2.$$

This is equivalent to

$$8abc+(2a^2+b^2+c^2)(b+c)\geq 2(a+b)(b+c)(c+a),$$

$$8abc+2a^2(b+c)+(b^2+c^2)(b+c)\geq 2[bc(b+c)+a^2(b+c)+a(b^2+c^2)+2abc],$$

$$(b+c-2a)(b-c)^2\geq 0$$

which is true because $a=\min\{a,b,c\}$. The conclusion follows. Equality occurs if and only if $a=b=c$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ioannis D. Sfikas, Athens, Greece.

O518. Let p be a prime congruent to 3 modulo 4. Alice and Bob have just filled in the cells of a table of size $2 \times p$ (2 rows, p columns). They proceeded thus: for every $m \in \{0, 1, \ldots, p-1\}$, in the m-th cell of the first row of the table Alice wrote the remainder that m^2 leaves when it is divided by p; for every $n \in \{0, 1, \ldots, p-1\}$, in the n-th cell of the second row of the table Bob wrote the remainder that n^4 leaves when it is divided by p. Prove that both rows of Alice and Bob's table contain the same numbers with the same multiplicities.

Proposed by José Hernández Santiago, Matemáticas UAGro

Solution by Daniel Lasaosa, Pamplona, Spain

In the first row, each nonzero quadratic residue modulo p appears exactly twice. Indeed, each one appears at least once because the second row contains all remainders of all possible nonzero residues modulo p. Considering m^2 and $(p-m)^2$, and since p is odd, each one appears at least twice. No quadratic residue may appear more than twice because if $r^2 \equiv s^2 \pmod{p}$, then p divides $r^2 - s^2 = (r-s)(r+s)$, and since $1 \le r, s \le p$, we must have either $s = r$ or $s = p - r$. This argument is also proof that there are exactly $\frac{p-1}{2}$ quadratic residues modulo p, such that if $r_1, r_2, \ldots, r_{\frac{p-1}{2}}$ are the quadratic residues, then $-r_1, -r_2, \ldots, -r_{\frac{p-1}{2}}$ are the quadratic non-residues, and together they form the $p-1$ nonzero residues modulo p, when $p \equiv 3 \pmod{4}$.

Consider now a third row, which we fill as follows: the first time that a quadratic residue r appears in the first row, we write r in the same position of the third row; the second (and last) time that a quadratic residue r appears in the first row, we write $-r$ in the same position of the third row. If r appears in the first row, note that the same position in the second row is filled with the remainder of r^2 modulo p, since $r \equiv n^2 \pmod{p}$, and that position of the second row contains the remainder of $n^4 \equiv r^2 \pmod{p}$. Since $r^2 = (-r)^2$, the second row is filled with the remainders modulo p of the squares of the corresponding elements in the third row.

But the third row contains exactly all nonzero residues modulo p, each one appearing exactly once; each quadratic residue r appears exactly the first time that r appeared in the first row, and each quadratic non-residue $-r$ appears exactly the second time that r appeared in the first row. Or the second row contains exactly the nonzero quadratic residues modulo p, each one appearing exactly twice, just as the first row. The conclusion follows.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania.

O519. Let a, b, c be positive numbers such that $a+b+c=ab+bc+ca$. Prove that

$$\frac{3}{1+a}+\frac{3}{1+b}+\frac{3}{1+c}-\frac{4}{(1+a)(1+b)(1+c)}\geq 4.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Titu Zvonaru, Comănești, Romania

After clearing the denominators, we obtain $2(a+b+c)+1\geq ab+bc+ca+4abc$,

$$a+b+c+a\geq 4abc$$

$$\frac{(a+b+c)(ab+bc+ca)^2}{(a+b+c)^2}+\frac{(ab+bc+ca)^3}{(a+b+c)^3}\geq 4abc$$

$$(a+b+c)^2(ab+bc+ca)^2+(ab+bc+ca)^3\geq 4abc(a+b+c)^3$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+3\sum_{cyc}a^3b^3\geq 2\sum_{cyc}a^4bc+\sum_{cyc}a^3b^2c+\sum_{cyc}a^3bc^2+\sum_{cyc}a^2b^2c^2$$

$$(a-b)^2(b-c)^2(c-a)^2+\sum_{cyc}ab(ab-bc)(ab-ca)+2\sum_{sym}(a^3b^3-a^3b^2c)\geq 0.$$

The last two sums are positive by Schur and Muirhead inequalities. The equality holds if and only if $a=b=c=1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Mohamed Ali, Houari Boumedien school, Algeria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ioannis D. Sfikas, Athens, Greece.

O520. Let x, y, z be positive integers such that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 6$$

and $\gcd(z, x) = 1$. Find the maximum value of $x + y + z$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Daniel Lasaosa, Pamplona, Spain

Since $6xy - yz - x^2 = \frac{xy^2}{z}$ is an integer, and z is coprime with x, then z divides y^2. Since $6yz - xz - y^2 = \frac{yz^2}{x}$ is an integer, and x is coprime with z, then x divides y. Let $y = uv$, where u and v are coprime, and such that all common prime factors from x and y are in u (and thus v is coprime with x), and all common factors from z and y are in v (and thus u is coprime with z). Then, z divides v^2 and x divides u. Let $u = ax$ and $v^2 = bz$, or $6av - abu - 1 = \frac{a^2vz}{u}$ is an integer, and since u is coprime with z and with v, then u divides a^2, and $a^2 = cu = cax$, for $a = cx$ and $u = cx^2$. The condition then rewrites as

$$\frac{1}{cxv} + \frac{bcx^2}{v} + \frac{v^2}{bx} = 6,$$

or using the AM-GM inequality,

$$6 > \frac{bcx^2}{v} + \frac{v^2}{bx} \geq 2\sqrt{cxv}, \qquad cxv < 3^2, \qquad cxv \leq 8.$$

Now, since c, x divide u, and v is coprime with u, then c, x are coprime with v. If $c = x = 1$, we have $b^2 - (6v-1)b + v^3 = 0$, with discriminant $(6v-1)^2 - 4v^3$, which must be a perfect square, and where $1 \leq v \leq 8$. For these values of v, the discriminant takes values $21, 89, 181, 273, 341, 361, 309, 161$, out of which the only perfect square is $361 = 19^2$, obtained for $v = 6$, and yielding $0 = b^2 - 35b + 216 = (b-8)(b-27)$. But b must divide $v^2 = 36$, or $b = 8$ and $b = 27$ are not valid solutions, and no solution exists when $c = x = 1$. Now, if $v > 4$ then $cx < 2$, whereas if $v = 4$, then $cx \leq 2$ with c, x odd since they must be coprime with 4. Or $v \geq 4$ results in the already analized case $c = x = 1$. All other cases are:

Case 1: $v = 1$ results in $b^2c^2x^3 + b + c = 6bcx$. Clearly b, c must divide each other, for $b = c$, or $c^3x^3 + 2 = 6cx$. Note that cx is even with $c^2x^2 < 6$, yielding $cx = 2$. But then

$$6 = \frac{1}{cxv} + \frac{bcx^2}{v} + \frac{v^2}{bx} = \frac{1}{2} + 4 + \frac{1}{2} = 5,$$

absurd. Or no solution exists with $v = 1$.

Case 2: $v = 2$ results in $cx \leq 4$, with c, x odd, for either $c = x = 1$ which we have already analized, or $cx = 3$. If $v = 2$ and $cx = 3$, then (c, x) is a permutation of $(1, 3)$, and $0 = 9b^2x^2 - 35bx + 24 = (bx-3)(9bx-8)$, whose only integral solution is $bx = 3$. Since b must divide 3 and $v^2 = 4$, it follows that $b = c = 1$ and $x = 3$, for $u = cx^2 = 9$, $y = uv = 18$, and $z = \frac{v^2}{b} = 4$. Indeed $\frac{3}{18} + \frac{18}{4} + \frac{4}{3} = \frac{6+162+48}{36} = 6$, with $x = 3$ and $z = 4$ coprime, and $x + y + z = 25$.

Case 3: $v = 3$ results in $cx \leq 2$, for either $c = x = 1$ which we have already analized, or $cx = 2$. If $v = 3$ and $cx = 2$, then (c, x) is a permutation of $(1, 2)$ and $0 = 4b^2x^2 - 35bx + 54 = (bx-2)(4bx-27)$, whose only integral solution is $bx = 2$. Since b must divide 2 and $v^2 = 9$, it follows that $b = c = 1$ and $x = 2$, for $u = cx^2 = 4$, $y = uv = 12$, and $z = \frac{v^2}{b} = 9$. Indeed $\frac{2}{12} + \frac{12}{9} + \frac{9}{2} = \frac{6+48+162}{36} = 6$, with $x = 2$ and $z = 9$ coprime, and $x + y + z = 23$.

Note therefore that the only triples (x, y, z) satisfying the given conditions are $(x, y, z) = (3, 18, 4)$ and $(x, y, z) = (2, 12, 9)$, and there can be no other one. The maximum is then $x + y + z = 25$.

Also solved by Dumitru Barac, Sibiu, Romania; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

O521. In the triangle ABC we denote by m_a, m_b, m_c the lengths of its medians and by w_a, w_b, w_c the lengths of the angle bisectors. Prove that

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \leq 1 + \frac{R}{r}$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Solution by Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA

Using the well known inequality [1]

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \geq \frac{s^2 + r^2 - 2Rr}{4Rr}$$

we rewrite the original expression as

$$\frac{s^2 + r^2 - 2Rr}{4Rr} \leq 1 + \frac{R}{r}$$

Simplifying,

$$s^2 \leq 4R^2 + 6Rr - r^2 \tag{1}$$

Given Euler's inequality $R \geq 2r$, this can be rewritten as

$$0 \leq R^2 - 2Rr \tag{2}$$

Adding inequalities (1) and (2) we obtain

$$s^2 \leq 5R^2 + 4Rr - r^2,$$

which is true [2] and the conclusion follows. Equality holds when $a = b = c$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romani; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, California, USA; Ioannis D. Sfikas, Athens, Greece.

[1] `http://www.ssmrmh.ro/wp-content/uploads/2020/04/RMM-TRIANGLE-MARATHON-1701-1800_compressed.pdf`, Problem 7667, p. 90-91.

[2] Recent Advances in Geometric Inequalities, D. S. Mitrinovic et al, 1989., p. 51.

O522. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{27abc}{4(a^3+b^3+c^3)}\ge\frac{21}{4}$$

Proposed by Marius Stanean, Zalau, Romania

Solution by Titu Zvonaru, Comănești, Romania

We wil prove a stronger inequality

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{9abc}{a^3+b^3+c^3}\ge 6.$$

It is known that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\ge 12. \tag{1}$$

We will prove that

$$\frac{abc}{a^3+b^3+c^3}+\frac{2}{3}\ge\frac{ab+bc+ca}{a^2+b^2+c^2} \tag{2}$$

The inequality (2) is equivalent to

$$3abc(a^2+b^2+c^2)+2(a^2+b^2+c^2)(a^3+b^3+c^3)\ge 3(a^3+b^3+c^3)(ab+bc+ca) \tag{3}$$

which can be rewritten as

$$(a^2+b^2+c^2-ab-bc-ca)[2(a^3+b^3+c^3)-ab(a+b)-bc(b+c)-ca(c+a)]>=0$$

and the conclusion follows.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioannis D. Sfikas, Athens, Greece.