The continuous solutions of Cauchy's functional equation using a simple geometric argument

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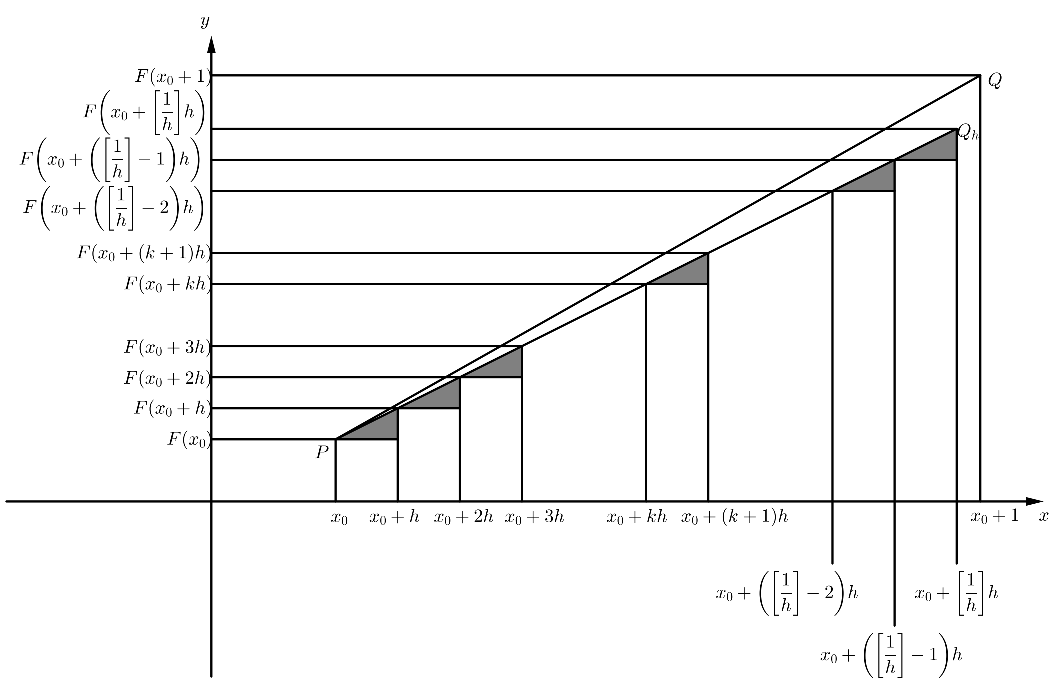
It is well-known that the continuous solutions $F\colon \mathbb{R} \to \mathbb{R}$ of Cauchy's additive functional equation:

$$F(A+B) = F(A) + F(B)$$

for all A, B are the functions $F(x) = \alpha x$. This classic calculus problem opens the door to the vast world of functional equations (for more details, please refer to the bibliography at the end of this note). In order to solve Cauchy's equation, we usually notice that $F(x) = xF(1)$, first for $x \in \mathbb{Z}$, then for $x \in \mathbb{Q}$. The general case follows from the density of $\mathbb{Q}$ in $\mathbb{R}$. In the new proof given here, we demonstrate and take advantage of the differentiability of the solutions, *even though there is no derivative in the equation and the solutions are only assumed to be continuous.* Thus, this proof provides an example of use of derivatives as a *tool.*

For any $h > 0$, since $F(x_0 + (k+1)h) - F(x_0 + kh) = F(h)$ does not depend on the integer k, all the colored right-angled triangles of the figure have the same catheti. So they are congruent, and their hypotenuses have the same slope. Thus they are subsets of the same line PQ_h, and the slope of PQ_h is the slope of any hypotenuse: $(F(x_0+h)-F(x_0))/h$.



On the other hand, it is easy to prove that, when $h \to 0$: $h[1/h] \to 1$, where $[x]$ denotes the largest integer less than or equal to x. In other words: $1/h$ and $[1/h]$ are equivalent when $1/h \to \infty$. By the continuity of F, this implies that $Q_h \to Q$. So when $h > 0$ tends to 0, the slope of PQ_h tends to the slope of PQ (which will turn out to be the graph of F):

$$\frac{F(x_0+h) - F(x_0)}{h} \to F(x_0+1) - F(x_0) = F(1)\cdot$$

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For $h < 0$, we have:

$$F(x_0 + |h|) - F(x_0) = F(|h|) = F(x_0) - F(x_0 - |h|),$$

$$\frac{F(x_0 + h) - F(x_0)}{h} = \frac{F(x_0) - F(x_0 - |h|)}{|h|} = \frac{F(x_0 + |h|) - F(x_0)}{|h|},$$

where $|h| > 0$. So when $h < 0$ tends to 0, we also have:

$$\frac{F(x_0 + h) - F(x_0)}{h} \to F(1) \cdot$$

Thus $F'(x_0) = F(1)$ and F has a constant derivative $F(1)$, which implies that:

$$F(x) = F(1) + \int_1^x F'(t)dt = F(1) + F(1)(x - 1) = F(1)x \cdot$$

References.

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