Junior problems

J523. Let a, b, c be real numbers. Prove that

$$(a-1)^2 + (b-1)^2 + (c-1)^2 \geq \frac{ab+bc+ca}{2} - 3.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
The inequality

$$(a-1)^2 + (b-1)^2 + (c-1)^2 \geq \frac{ab+bc+ca}{2} - 3$$

is equivalent to

$$(a-2)^2 + (b-2)^2 + (c-2)^2 + \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 \geq 0,$$

which is clearly true. Equality holds if and only if $a = b = c = 2$.

Also solved by Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Murat Chashemov, Dashoguz, Turkmenistan; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Israel Castillo Pilco, Huaral, Peru; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Joel Schlosberg, Bayside, NY, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Daniel Văcaru, Piteşti, Romania; Jennisha Sunil Agrawal, DDPS, Gujarat, India; Lorenzo Benedetti, Genoa, Italy; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Michail Prousalidis, Evangeliki Model High School of Smyrna Athens, Greece; Ioannis D. Sfikas, Athens, Greece; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Booyeon Brian Choi, Middlesex School, Concord, MA, USA; Polyahedra, Polk State College, FL, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Taes Padhihary, Disha Delphi Public School, India; Nguyen Viet Hung, Hanoi University of Science, Vietnam.

J524. Prove that

$$\sum_{1\le i<j\le n}\frac{i+2ij+j}{\sqrt{(i+1)(j+1)}}<\frac{n\left(n^2-1\right)}{2}$$

Proposed by Mihaly Bencze, Brasov, Romania

Solution by Polyahedra, Polk State College, FL, USA

By the Cauchy-Schwarz inequality, for $n\ge 2$,

$$\left(\sum_{i=1}^{n}\sqrt{i+1}\right)^2<n\sum_{i=1}^{n}(i+1)=\frac{n^2(n+3)}{2}$$

and

$$\left(\sum_{i=1}^{n}\sqrt{i+1}\right)\left(\sum_{i=1}^{n}\frac{1}{\sqrt{i+1}}\right)>\left(\sum_{i=1}^{n}\sqrt{i+1}\cdot\frac{1}{\sqrt{i+1}}\right)^2=n^2.$$

Therefore,

$$\begin{aligned}\sum_{1\le i<j\le n}\frac{i+2ij+j}{\sqrt{(i+1)(j+1)}}&=\sum_{1\le i<j\le n}\left(i\sqrt{\frac{j+1}{i+1}}+j\sqrt{\frac{i+1}{j+1}}\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}i\sqrt{\frac{j+1}{i+1}}-\sum_{k=1}^{n}k\\&=\left(\sum_{j=1}^{n}\sqrt{j+1}\right)\sum_{i=1}^{n}\left(\sqrt{i+1}-\frac{1}{\sqrt{i+1}}\right)-\frac{n(n+1)}{2}\\&<\frac{n^2(n+3)}{2}-n^2-\frac{n(n+1)}{2}=\frac{n(n^2-1)}{2}.\end{aligned}$$

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Booyeon Brian Choi, Middlesex School, Concord, MA, USA; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Daniel Văcaru, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J525. Prove that positive integers a, b, c are consecutive in some order if and only if

$$a^3 + b^3 + c^3 = 3(a + b + c + abc).$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan
Since $a, b, c > 0$,

$$\begin{aligned} & a^3 + b^3 + c^3 = 3(a + b + c + abc) \\ \Leftrightarrow\ & (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca - 3) = 0 \\ \Leftrightarrow\ & (a - b)^2 + (b - c)^2 + (c - a)^2 = 6 \\ \Leftrightarrow\ & \{|a - b|, |b - c|, |c - a|\} = \{1, 1, 2\}, \end{aligned}$$

we are done.

Also solved by Polyahedra, Polk State College, FL, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Israel Castillo Pilco, Huaral, Peru; Nakis Konstantinos Pantelis, Ionidios Model High School, Greece; Booyeon Brian Choi, Middlesex School, Concord, MA, USA; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Taes Padhihary, Disha Delphi Public School, India; Murat Chashemov, Dashoguz, Turkmenistan; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Piteşti, Romania; Jennisha Sunil Agrawal, DDPS, Gujarat, India; Joel Schlosberg, Bayside, NY, USA; Lorenzo Benedetti, Genoa, Italy; Ioannis D. Sfikas Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Evripides P. Nastou, 6th High School, Nea Smyrni, Greece.

J526. Let ABC be a triangle with circumcenter O, incenter I, and excenters I_a, I_b, I_c. Prove that

$$OI^2 + OI_a^2 + OI_b^2 + OI_c^2 = 12R^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Polyahedra, Polk State College, FL, USA

Let r, r_a, r_b, r_c be the radii of the incircle and the excircles. Suppose that AI intersects the circumcircle of $\triangle ABC$ at M. It is well known (by angle-chasing) that $MI = MB = MC = MI_a$. Therefore,

$$AI \cdot IM = AM \cdot BM - BM^2 = \frac{2[ABM] - 2[IBM]}{\sin \angle AMB} = \frac{2[ABI]}{\sin C} = \frac{cr}{\sin C} = 2Rr.$$

By the power of a point, $AI \cdot IM = R^2 - OI^2$ and $AI_a \cdot I_aM = OI_a^2 - R^2$, so $OI^2 = R^2 - 2Rr$ and

$$OI_a^2 = R^2 + AI_a \cdot I_aM = R^2 + \frac{s}{s-a} AI \cdot IM = R^2 + 2Rr_a.$$

The claimed identity now follows from the well-known identity that $r_a + r_b + r_c - r = 4R$, which is also easy to obtain from

$$rs\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}\right) = \frac{rsabc}{s(s-a)(s-b)(s-c)} = \frac{abc}{[ABC]} = 4R.$$

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Taes Padhihary, Disha Delphi Public School, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Piteşti, Romania; Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA; Ioannis D. Sfikas, Athens, Greece; Todor Zaharinov, Sofia, Bulgaria; Telemachus Baltsavias, Keramies Junior High School ,Kefalonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA .

J527. Find all n for which $2\dots225$ (n twos) is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Clearly, $n = 1$ and $n = 2$ are solutions. For $n \geq 3$ we have

$$\begin{aligned} 2...25 \text{ (} n \text{ twos)} = 25 \cdot 8...89 \text{ (} n-2 \text{ eights)} = \\ 25 \cdot (1 + 8 \cdot 1...1 \text{ (} n-1 \text{ ones)}) = \\ \left(\frac{25}{9}\right) \cdot 80...01 \text{ (} n-1 \text{ zeros)}, \end{aligned}$$

so 80...01 ($n-1$ zeros) must be a perfect square. It follows that $8 \cdot 10^{n-1} + 1 = (2k+1)^2$, for some positive integer k, implying $k(k+1) = 2^n \cdot 5^{n-1}$. From here $5^{n-1} \mid k$ or $k+1$, hence $k \geq 5^{n-1} - 1$ and $2^n \geq 5^{n-1} - 1$, impossible for $n \geq 3$. Thus $n = 1$ and $n = 2$ are the only solutions.

Second solution by Joel Schlosberg, Bayside, NY, USA

For a positive integer n, if

$$\underbrace{2\dots22}_{n \text{ twos}}5 = \frac{2}{9}\cdot\underbrace{9\dots9}_{n+1 \text{ nines}}+3 = \frac{2}{9}(10^{n+1}-1)+3 = \frac{2\cdot10^{n+1}+25}{9}$$

equals a perfect square s^2 for $s \in \mathbb{N}$, then

$$(3s-5)(3s+5) = 9s^2-25 = 2^{n+2}5^{n+1}.$$

Since their difference is even, $3s-5$ and $3s+5$ are either both odd or both even; since their product is even, they are both even. Since the prime number 5 divides the product of $3s-5$ and $3s+5$, by Proposition 30 in Book VII of Euclid's *Elements*, 5 divides $3s-5$ or $3s+5$; since their difference is 10, $5 \mid 3s\pm5$ implies that $5 \mid 3s\mp5$. Then

$$\frac{3s-5}{10}\cdot\frac{3s+5}{10} = 2^n5^{n-1}$$

where $\frac{3s-5}{10}$ and $\frac{3s+5}{10}$ are consecutive integers, and so must be relatively prime. By unique factorization,

$$\left\{\frac{3s-5}{10},\frac{3s+5}{10}\right\} = \{2^n, 5^{n-1}\}.$$

For $n \geq 3$,

$$5^{n-1}-2^n > 4^{n-1}-2^n = \left(2^{n-1}-1\right)^2-1 \geq 8,$$

so 2^n and 5^{n-1} cannot be consecutive integers.

Since $25 = 5^2$ and $225 = 15^2$, the positive integers n such that $\underbrace{2\dots22}_{n \text{ twos}}5$ is a perfect square are $n = 1$ and $n = 2$.

Also solved by Lorenzo Benedetti, Genoa, Italy; Polyahedra, Polk State College, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Booyeon Brian Choi, Middlesex School, Concord, MA, USA; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Taes Padhihary, Disha Delphi Public School, India; Daniel Văcaru, Piteşti, Romania; Sailalitha Kodukula, Archimedean Middle Conservatory, Miami, FL, USA; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J528. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\sum_{cyc} \frac{a^2 + b^2}{a + b + 2} \geq \frac{3(a + b + c - 1)}{4}.$$

Proposed by Mihaela Berindeanu, București, Romania

Solution by Arkady Alt, San Jose, CA, USA

By the Cauchy-Schwarz inequality

$$\sum_{cyc} \frac{a^2 + b^2}{a + b + 2} = \sum_{cyc} \frac{a^2}{a + b + 2} + \sum_{cyc} \frac{b^2}{a + b + 2} \geq$$

$$\frac{(a + b + c)^2}{\sum_{cyc} (a + b + 2)} + \frac{(a + b + c)^2}{\sum_{cyc} (a + b + 2)} = \frac{2(a + b + c)^2}{2(a + b + c) + 6}$$

and

$$\frac{(a + b + c)^2}{a + b + c + 3} \geq \frac{3(a + b + c - 1)}{4}.$$

Now, let $s := a + b + c$. Then the latter inequality becomes

$$\frac{s^2}{s + 3} \geq \frac{3(s - 1)}{4} \iff 4s^2 \geq 3(s - 1)(s + 3) \iff (s - 3)^2 \geq 0.$$

Note: $ab + bc + ca = 3$ is not needed for the proof.

Also solved by Nguyen Viet Hung, Hanoi University of Science, Vietnam; Polyahedra, Polk State College, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Israel Castillo Pilco, Huaral, Peru; Daniel Văcaru, Piteşti, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Taes Padhihary, Disha Delphi Public School, India; Murat Chashemov, Dashoguz, Turkmenistan; Dumitru Barac, Sibiu, Romania; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

S523. Let a, b, c in $[1, 8]$. Prove that

$$\left(2-\frac{a}{b^2}\right)\left(2-\frac{b}{c^2}\right)\left(2-\frac{c}{a^2}\right) \le abc.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey

Let $x = \dfrac{1}{a}$, $y = \dfrac{1}{b}$ and $z = \dfrac{1}{c}$. Then it suffices to prove that

$$(2x - y^2)(2y - z^2)(2z - x^2) \le 1, \text{ for } x, y, z \in \left[\frac{1}{8}, 1\right].$$

Since $\dfrac{2-\frac{a}{b^2}}{a} = \dfrac{2}{a} - \dfrac{1}{b^2} = 2x - y^2$ and so on. Now let $u = 2x - y^2$,$v = 2y - z^2$ and $w = 2z - x^2$. If at least one of u, v, w is equal to 0, then it is true. So we can assume that $u, v, w \ne 0$

If 3 or 1 of u, v, w is negative, then it is trivial, because then $(2x - y^2)(2y - z^2)(2z - x^2)$ is negative and less than 1.

If 2 of u, v, w are negative, then assume $u < 0, v < 0, w > 0$. This means

$$y^2 > 2x, z^2 > 2y, 2z > x^2$$

So

$$1 \ge z^4 = (z^2)^2 > (2y)^2 = 4y^2 > 8x \Rightarrow \frac{1}{8} > x, \text{ a contradiction.}$$

If u, v, w are all positive, because of

$$x^2 - 2x + 1 \ge 0 \Rightarrow 1 \ge 2x - x^2 \Rightarrow 3 \ge (2x - x^2) + (2y - y^2) + (2z - z^2),$$

we have

$$1 \ge \frac{(2x - x^2) + (2y - y^2) + (2z - z^2)}{3} = \frac{u + v + w}{3} \ge \sqrt[3]{uvw} \Rightarrow 1 \ge uvw.$$

Also solved by Fahreezan Sheraz Diyaldin Sleman, Special Region of Yogyakarta, Indonesia; Arkady Alt, San Jose, CA, USA; Doniyor Yazdonov, National University of Uzbekistan, Tashkent, Uzbekistan.

S524. Find all systems (x, y, z, t) of real numbers such that

$$x(y+z+t)^2 = y(x+z+y)^2 = z(x+z+y)^2 = t(x+z+y)^2$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Joel Schlosberg, Bayside, NY, USA

For such a system (x, y, z, t),

$$0 = x(y+z+t)^2 - y(x+z+t)^2 = (x-y)(z^2+2tz+t^2-xy)$$

and

$$0 = z(x+y+t)^2 - t(x+y+z)^2 = (z-t)(x^2+2xy+y^2-tz)$$

so if $x \neq y$ and $z \neq t$,

$$0 = 4(z^2+2tz+t^2-xy) + 4(x^2+2xy+y^2-tz)$$
$$= 3(x+y)^2 + (x-y)^2 + 3(z+t)^2 + (z-t)^2 > 0$$

a contradiction. By the same reasoning, $y = z$ or $t = x$. Therefore, at least three of x, y, z, t are equal. If $x = y = z$, then

$$0 = x(y+z+t)^2 - t(x+y+z)^2 = x(2x+t)^2 - t(3x)^2 = x(x-t)(4x-t)$$

so $(x, y, z, t) = (r, r, r, r), (0, 0, 0, r)$ or $(r, r, r, 4r)$ for some real r. Conversely, for any real r, all such systems satisfy the given equalities. By the same reasoning for the cases $y = z = t$, $z = t = x$ and $t = x = y$, the solutions of the given equalities are given by $\{x, y, z, t\} = \{r, r, r, r\}, \{0, 0, 0, r\}$ and $\{r, r, r, 4r\}$ for real r.

Also solved by Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Nakis Konstantinos Pantelis, Ionidios Model High School, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S525. Find the maximum of $(x-2)(y+1)$ over all real numbers x and y satisfying $3x^2+4xy+5y^2=1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Let $E(x,y)=(x-2)(y+1)$. We have

$$\begin{aligned}1-2E(x,y) &= 3x^2+4xy+5y^2-2(xy+x-2y-2)= \\ &(x^2+2xy+y^2)+2\left(x^2-x+\frac{1}{4}\right)+4\left(y^2+y+\frac{1}{4}\right)+\frac{5}{2}= \\ &(x+y)^2+2\left(x-\frac{1}{2}\right)^2+4\left(y+\frac{1}{2}\right)^2+\frac{5}{2}\geq\frac{5}{2},\end{aligned}$$

with equality if and only if $x=\frac{1}{2}$ and $y=-\frac{1}{2}$. Hence, the maximum of $E(x,y)$ is $-\frac{3}{4}$.

Also solved by Joel Schlosberg, Bayside, NY, USA; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Todor Zaharinov, Sofia, Bulgaria.

S526. Let x, y, z be nonnegative real numbers. Prove that

$$x^3 + y^3 + z^3 \geq \sqrt{\frac{1}{3}\prod_{cyc}(2x^2 - xy + 2y^2)} \geq \sum_{cyc} xy(x+y) - 3xyz.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

For inequality on the left hand side, without loss of generality, we may assume that $z = \min\{x, y, z\}$. Squaring both sides and expanding, we need to prove that

$$3(x^3 + y^3 + z^3)^2 \geq (2x^2 - xy + 2y^2)(2y^2 - yz + 2z^2)(2z^2 - zx + 2x^2),$$

or

$$3\sum_{cyc} x^6 - 8\sum_{cyc} x^2y^2(x^2 + y^2) + 4xyz\sum_{cyc} x^3 + 10\sum_{cyc} x^3y^3 + 2\sum_{cyc} xz(x^2y^2 + y^2z^2) - 15x^2y^2z^2 \geq 0,$$

that is

$$3\sum_{cyc}(x^6 - y^3z^3) - 8\sum_{cyc} x^2y^2(x-y)^2 + 4xyz\left(\sum_{cyc} x^3 - 3xyz\right) - 3\left(\sum_{cyc} x^3y^3 - 3x^2y^2z^2\right)$$
$$+2xyz\sum_{cyc} z(x-y)^2 \geq 0.$$

We have the following SOS-Schur representations,

$$\sum_{cyc}(x^6 - y^3z^3) = (x^2 + xy + y^2)^2(x-y)^2 + (y^2 + yz + z^2)(z^2 + zx + x^2)(x-z)(y-z),$$
$$\sum_{cyc} x^2y^2(x-y)^2 = (x^2y^2 + y^2z^2 + z^2x^2 + z^2(xy - zx - zy))(x-y)^2 + z^2(x^2 + y^2)(x-z)(y-z),$$
$$\sum_{cyc} x^3 - 3xyz = (x+y+z)(x-y)^2 + (x+y+z)(x-z)(y-z),$$
$$\sum_{cyc} x^3y^3 - 3x^2y^2z^2 = (xy + yz + zx)z^2(x-y)^2 + (xy + yz + zx)xy(x-z)(y-z)),$$
$$\sum_{cyc} z(x-y)^2 = 2z(x-y)^2 + (x+y)(x-z)(y-z).$$

So, our inequality can be written in SOS-Schur form as follows

$$A(x,y,z)(x-y)^2 + B(x,y,z)(x-z)(y-z) \geq 0,$$

where

$$\begin{aligned}
A(x,y,z) =& 3(x^2 + xy + y^2)^2 - 8(x^2y^2 + y^2z^2 + z^2x^2 + z^2(xy - zx - zy)) \\
&+ 4xyz(x+y+z) - 3z^2(xy + yz + zx) + 4xyz^2 \\
=& 3(x^4 + y^4) + 6xy(x^2 + y^2) + x^2y^2 - 8z^2(x^2 + y^2) + xyz(4x + 4y - 3z) + 5(x+y)z^3 \\
\geq& 9xy(x^2 + y^2) - 8z^2(x^2 + y^2) + x^2y^2 + xyz(4x + 4y - 3z) + 5(x+y)z^3 \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
B(x,y,z) =& 3(y^2 + yz + z^2)(z^2 + zx + x^2) - 8z^2(x^2 + y^2) + 4xyz(x+y+z) \\
&- 3xy(xy + yz + zx) + 2xyz(x+y) \\
=& z\left(6x^2y + 6y^2x - 5x^2z - 5y^2z + 7xyz + 3(x+y)z^2 + 3z^3\right) \geq 0.
\end{aligned}$$

Our proof is complete. Equality holds when $x = y = z$ or $x = y,\ z = 0$.

We will focus on the right hand side. For a start, let's note that

$$2x^2 - xy + 2y^2 = \frac{3(x+y)^2 + 5(x-y)^2}{4},$$

and

$$\begin{aligned}(2y^2 - &yz + 2z^2)(2z^2 - zx + 2x^2)\\ &= 4z^4 - 2(x+y)z^3 + (4x^2 + xy + 4y^2)z^2 - 2xy(x+y)z + 4x^2y^2\\ &= 4z^4 + 4x^2y^2 - 2(x+y)z(z^2 + xy) + (4x^2 + xy + 4y^2)z^2\\ &= 4(z^2 + xy)^2 - 2(z^2 + xy)(x+y)z + \frac{(x+y)^2z^2}{4} + \frac{15}{4}z^2(x-y)^2\\ &= \frac{1}{4}(4z^2 + 4xy - xz - yz)^2 + \frac{15}{4}z^2(x-y)^2.\end{aligned}$$

By the Cauchy-Schwarz Inequality,

$$\begin{aligned}\prod_{cyc}(2x^2 - xy + 2y^2) &= \frac{1}{16}\left(3(x+y)^2 + 5(x-y)^2\right)\left((4z^2 + 4xy - xz - yz)^2 + 15z^2(x-y)^2\right)\\ &\geq \frac{3}{16}\left[(x+y)(4z^2 + 4xy - xz - yz) + 5z(x-y)^2\right]^2\\ &= \frac{3}{16}\left[4(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 - 3xyz)\right]^2\\ &= 3\left[xy(x+y) + yz(y+z) + zx(z+x) - 3xyz\right]^2.\end{aligned}$$

Equality holds when $x = y = z$ or $x = y,\ z = 0$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S527. Let ABC be a triangle with $AB = AC$ and $\angle A = 90°$. Points E and F are given inside the angle $\angle BAC$ such that $\angle EAF = \angle EAB + \angle FAC$ and $BE \| CF$. Prove that

$$EF^2 = BE^2 + CF^2.$$

Proposed by Mihai Miculita, Oradea, Romania

Solution by Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey

Lets take a point D, inside the angle BAC, such that $AB = AC = AD$ and $\angle DAF = \angle CAF$, $\angle DAE = \angle BAE$. This is possible since the given information. Then this point D is the reflections of C and B, over AF and AE, respectively. So $FD = FC$ and $ED = EB$.

Also a single angle chasing gives us

$$\angle FDE = \angle FCA + \angle EBA = \angle BCA + \angle CBA = 90°.$$

So the triangle FDE is right triangle. Hence

$$EF^2 = FD^2 + ED^2 = FC^2 + EB^2.$$

Also solved by Taes Padhihary, Disha Delphi Public School, India; Fahreezan Sheraz Diyaldin Sleman, Special Region of Yogyakarta, Indonesia; Ervin Macić, Bosnia and Herzegovina, Vogosca, Sarajevo, Bosnia and Herzegovina; Corneliu Mănescu-Avram, Ploieşti, Romania; Murat Chashemov, Dashoguz, Turkmenistan; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S528. Let a, b, c be positive real numbers such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{11}{a+b+c}.$$

Find the minimum of

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey

We have $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=11 \Rightarrow \sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{a}{c}=8$. Let $\sum_{cyc}\frac{a}{b}=x$, $\sum_{cyc}\frac{a}{c}=y$, $x+y=8$.

$$\begin{aligned}\sum_{cyc}\frac{a^4}{b^4}&=\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2-2\left(\sum_{cyc}\frac{a^2}{c^2}\right)\\&=\left(\left(\sum_{cyc}\frac{a}{b}\right)^2-2\left(\sum_{cyc}\frac{a}{c}\right)\right)^2-2\left(\left(\sum_{cyc}\frac{a}{c}\right)^2-2\left(\sum_{cyc}\frac{a}{b}\right)\right)\\&=(x^2-2y)^2-2(y^2-2x)\end{aligned}$$

Similarly, we have

$$\sum_{cyc}\frac{a^4}{c^4}=(y^2-2x)^2-2(x^2-2y)$$

Then

$$\begin{aligned}\sum_{cyc}\frac{a^4}{b^4}+\sum_{cyc}\frac{a^4}{c^4}&=(x^2-2y)^2-2(y^2-2x)+(y^2-2x)^2-2(x^2-2y)\\&=(x^2-2y-1)^2+(y^2-2x-1)^2-2\\&\geq\frac{1}{2}(x^2-2x+y^2-2y-2)^2-2\\&=\frac{1}{2}((x-1)^2+(y-1)^2-4)^2-2\\&\geq\frac{1}{2}\left(\frac{1}{2}((x-1)+(y-1))^2-4\right)^2-2\\&=\frac{1}{2}\left(\frac{1}{2}(x+y-2)^2-4\right)^2-2\\&=\frac{1}{2}\left(\frac{1}{2}6^2-4\right)^2-2\\&=96.\end{aligned}$$

Finally,

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)=\sum_{cyc}\frac{a^4}{b^4}+\sum_{cyc}\frac{a^4}{c^4}+3\geq 99.$$

Equality occurs when $a=1, b=1, c=\frac{3}{2}-\frac{\sqrt{5}}{2}$.

Also solved by Israel Castillo Pilco, Huaral, Peru; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioannis D. Sfikas, Athens, Greece; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U523. Prove that for each nonegative integer n,

$$\prod_{k=1}^{10^n-1}\left(\frac{4(2k+1)}{2k^2-2k+1}+1\right)$$

is a positive integer whose sum of digits is 5.

Proposed by Titu Andreescu, Unversity of Texas at Dallas, USA

Solution by the author

$$\frac{4(2k+1)}{2k^2-2k+1}+1=\frac{2k^2+6k+5}{2k^2-2k+1}$$

Because

$$(2k^2-2k+1)(2k^2+2k+1)=(2k^2+1)^2-(2k)^2=4k^4+1$$

and

$$(2k^2+6k+5)(2k^2+2k+1)=4(k^4+4k^3+6k^2+4k+1)+1=4(k+1)^4+1,$$

the product becomes

$$\prod_{k=0}^{10^n-1}\frac{4(k+1)^4+1}{4k^4+1}.$$

and telescopes to $4\times(10^n-1+1)^4+1=400...01$, with $4n-1$ zeros. Hence the conclusion.

Also solved by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Booyeon Brian Choi, Middlesex School, Concord, MA, USA; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Murat Chashemov, Dashoguz, Turkmenistan; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Davrbek Oltiboev, National University of Uzbekistan, Tashkent, Uzbekistan; Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Maiteyo Bhattacharjee, IACS, Kolkata; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U524. Prove that for all integers $n > 2$,

$$2 - \frac{1}{n!} < \left(1 + \frac{1}{2!}\right)\left(1 + \frac{2}{3!}\right)\dots\left(1 + \frac{n-1}{n!}\right) < 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Let us denote $L_n = 2 - \frac{1}{n!}$ and $M_n = \left(1 + \frac{1}{2!}\right)\left(1 + \frac{2}{3!}\right)\dots\left(1 + \frac{n-1}{n!}\right)$. For $n = 3$, $L_3 = \frac{11}{6}$, and $M_n = 2$, so $L_3 < M_3 < 3$.

Let us suppose that the left-hand side inequality proposed hold for $n > 2$. In order to prove that it also hold for $n + 1$ it is enough to see that

$$\begin{aligned} 2 - \frac{1}{(n+1)!} - 2 + \frac{1}{n!} &\leq M_n\left(1 + \frac{n}{(n+1)!} - 1\right) \\ \frac{1}{n!}\left(1 - \frac{1}{n+1}\right) &\leq M_n \cdot \frac{n}{(n+1)!} \\ \frac{n}{(n+1)!} &\leq M_n \cdot \frac{n}{(n+1)!}, \end{aligned}$$

which it is true because $M_n > 1$.

In order to prove the right-hand side proposed inequality it is enough to see that $\prod_{n=2}^{\infty}\left(1 + \frac{n-1}{n!}\right) < 3$, since $\{M_n\}_{n\geq 3}$ is and increasing sequence. We may proceed as follows

$$\ln \prod_{n=2}^{\infty}\left(1 + \frac{n-1}{n!}\right) = \sum_{n=2}^{\infty} \ln\left(1 + \frac{n-1}{n!}\right) < \sum_{n=2}^{\infty} \frac{n-1}{n!} = 1$$

because $\ln(1+x) < x$ for $x > 0$ and $\sum_{n=2}^{\infty} \frac{n-1}{n!} x^n = e^x(x-1) + 1$. Therefore, $\prod_{n=2}^{\infty}\left(1 + \frac{n-1}{n!}\right) < e < 3$ and the problem is done.

Second solution by Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India

Suppose for $k > 1$, $\alpha_1, \alpha_2 \dots \alpha_k$ are positive real numbers, then

$$\begin{aligned}(1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_k) &= 1+\alpha_1+\alpha_2+\dots\alpha_k+\prod_{1\le i<j\le n}\alpha_i\alpha_j+\dots\\ &\ge 1+\alpha_1+\alpha_2+\dots\alpha_k+\prod_{1\le i<j\le n}\alpha_i\alpha_j\\ &> 1+\alpha_1+\alpha_2+\dots\alpha_k\end{aligned}$$

(The $k > 1$ condition was necessary to ensure that cross-terms arise in the expansion). Applying this with $\alpha_i = \dfrac{i}{(i+1!)}$, we get

$$\begin{aligned}\prod_{i=1}^{n-1}\left(1+\frac{i}{(i+1)!}\right) &> 1+\sum_{i=1}^{n-1}\frac{i}{(i+1)!}\\ &= 1+\sum_{i=1}^{n-1}\left(\frac{1}{i!}-\frac{1}{(i+1)!}\right)\\ &= 1+\frac{1}{1!}-\frac{1}{n!}\\ &= 2-\frac{1}{n!}\end{aligned}$$

By AM-GM inequality, we have

$$\begin{aligned}\left(\prod_{i=1}^{n-1}\left(1+\frac{i}{(i+1)!}\right)\right)^{\frac{1}{n-1}} &\le \frac{1}{n-1}\sum_{i=1}^{n-1}\left(1+\frac{i}{(i+1)!}\right)\\ &= \frac{n-1+\sum_{i=1}^{n-1}\frac{i}{(i+1)!}}{n-1}\\ &= \frac{n-2+1+\sum_{i=1}^{n-1}\frac{i}{(i+1)!}}{n-1}\\ &= \frac{n-2+2-\frac{1}{n!}}{n-1} \quad \text{(evaluated earlier)}\\ &= \frac{n-1+1-\frac{1}{n!}}{n-1}\\ &\le \frac{n-1+1}{n-1}\\ &= 1+\frac{1}{n-1}\end{aligned}$$

Hence,

$$\prod_{i=1}^{n-1}\left(1+\frac{i}{(i+1)!}\right) \le \left(1+\frac{1}{n-1}\right)^{n-1}$$

Observe that $(1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_n)(1+\alpha_{n+1})-(1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_n) = (1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_n)\alpha_{n+1} \geq 0$, hence the given sequence is monotonically increasing. Therefore, for every $m \geq n$,

$$\prod_{i=1}^{n-1}\left(1+\frac{i}{(i+1)!}\right) \leq \prod_{i=1}^{m-1}\left(1+\frac{i}{(i+1)!}\right) \leq \left(1+\frac{1}{m-1}\right)^{m-1}$$

Taking the limit as $m \to \infty$, we get

$$\prod_{i=1}^{n-1}\left(1+\frac{i}{(i+1)!}\right) \leq e < 3$$

Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Davrbek Oltiboev, National University of Uzbekistan, Tashkent, Uzbekistan; Booyeon Brian Choi, Middlesex School, Concord, MA, USA; Emirhan Yagcioglu, İhsan Doğramacı Bilkent University, Turkey; Taes Padhihary, Disha Delphi Public School, India; Fahreezan Sheraz Diyaldin Sleman, Special Region of Yogyakarta, Indonesia; Toyesh Prakash Sharma, St. C.F Andrews School, Agra, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Piteşti, Romania; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U525. Let $P(x)$ be a polynomial of degree d with integer coefficients and let $d_1, \dots, d_k$ be distinct integers. Prove that for any positive integer $k \le d$ there are unique integers $a_1, \dots, a_k$, not all zero, with $\gcd(a_1, \dots, a_k) = 1$ and such that the polynomial $a_1 P(x + d_1) + \cdots + a_k P(x + d_k)$ has degree $d - k + 1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

First solution by Li Zhou, Polk State College, USA

By Taylor's expansion centered at x, we have

$$P(x+c) = \sum_{i=0}^{d} \frac{P^{(i)}(x)}{i!} c^i,$$

and thus

$$b_1 P(x+d_1) + \cdots + b_k P(x+d_k) = \sum_{i=0}^{d} \frac{b_1 d_1^i + \cdots + b_k d_k^i}{i!} P^{(i)}(x).$$

Let

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}, \ D = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_k \\ \vdots & \vdots & \ddots & \vdots \\ d_1^{k-1} & d_2^{k-1} & \cdots & d_k^{k-1} \end{bmatrix}, \ E = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Setting $DB = E$ and using Cramer's rule and Vandermonde's determinants, we get $b_i = 1/q_i$ for $1 \le i \le k$, where $q_i = \prod_{j \in \{1,\dots,k\} \setminus \{i\}} (d_j - d_i)$. Finally, let $a_i = b_i \operatorname{lcm}(q_1, \dots, q_k)$, then $a_i \in \mathbb{Z} \setminus \{0\}$ for each i, $\gcd(a_1, \dots, a_k) = 1$, and $a_1 P(x + d_1) + \cdots + a_k P(x + d_k)$ has degree $d - k + 1$. This $(a_1, \dots, a_k)$ is clearly unique up to a sign.

Second solution by the author

First, assume that the degere of polynomial $Q(x) = a_1P(x+d_1) + \cdots + a_kP(x+d_k)$ is at most $d-k$. Therefore,

$$Q(x) = (a_1 + \cdots + a_k)P(x) + (a_1d_1 + \cdots + a_kd_k)P'(x) + \\ + \frac{1}{2!}(a_1d_1^2 + \cdots + a_kd_k^2)P''(x) + \cdots + \frac{1}{(k-1)!}(a_1d_1^{k-1} + \cdots + a_kd_k^{k-1})P^{(k-1)}(x) + R(x).$$

Where $R(x)$ is a polynomial of degree at most $d-k$. Therefore, we should have

$$a_1 + \cdots + a_k = a_1d_1 + \cdots + a_kd_k = \cdots = a_kd_1^{k-1} + \cdots + a_kd_k^{k-1} = 0.$$

Now, consider the polynomial $f_1(x) = (x-d_2)\dots(x-d_k) = x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_0$. Multiplying the first equation by c_0, the second by c_1 and the last one by $c-2$ and adding them up results into

$$a_1f_1(d_1) + a_2f_1(d_2) + \cdots + a_kf_1(d_k) = 0.$$

Since $f_1(d_2) = \cdots = f_1(d_k) = 0, f_1(d_1) \neq 0$, we find that $a_1 = 0$. Analogously, $a_2 = \cdots = a_k = 0$. Absurd.

Now, we prove that there are unique (apart from sign) non-zero integers $a_1, \dots, a_k, \gcd(a_1, \dots, a_k) = 1$ such that

$$a_1, \dots, a_k = a_1d_1 + \cdots + a_kd_k = \cdots = a_1d_1^{k-2} + \cdots + a_kd_2^{k-2} = 0$$

Assume that $a_kd_1^{k-1} + \cdots + a_kd_k^{k-1} = S(a_1, \dots, a_k)$. By the same logic we find that

$$a_1f_1(d_1) + a_2f_1(d_2) + \cdots + a_kf_1(d_k) = S(a_1, \dots, a_k).$$

Hence, $a_1f_1(d_1) = S(a_1, \dots, a_k)$. With $S(a_1, \dots, a_k)$ being a linear polynomial of $a_1, \dots, a_k$ with integer coefficients. Therefore, $a_1 = \dfrac{S(a_1, \dots, a_k)}{f_1(d_1)}, f_1(d_1) \in \mathbb{Q}$.

Next, for each choice of $S(a_1, \dots, a_k)$ we can find $a_1, \dots, a_k$. Using the same argument we find that there is a non-trivial solution $(b_1, \dots, b_k)$ with $b_1, \dots, b_k \in \mathbb{Q}$. Further, all the solutions can be obtained by multiplying this solution by $r \in mathbbQ$. We choose to multiply $(b_1, \dots, b_k)$ by the least common multiple of the denominators of $a_1, \dots, a_k$, that is, we get to the solution $(a_1, \dots, a_k)$ such that $g_1, \dots, g_k$ is integer. We can divide this by the greatest common divisor of $g_1, \dots, g_k$ to get the integer solution $a_1, \dots, a_k, \gcd(a_1, \dots, a_k) = 1$.

Finally, if for example $a_1 = 0$, we eentually arrive at the system

$$a_2 + \cdots + a_k = a_2d_2 + \cdots + a_kd_k = \cdots = a_2d_2^{k-2} + \cdots + a_kd_k^{k-2} = 0,$$

which gives the solution $a_2 = \cdots = a_k = 0$ and we are done.

Note: The same argument remains true for the field $\mathbb{Z}_p$.

U526. Evaluate

$$\int_0^2 \frac{1}{t} \arctan \frac{3t}{t^2+4}\, dt.$$

Proposed by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Note

$$\arctan t - \arctan \frac{t}{4} = \arctan \frac{t - \frac{t}{4}}{1 + \frac{t^2}{4}} = \arctan \frac{3t}{t^2+4}.$$

Therefore,

$$\int_0^2 \frac{1}{t} \arctan \frac{3t}{t^2+4}\, dt = \int_0^2 \frac{1}{t} \left(\arctan t - \arctan \frac{t}{4} \right) dt.$$

Now,

$$\int_0^2 \frac{1}{t} \arctan \frac{t}{4}\, dt = \int_0^{1/2} \frac{1}{t} \arctan t\, dt,$$

so

$$\int_0^2 \frac{1}{t} \arctan \frac{3t}{t^2+4}\, dt = \int_{1/2}^2 \frac{1}{t} \arctan t\, dt.$$

By integration by parts,

$$\begin{aligned}
\int_{1/2}^2 \frac{1}{t} \arctan t\, dt &= \ln t \arctan t \Big|_{1/2}^2 - \int_{1/2}^2 \frac{\ln t}{1+t^2}\, dt \\
&= \ln 2 \arctan 2 - \ln \frac{1}{2} \arctan \frac{1}{2} - \int_{1/2}^2 \frac{\ln t}{1+t^2}\, dt \\
&= \ln 2 \left(\arctan 2 + \arctan \frac{1}{2} \right) - \int_{1/2}^2 \frac{\ln t}{1+t^2}\, dt \\
&= \frac{\pi \ln 2}{2} - \int_{1/2}^2 \frac{\ln t}{1+t^2}\, dt.
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_{1/2}^2 \frac{\ln t}{1+t^2}\, dt &= \int_{1/2}^1 \frac{\ln t}{1+t^2}\, dt + \int_1^2 \frac{\ln t}{1+t^2}\, dt \\
&= -\int_1^2 \frac{\ln t}{1+t^2}\, dt + \int_1^2 \frac{\ln t}{1+t^2}\, dt \\
&= 0,
\end{aligned}$$

so

$$\int_0^2 \frac{1}{t} \arctan \frac{3t}{t^2+4}\, dt = \frac{\pi \ln 2}{2}.$$

Also solved by Davrbek Oltiboev, National University of Uzbekistan, Tashkent, Uzbekistan; Joe Simons Utah Valley University Orem, UT, USA; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Toyesh Prakash Sharma, St. C.F Andrews School, Agra, India.

U527. Find the smallest constant C such that

$$\sum_{k=1}^{n} \frac{k}{k^4+4} < C.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Henry Ricardo, Westchester Area Math Circle

Let $f(k) = 1/(k^2-2k+2)$, then $f(k+2) = 1/(k^2+2k+2)$, we have a telescoping series:

$$\begin{aligned}\sum_{k=1}^{n} \frac{k}{k^4+4} &= \frac{1}{4}\sum_{k=1}^{n}(f(k)-g(k)) \\ &= \frac{1}{4}\sum_{k=1}^{n}(f(k)-f(k+2)) \\ &= \frac{1}{4}\left(1+\frac{1}{2}-f(n+2)-f(n+1)\right) \\ &= \frac{3}{8} - \frac{1}{4}\left(\frac{1}{n^2+2n+2} + \frac{1}{n^2+1}\right),\end{aligned}$$

so that $\sup_{n\in\mathbb{N}}\left\{\sum_{k=1}^{n}\frac{k}{k^4+4}\right\} = \frac{3}{8}.$

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Israel Castillo Pilco, Huaral, Peru; Booyeon Brian Choi, Middlesex School, Concord, MA, USA; Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Taes Padhihary, Disha Delphi Public School, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Kelvin Kim, Bergen Catholic High School, Oradell, NJ, USA; Davrbek Oltiboev, National University of Uzbekistan, Tashkent, Uzbekistan; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Dumitru Barac, Sibiu, Romania; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Telemachus Baltsavias, Keramies Junior High School ,Kefalonia, Greece; Arkady Alt, San Jose, CA, USA.

U528. Consider the double sequence

$$s(m,n) := \frac{m-1}{n}\sum_{k=1}^{n}\frac{\left\lfloor \frac{3k}{n}(m-1)\right\rfloor + 3}{\left(\left\lfloor 1+\frac{k}{n}(m-1)\right\rfloor\right)!}.$$

Find $\lim_{n\to\infty}\lim_{m\to\infty} s(m,n)$ and $\lim_{m\to\infty}\lim_{n\to\infty} s(m,n)$.

Proposed by Besfort Shala, University of Primorska, Koper, Slovenia

Solution by the author

We will show that the first limit is zero and that the second is $4e-1$, showing that this is yet another example where limiting processes do not commute. We start off with

$$0 \le s(m,n) = \sum_{k=1}^{n}\frac{(m-1)\left(\left\lfloor \frac{3k}{n}(m-1)\right\rfloor + 3\right)}{n\left(\left\lfloor 1+\frac{k}{n}(m-1)\right\rfloor\right)!} < \sum_{k=1}^{n}\frac{(m-1)\left(\frac{3k(m-1)}{n}+3\right)}{n\Gamma\left(1+\frac{k}{n}(m-1)-1+1\right)},$$

for large enough m, where we used the inequalities $x-1 < \lfloor x\rfloor \le x$ and the fact that the Gamma function Γ is increasing for large enough arguments. Letting $m\to\infty$, we have

$$\begin{aligned} 0 \le \lim_{m\to\infty} s(m,n) &\le \lim_{m\to\infty}\sum_{k=1}^{n}\frac{(m-1)\left(\frac{3k(m-1)}{n}+3\right)}{n\Gamma\left(1+\frac{k}{n}(m-1)\right)} \\ &= \sum_{k=1}^{n}\lim_{m\to\infty}\frac{(m-1)\left(\frac{3k(m-1)}{n}+3\right)}{n\Gamma\left(1+\frac{k}{n}(m-1)\right)} \\ &= 0, \end{aligned}$$

since $\Gamma\left(1+\frac{k}{n}(m-1)\right)$ in the denominator dominates the quadratic in m in the numerator as $m\to\infty$, for any $k\in\{1,2,\dots,n\}$ (note that n is fixed here). So,

$$\lim_{n\to\infty}\lim_{m\to\infty} s(m,n) = \lim_{n\to\infty} 0 = 0.$$

On the other hand, for a fixed m, we recognize $\lim_{n\to\infty} s(m,n)$ as a limit of Riemann sums over the interval $[1,m]$ with sample points $x_k = 1+\frac{k}{n}(m-1)$ of the function $f(x) = \frac{\lfloor 3x\rfloor}{(\lfloor x\rfloor)!}$. Therefore,

$$\lim_{m\to\infty}\lim_{n\to\infty} s(m,n) = \lim_{m\to\infty}\int_{[1,m]}\frac{\lfloor 3x\rfloor}{(\lfloor x\rfloor)!}\mathrm{d}x = \int_{[1,\infty)}\frac{\lfloor 3x\rfloor}{(\lfloor x\rfloor)!}\mathrm{d}x = \sum_{k=1}^{\infty}\int_{[k,k+1]}\frac{\lfloor 3x\rfloor}{k!}\mathrm{d}x =$$

$$= \sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{i=0}^{2}\int_{k+i/3}^{k+(i+1)/3}(3k+i)\mathrm{d}x\right) = \sum_{k=1}^{\infty}\frac{1}{k!}(3k+1) = 3\sum_{k=1}^{\infty}\frac{1}{(k-1)!} + \sum_{k=1}^{\infty}\frac{1}{k!}$$

$$= 3e + (e-1) = 4e-1.$$

We are done!

Olympiad problems

O523. Let $C_1(O_1, r_1)$ and $C_2(O_2, r_2)$ be two external circles, the line O_1O_2 intersects the circumference of C_1 at a point A and $d = O_1O_2 - r_1 - r_2 \geq 0$. Prove that for any points $M \in C_1$ and $N \in C_2$ the inequality $MN \geq MA$ holds true if and only if $r_1r_2^2 \leq d(d+r_2)(d+2r_2)$.

Proposed by Oleg Muskarov, Sofia, Bulgaria

First solution by Li Zhou, Polk State College, USA

n the statement, A should be between O_1 and O_2. Let $t = \frac{1}{2}\angle O_2O_1M \in [0, \frac{\pi}{2}]$, then $MA = 2r_1 \sin t$ and

$$MO_2 = \sqrt{(d+r_2+r_1)^2 + r_1^2 - 2r_1(d+r_2+r_1)\cos 2t}.$$

Hence, $MO_2 \geq r_2 + MA$ if and only if

$$(d+r_2+r_1)^2 + r_1^2 - 2r_1(d+r_2+r_1)(1-2\sin^2 t) \geq r_2^2 + 4r_1r_2\sin t + 4r_1^2\sin^2 t,$$

which simplifies to

$$\left(\sin t - \frac{r_2}{2(d+r_2)}\right)^2 + \frac{d(d+r_2)(d+2r_2) - r_1r_2^2}{4r_1(d+r_2)^2} \geq 0.$$

Therefore, $MN \geq MA$ for all M and N, if and only if $MO_2 \geq r_2 + MA$ for all M (that is, for all t), if and only if $r_1r_2^2 \leq d(d+r_2)(d+2r_2)$.

Second solution by the author

Let $M \in C_1$ be a fixed point. Set $MA = t$ and $\angle MAO_1 = \alpha$. Then $-90° < \alpha \le 90°$, and $0 \le t \le 2r_1 \cos\alpha$. Let $N \in C_2$ be an arbitrary point and denote by M_0 the intersection point of the line MO_2 and the circumference of C_2. Then $MN \ge MM_0$ and we have to prove that $MM_0 \ge MA$ for every point $M \in C_1$ if and only if $r_1 r_2^2 \le d(d + r_2)(d + 2r_2)$. To do this we use the cosine law for triangle MAO_2 and obtain

$$MO_2 = \sqrt{t^2 + (d + r_2)^2 + 2t(d + r_2)\cos\alpha}.$$

Having in mind that $MM_0 = MO_2 - r_2$ and $MA = t$ it follows that the inequality $MM_0 \ge MA$ is equivalent to

$$\sqrt{t^2 + (d + r_2)^2 + 2t(d + r_2)\cos\alpha} \ge t + r_2.$$

Squaring both sides of this inequality we see easily that it is equivalent to the inequality

$$\frac{d^2 + 2dr_2}{d + r_2} \ge 2t\left(\frac{r_2}{d + r_2} - \cos\alpha\right) \qquad (*)$$

for all $-90° < \alpha \le 90°$ and $0 \le t \le 2r_1 \cos\alpha$. The inequality (*) is obviously true for

$$\cos\alpha \ge \frac{r_2}{d + r_2}$$

and we may assume that

$$0 \le \cos\alpha \le \frac{r_2}{d + r_2}.$$

In this case we have

$$2t\left(\frac{r_2}{d + r_2} - \cos\alpha\right) \le 4r_1 \cos\alpha\left(\frac{r_2}{d + r_2} - \cos\alpha\right) \le \frac{r_1 r_2^2}{(d + r_2)^2}.$$

This shows that (*) holds true for all $-90° < \alpha \le 90°$ and $0 \le t \le 2r_1 \cos\alpha$ if and only if

$$\frac{d^2 + 2dr_2}{d + r_2} \ge \frac{r_1 r_2^2}{(d + r_2)^2}$$

which can be written as

$$r_1 r_2^2 \le d(d + r_2)(d + 2r_2).$$

O524. Let x, y, z be positive real numbers such that $x^4 + y^4 + z^4 = 3$. Prove that

$$\sqrt{\frac{yz}{7-2x}} + \sqrt{\frac{zx}{7-2y}} + \sqrt{\frac{xy}{7-2z}} \le \frac{3\sqrt{5}}{5}.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietman

Solution by the author

From the AM-GM inequality we get

$$\begin{aligned} 7-2x = 3-2x+4 = x^4+y^4+z^4-2x+4 = \\ (x^4-2x^2+1)+(x^2-2x+1)+x^2+(y^4+z^4+1+1) = \\ (x^2-1)^2+(x-1)^2+x^2+(y^4+z^4+1+1) \ge \\ x^2+4\sqrt[4]{y^4z^4} = x^2+4yz \end{aligned}$$

$$\Rightarrow 7-2x \ge x^2+4yz \Leftrightarrow \frac{1}{7-2x} \le \frac{1}{x^2+4yz} \Leftrightarrow \frac{yz}{7-2x} \le \frac{yz}{x^2+4yz} \Leftrightarrow \sqrt{\frac{yz}{7-2x}} \le \sqrt{\frac{yz}{x^2+4yz}}.$$

Analogously for the permutations.

Hence, $P = \sqrt{\frac{yz}{7-2x}} + \sqrt{\frac{zx}{7-2y}} + \sqrt{\frac{xy}{7-2z}} \le \sqrt{\frac{yz}{x^2+4yz}} + \sqrt{\frac{zx}{y^2+4zx}} + \sqrt{\frac{xy}{z^2+4xy}}$

Then,

$$P \le \sqrt{3\left(\frac{yz}{x^2+4yz} + \frac{zx}{y^2+4zx} + \frac{xy}{z^2+4xy}\right)} \tag{1}$$

By Cauchy-Schwarz inequality we have

$$\frac{x^2}{x^2+4yz} + \frac{y^2}{y^2+4zx} + \frac{z^2}{z^2+4xy} \ge \frac{(x+y+z)^2}{x^2+4yz+y^2+4zx+z^2+4xy} = \frac{(x+y+z)^2}{(x+y+z)^2+2(xy+yz+zx)}. \tag{2}$$

Using inequality $xy+yz+zx \le \frac{(x+y+z)^2}{3}$ and (2) yields

$$\frac{x^2}{x^2+4yz} + \frac{y^2}{y^2+4zx} + \frac{z^2}{z^2+4xy} \ge \frac{(x+y+z)^2}{(x+y+z)^2+2\frac{(x+y+z)^2}{3}} = \frac{3}{5} \Leftrightarrow$$

$$\left(1-\frac{x^2}{x^2+4yz}\right) + \left(1-\frac{y^2}{y^2+4zx}\right) + \left(1-\frac{z^2}{z^2+4xy}\right) \le 3-\frac{3}{5} = \frac{12}{5} \Leftrightarrow$$

$$\frac{yz}{x^2+4yz} + \frac{zx}{y^2+4zx} + \frac{xy}{z^2+4xy} \le \frac{3}{5}. \tag{3}$$

From (1) and (3) it follows that

$$P \le \sqrt{3 \cdot \frac{3}{5}}$$

and the conclusion follows. Equality occurs if and only if $x = y = z = 1$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ioannis D. Sfikas, Athens, Greece.

O525. For any two number sets A and B define their sum as $A+B=\{a+b : a\in A \text{ and } b\in B\}$, and their difference as $A-B=\{a-b : a\in A \text{ and } b\in B\}$. Let $P=\{2,3,5,7,11,\ldots\}$ be the set of primes and $S=\{0,1,8,16,27,64,81,125,216,256,\ldots\}$ be the set of perfect cubes and fourth powers. Prove whether or not there are infinitely many positive integers not in

$$(P+S)\cup(P-S)\cup(S-P).$$

Proposed by Li Zhou, Polk State College, Winter Haven, FL

Solution by the author

Yes. Take any $a=64n^{12}$ with $n\equiv 3 \pmod{7\cdot 103}$. Assume that m is any nonnegative integer.

First, by Germain's factoring,

$$a+m^4=4\left(2n^3\right)^4+m^4=\left(8n^6+4n^3m+m^2\right)\left(4n^6+\left(2n^3-m\right)^2\right),$$

which cannot be prime. Also, if $a-m^4=\left(8n^6-m^2\right)\left(8n^6+m^2\right)$ is prime, then we must have $\left(2n^2\right)^3-m^2=1$. But it is well known that $x^3-y^2=1$ only has the solution $(x,y)=(1,0)$, an impossibility. Similarly, if $m^4-a=\left(m^2-8n^6\right)\left(m^2+8n^6\right)$ is prime, then we must have $m^2-\left(2n^2\right)^3=1$. But it is also well known that $x^2-y^3=1$ only has the solutions $(x,y)=(0,-1)$, $(\pm1,0)$ and $(\pm3,2)$, all impossibilities for our choice of n.

Next,

$$a+m^3=\left(4n^4+m\right)\left(12n^8+\left(2n^4-m\right)^2\right),$$

which cannot be prime. Now if $a-m^3=\left(4n^4-m\right)\left(16n^8+4n^4m+m^2\right)$ is prime, then we must have $4n^4-m=1$, thus

$$16n^8+4n^4m+m^2=48n^8-12n^4+1\equiv 48\cdot 3^8-12\cdot 3^4+1\equiv 0 \pmod 7,$$

which cannot be prime. Finally, if $m^3-a=\left(m-4n^4\right)\left(m^2+4mn^4+16n^8\right)$ is prime, then we must have $m-4n^4=1$, so

$$m^2+4mn^4+16n^8=48n^8+12n^4+1\equiv 48\cdot 3^8+12\cdot 3^4+1\equiv 0 \pmod{103},$$

which cannot be prime. Therefore, all such a are not in $(P+S)\cup(P-S)\cup(S-P)$.

Note: After proposing this problem I have also proved the following generalization: Let P be the set of primes and for any $k\ge 3$, let $S_k=\{m^i : m\ge 0 \text{ and } 3\le i\le k\}$. There are infinitely many positive integers not in $(P-S_k)\cup(P+S_k)\cup(S_k-P)$.

O526. Let x, y, z be nonnegative real numbers such that $xy + yz + zx = 3$. Prove that

$$(x^2 + y^2 + z^2 + 1)^3 \geq (x^3 + y^3 + z^3 + 5xyz)^2.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Todor Zaharinov, Sofia, Bulgaria

The inequality can be written as

$$A = \left(\sum_{cyc} x^2 + 1\right)^3 - \left(\sum_{cyc} x^3 + 5xyz\right)^2 \geq 0$$

Let us denote:

$$p = x + y + z \geq 0$$
$$q = xy + yz + zx = 3$$
$$r = xyz \geq 0$$

Then we have:

$$\sum_{cyc} x^2 = p^2 - 2q$$
$$\sum_{cyc} x^3 = p^3 - 3pq + 3r$$

We can rewrite the inequality as follows

$$A = \left(p^2 - 2q + 1\right)^3 - \left(p^3 - 3pq + 3r + 5r\right)^2 \geq 0$$
$$A = (p^2 - 5)^3 - (p^3 - 9p + 8r)^2 =$$
$$= -125 - 6p^2 + 3p^4 - 16(p-3)p(3+p)r - 64r^2$$

Clearly

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0$$
$$\Leftrightarrow \quad 2\sum_{cyc} x^2 - 2\sum_{cyc} xy \geq 0$$
$$\Leftrightarrow \quad 2(p^2 - 2q) - 2q \geq 0 \quad \Leftrightarrow \quad 2p^2 \geq 6q = 18$$
$$p \geq 3$$

(equality occurs iff $x = y = z$).

Since AM-GM inequality we have

$$q = xy + yz + zx \geq 3\sqrt[3]{x^2y^2z^2} = 3\sqrt[3]{r^2}$$
$$\Leftrightarrow \quad r^2 \leq \frac{q^3}{27} = 1$$
$$0 \leq r \leq 1$$

(equality $r = 1$ occurs iff $x = y = z$).

Since $0 \le r \le 1$ and $p \ge 3$ it follows that:

$$\begin{aligned} A &= -125 - 6p^2 + 3p^4 - 16(p-3)p(3+p)r - 64r^2 \ge \\ &\ge -125 - 6p^2 + 3p^4 - 16(p-3)p(3+p).1 - 64.1 = \\ &= -189 + 144p - 6p^2 - 16p^3 + 3p^4 = \\ &= (p-3)^2(3p^2 + 2p - 21) \ge 0 \end{aligned}$$

as required.

Equality occurs if and only if $p = 3, q = 3, r = 1$, and then $x = y = z = 1$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Dumitru Barac, Sibiu, Romania; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

O527. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonconstant function such that

$$max\{f(x+y),f(x-y)\}=f(x)f(y)$$

for all $x,y\in\mathbb{R}$. Prove that $f(x)\geq 1$ for all $x\in\mathbb{R}$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Plugging $x=y=0$ yields to $f(0)\in\{0,1\}$. If $f(0)=0$ then $f(x)=f(x)f(0)=0$. Impossible. Therefore, $f(0)=1$. Putting $x=0$ we find that $max\{f(y),f(-y)\}=f(y)$ resulting into $f(y)=f(-y)$. Setting $x=y$ we find that $max\{f(2x),1\}=f(x)^2$. Setting $y=2x$ gives us $max\{f(3x),f(x)\}=f(x)f(2x)$. Finally, if $y=3x$, $max\{f(4x),f(2x)\}=f(x)f(3x)$.

Now, we find that $f(x)^2\geq 1$, that is, $|f(x)|\geq 1$. If $f(2x)<0$ then $f(x)=\pm 1$, hence, $f(x)f(2x)\neq f(x)$, that is, $f(3x)=f(x)f(2x)>f(x)$ and $f(x)=-1,f(3x)=-f(2x)>0$, $f(4x)\leq f(x)f(3x)=-f(3x)<0$. In conclusion, of $f(2x)<0$, then

$$f(x)=-1,f(3x)>0,f(2x)<0.$$

Now,let us prove the following lemma.

Lemma: $f(x)$ and $f(2x)$ have the same sign. *Proof:* If $f(x)<0$, then setting $x=2z$ for some real z, gives us $f(2z)<0,f(2x)=f(4z)<0$. If $f(x)>0$ and $f(2x)<0$, then $f(x)=-1$. Absurd. This completes the proof.

Now, assume that $f(x)<0$ for some x. Then $f(2x)<0$ and $f(3x)>0$. On the other hand, if $f(x)>0$ for some x, then $f(2x)>0$ and $f(4x)>0$. Since $max\{f(4x),f(2x)\}=f(x)f(3x)$, we find that $f(x)f(3x)>0$. In either case $f(3x)>0$. Assume that $f(x)<0$ for some x. Setting $x=3t$ we get $f(3t)<0$, which contradicts out observation. Hence, from $|f(x)|\geq 1$ we derive that $f(x)\geq 1$ for all x.

Note: By setting $g(x)=\log f(x)$ we find that

$$max\{g(x+y),g(x-y)\}=g(x)+g(y).$$

Thus, we can prove that $g(x)=|A(x)|$ for some additive function $A(x)$.

Also solved by Ioannis D. Sfikas, Athens, Greece.

O528. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) \mid f(m)$ and

$$f(mn)f(\gcd(m,n)) = \gcd(m,n)f(m)f(n),$$

for all $m, n \in \mathbb{N}$.

Proposed by Besfort Shala, University of Primorska, Koper, Slovenia

First solution by the author

Assume first that $\gcd(m,n) = 1$. Plugging in such m, n into the functional equation, we obtain

$$f(mn) = \frac{f(m)f(n)}{f(1)}$$

which can be rewritten as

$$\frac{f(mn)}{f(1)} = \frac{f(m)}{f(1)} \cdot \frac{f(n)}{f(1)},$$

meaning that the function $g : \mathbb{N} \to \mathbb{N}$ defined by $f(1)g(m) = f(m)$ is multiplicative. It is easy to check that g also satisfies the given functional equation. So, we will first find all multiplicative solutions. Note that $g(1) = 1$ and we only need to define g at points p^k for p a prime number and $k \in \mathbb{N}$. However, letting $m = n = p$, we obtain

$$g(p^2)g(p) = pg(p)^2 \implies g(p^2) = pg(p).$$

Plugging in $m = p^2, n = p$, we obtain

$$g(p^3)g(p) = pg(p^2)g(p) = p^2g(p)^2 \implies g(p^3) = p^2g(p).$$

By induction, $g(p^k) = p^{k-1}g(p)$ for all primes p and $k \in \mathbb{N}$. This further simplifies our problem as it is now enough to define $g(p)$ only at primes p. We will show that, in fact, any positive integer value can be assigned to $g(p)$. For each prime p, let $g(p) = a_p \in \mathbb{N}$ for some arbitrary natural number a_p. Let m, n be arbitrary natural numbers, both greater than 1 (if $m = 1$ or $n = 1$, there is nothing to check) with prime divisors in the set $\{p_1, p_2, \dots, p_l\}$ so that we can write $m = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_l^{\alpha_l}$ and $n = p_1^{\beta_1}p_2^{\beta_2}\cdots p_l^{\beta_l}$ with $\alpha_i, \beta_i \geq 0$ but $\alpha_i + \beta_i > 0$ for all i. Plugging in m, n into the functional equation and denoting $\min\{\alpha_i, \beta_i\}$ by m_i gives (note that $\gcd(m,n) = p_1^{m_1}p_2^{m_2}\cdots p_l^{m_l}$)

$$\left(\prod_{i=1}^{l} p_i^{\alpha_i+\beta_i-1}a_{p_i}\right)\left(\prod_{m_i\neq 0} p_i^{m_i-1}a_{p_i}\right) = \left(\prod_{\alpha_i\neq 0} p_i^{\alpha_i-1}a_{p_i}\right)\left(\prod_{\beta_i\neq 0} p_i^{\beta_i-1}a_{p_i}\right)\left(\prod_{m_i\neq 0} p_i^{m_i}\right).$$

Firstly, note that $p_i^{m_i-1}$ cancels with $p_i^{m_i}$ when $m_i \neq 0$, leaving $\prod p_i$ on the right hand side. Then, for a fixed prime p_i, a_{p_i} appears exactly once or twice in the left hand side, appearing twice if and only if p_i divides both m, n. The same holds for the right hand side, hence all a_{p_i}'s cancel. Finally, for p_i which divide both m, n (consider the following arguments after the first cancellation), $v_{p_i}(LHS) = \alpha_i + \beta_i - 1 = (\alpha_i - 1) + (\beta_i - 1) + 1 = v_{p_i}(RHS)$ whereas for p_i that only divides m (note that this means $\beta_i = 0$ hence also $m_i = 0$), $v_{p_i}(LHS) = \alpha_i - 1 = v_{p_i}(RHS)$. Similarly for p_i which only divides n. We conclude that everything cancels out, hence the given functional equation is satisfied for such $g(m)$. As $f(m) = cg(m)$ also satisfies the functional equation for any $c \in \mathbb{N}$, we conclude that all solutions are of this form, where $g(m)$ is a multiplicative function with $g(p^k) = p^{k-1}g(p)$ at prime powers p^k and $g(p)$ being an arbitrary natural number.

Second solution by Joel Schlosberg, Bayside, NY, USA

We prove by induction on n that

$$f(n) = nf(1) \prod_{\text{prime } p|n} \frac{f(p)}{pf(1)}.$$

If n is 1 or prime, the right-hand side trivially reduces to $f(n)$.

If composite n has a prime divisor q for which $q^2 \mid n$, by induction

$$\begin{aligned} f(n) &= \frac{\gcd(q, n/q) f(q) f(n/q)}{f(\gcd(q, n/q))} = qf(n/q) \\ &= q \cdot \frac{n}{q} f(1) \prod_{\text{prime } p|(n/q)} \frac{f(p)}{pf(1)} = nf(1) \prod_{\text{prime } p|n} \frac{f(p)}{pf(1)}. \end{aligned}$$

If composite n has a prime divisor q for which $q^2 \nmid n$, by induction

$$\begin{aligned} f(n) &= \frac{\gcd(q, n/q) f(q) f(n/q)}{f(\gcd(q, n/q))} = \frac{f(q)}{f(1)} f(n/q) \\ &= \frac{f(q)}{f(1)} \cdot \frac{n}{q} \cdot f(1) \prod_{\text{prime } p|(n/q)} \frac{f(p)}{pf(1)} \\ &= nf(1) \cdot \frac{f(q)}{qf(1)} \prod_{\substack{\text{prime } p \neq q \\ p|n}} \frac{f(p)}{pf(1)} \\ &= nf(1) \prod_{\text{prime } p|n} \frac{f(p)}{pf(1)}. \end{aligned}$$

On the other hand, as long as $f(1), \frac{f(p)}{f(1)} \in \mathbb{N}$ for all primes p,

$$f(n) = nf(1) \prod_{\text{prime } p|n} \frac{f(p)}{pf(1)}$$

defines a function satisfying the given conditions, since by unique factorization

$$\frac{m}{f(1)} = \frac{m}{\prod_{\text{prime } p|m} p} \cdot \prod_{\text{prime } p|m} \frac{f(p)}{f(1)}$$

is the product of positive integers and so is a positive integer, and

$$\begin{aligned} f(mn) &= \frac{\gcd(m, n) mf(1) \prod_{\text{prime } p|m} \frac{f(p)}{pf(1)} \cdot nf(1) \prod_{\text{prime } p|n} \frac{f(p)}{pf(1)}}{\gcd(m, n) f(1) \prod_{\substack{\text{prime } p \\ p|m \text{ and } p|n}} \frac{f(p)}{pf(1)}} \\ &= mnf(1) \prod_{\substack{\text{prime } p \\ p|m \text{ and } p\nmid n}} \frac{f(p)}{pf(1)} \prod_{\text{prime } p|n} \frac{f(p)}{pf(1)} \\ &= mnf(1) \prod_{\text{prime } p|mn} \frac{f(p)}{pf(1)}. \end{aligned}$$

Also solved by Rathindra Nath Karmakar, Indian Statistical Institute, Bangalore, India; Taes Padhihary, Disha Delphi Public School, India.