Counting with the Aid of Cyclotomic Polynomials

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Abstract

The purpose of this article is to present a solution to a counting problem, which is created by the author, in case *n* is power of a prime number. The problem then remains open to be solved for positive integer values of *n* with at least two different prime factors. Since cyclotomic polynomials will be used as generating functions, the way of solution resembles the problem of *Sicherman Dice.*

Introduction of the Problem

In how many ways can the numbers from 1 to n^2 be written onto the cells of an n x n chess board such that the numbers written on any n cells, any two of which are neither in the same row nor in the same column, sum up to a constant (the value of which we will actually observe to be $\frac{1+2+...+n^2}{n}=\frac{n\left(n^2+1\right)}{2}$ towards the end of the paper)?

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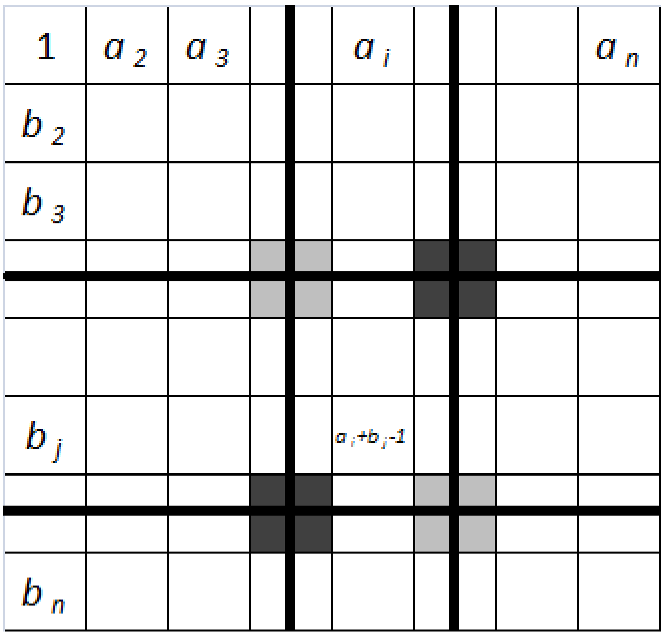
Partial Solution



Due to the fact that the same $n-2$ numbers can be chosen from other rows and columns such that no two numbers are in the same row or in the same column, the condition in the problem is equivalent to the following: For any rectangle formed by the intersection of two rows and two columns, the numbers on each pair of opposite corners have the same sum*.

When the entries of two rows or two columns are interchanged, the chess board keeps fulfilling the required condition. Hence, we may assume without loss of generality that 1 is located on the upper – left corner of the chess board. In addition, the first row consists of numbers $1 = a_1 < a_2 < a_3 < ... < a_n$ and the first column consists of numbers $1 = b_1 < b_2 < b_3 < ... < b_n$ in that order. The outcome will be multiplied by $(n!)^2$ in the end that is the number of permutations of the rows and columns.

The entries on the other cells of the chess board are now uniquely determined: Owing to *, the number $a_i + b_j - 1$ should be written in the intersection of the *i.* column and the *j.* row. The main condition left to account for is getting numbers from 1 to n^2 exactly once.

Define the polynomials $f(x) = 1 + x^{a_2-1} + x^{a_3-1} + ... + x^{a_n-1}$ and $g(x) = 1 + x^{b_2-1} + x^{b_3-1} + ... + x^{b_n-1}$

$$f(x) \cdot g(x) = \sum_{1 \le i,j \le n} x^{a_i+b_j-2} = x^0 + x^1 + x^2 + ... + x^{n^2-1} = \frac{x^{n^2}-1}{x-1}$$

Therefore, the number of such chess boards is equal to the number of pairs of polynomials (f, g) satisfying the following conditions:

(i) All of the coefficients of *f* and *g* are 1.

(ii) $f(1) = g(1) = n$

(iii) $f(x) \cdot g(x) = \dfrac{x^{n^2}-1}{x-1}$

(iv) *f* and *g* do not contain terms of same degree, excluding 1.

Suppose now *n* is power of a prime number, $n = p^\alpha$.

$$f(x) \cdot g(x) = \frac{x^{p^{2\alpha}}-1}{x-1} = \frac{x^{p^{2\alpha}}-1}{x^{p^{2\alpha-1}}-1} \cdot \frac{x^{p^{2\alpha-1}}-1}{x^{p^{2\alpha-2}}-1} \cdots \frac{x^{p^2}-1}{x^p-1} \cdot \frac{x^p-1}{x-1}$$

$\dfrac{x^{p^k}-1}{x^{p^{k-1}}-1} = \Phi_{p^k}(x)$ is the p^k. cyclotomic polynomial, thus irreducible.

$$f(x) \cdot g(x) = \underbrace{\left(1 + x + x^2 + ... + x^{p-1}\right)}_{\Phi_p(x)} \underbrace{\left(1 + x^p + x^{2p} + ... + x^{(p-1)p}\right)}_{\Phi_{p^2}(x)} \cdots \underbrace{\left(1 + x^{p^{2\alpha-1}} + x^{2p^{2\alpha-1}} + ... + x^{(p-1)p^{2\alpha-1}}\right)}_{\Phi_{p^{2\alpha}}(x)}$$

is the product of 2α irreducible polynomials. $\Phi_{p^k}(1) = p$, for all $1 \le k \le 2\alpha$. In order that the condition (ii) $f(1) = g(1) = p^\alpha$ is satisfied, *f* should be the product of α of those cyclotomic polynomials and *g* should be the product of other α of those cyclotomic polynomials.

It suffices to prove that the conditions (i) and (iv) also hold in this case. There are two different approaches to do this, one using modular arithmetics and one using simple inequalities.
First of all, assume that $f(x) = \prod_{i \in S} \Phi_{p^i}(x)$, where *S* is a subset of $\{1, 2, 3, ..., 2\alpha\}$ containing exactly α elements.

First Approach: Using Modular Arithmetics

(i) Let us show by induction on *d* that the coefficients of the polynomial obtained by multiplying $1 \le d \le \alpha$ of those α cyclotomic polynomials are all 1. This is obviously true when $d = 1$. Suppose the statement holds for any d cyclotomic polynomials. It is enough to prove that the statement holds for any $d+1$ cyclotomic polynomials as well. Let $\Phi_{p^m}(x)$ be the cyclotomic polynomial of the smallest degree under those $d+1$ cyclotomic polynomials. Due to the induction statement, the product of other *d* cyclotomic polynomials is a polynomial with all coefficients 1. Moreover, according to the definition of *m*, the exponents of all of its terms are divisible by p^{m+1}. Now, suppose to the contrary that the product

$$\left(1 + x^{p^m} + x^{2p^m} + \ldots + x^{(p-1)p^m}\right)\left(1 + \sum x^{\theta}\right)$$

includes a term with a coefficient at least 2. Then, the same exponential number appears at least twice while expanding, which means

$$i \cdot p^m + \theta_1 = j \cdot p^m + \theta_2$$

where $0 \le i \ne j \le p-1$ and $\theta_1 \ne \theta_2$ are positive integers divisible by p^{m+1}.

$$\theta_1 - \theta_2 = (j-i)\,p^m$$

$p^{m+1} \mid \theta_1 - \theta_2 \Rightarrow \; p^{m+1} \mid (j-i)\,p^m \Rightarrow \; p \mid j-i$, but $1 \le |j-i| \le p-1$, which is a contradiction.

(iv) Suppose to the contrary that the polynomials *f* and *g* have a term of same degree in common, excluding 1. One of the polynomials *f* and *g* does not contain the cyclotomic polynomial $\Phi_p(x)$ and the exponents of the terms of this polynomial are all divisible by p^2. Therefore, the term 1 should be picked from the cyclotomic polynomial $\Phi_p(x)$ to obtain this common exponent divisible by p^2 in the other polynomial too. Considering in modulo p^3 in the same way, the term 1 should be picked from the cyclotomic polynomial $\Phi_{p^2}(x)$ as well. Proceeding similarly, the term 1 should be picked from all cyclotomic polynomials $\Phi_p(x), \Phi_{p^2}(x), \ldots, \Phi_{p^{2\alpha}}(x)$. Consequently, the only common term of the polynomials *f* and *g* is 1, which is a contradiction.

Second Approach: Using Simple Inequalities

Lemma: Note that the following inequality holds for all positive integers β:

$$(p-1) + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^{\beta-1} < p^{\beta}$$

which directly follows from

$$\begin{aligned}(p-1) + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^{\beta-1} &= (p-1)\left(1 + p + p^2 + \ldots + p^{\beta-1}\right)\\ &= (p-1) \cdot \frac{p^{\beta}-1}{p-1}\\ &= p^{\beta} - 1 < p^{\beta}.\end{aligned}$$

(i) Suppose to the contrary that the coefficient of one of the terms of the polynomial *f* is at least 2 (the coefficients of the polynomial *f* are obviously positive). In other words, the product of two different α–tuples, obtained by multiplying one term from each cyclotomic polynomial $\Phi_{p^i}(x)$, where $i \in S$, provides the same exponent. Let *r* be the maximal number in S such that different terms are picked from the cyclotomic polynomial $\Phi_{p^r}(x)$. The difference between the exponents of those terms are at least p^r. In order to obtain the same exponent in the end, this gap should be closed, which is however not possible according to the lemma, even if the smallest term 1 is picked from all other cyclotomic polynomials to be multiplied with the greater term and the greatest term $x^{(p-1)p^i}$ $(1 \le i \le r-1)$ is picked from all other cyclotomic polynomials to be multiplied with the smaller term.

(iv) Suppose to the contrary that the polynomials *f* and *g* have a term of same degree in common, excluding 1. The term 1 should be chosen from the cyclotomic polynomial $\Phi_{p^{2\alpha}}(x)$ to obtain such a common term (here it does not matter which polynomial the cyclotomic polynomial $\Phi_{p^{2\alpha}}(x)$ belongs to). Otherwise, according to the lemma, it would not be possible to close the gap between the exponents, which is at least $p^{2\alpha}$ at this point, even if the greatest terms $x^{(p-1)p^i}$ $(1 \le i \le 2\alpha-1)$ are chosen from all cyclotomic polynomials of the other polynomial. In the same way, the term 1 should be chosen from $\Phi_{p^{2\alpha-1}}(x)$ as well. Proceeding similarly, the term 1 should be chosen from all cyclotomic polynomials $\Phi_{p^{2\alpha}}(x), \Phi_{p^{2\alpha-1}}(x), \ldots, \Phi_p(x)$. Consequently, the only common term of polynomials *f* and *g* is 1, which is a contradiction.

Finally, α of 2α cyclotomic polynomials can be chosen in $C(2\alpha, \alpha)$ different ways to determine the polynomial *f*. The polynomial *g* is then determined automatically through the product of the other unchosen α cyclotomic polynomials. Hence, the answer is $\frac{(2\alpha)!}{(\alpha!)^2} \cdot \left((p^\alpha)!\right)^2$ for $n = p^\alpha$.

The Case *n* = 6

The number of solutions in case $n=6$ that is the smallest positive integer with two different prime factors is additionally counted below through case work:

$$f(x)\cdot g(x)=\frac{x^{36}-1}{x-1}$$
$$=(x+1)(x^2+1)(x^2+x+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$$

is the product of 8 irreducible polynomials, which are the 2nd, 4th, 3rd, 9th, 6th, 12th, 18th and 36th cyclotomic polynomials, respectively.

$\Phi(x)$	$x+1$	x^2+1	x^2+x+1	x^6+x^3+1	x^2-x+1	x^4-x^2+1	x^6-x^3+1	$x^{12}-x^6+1$
$\Phi(1)$	2	2	3	3	1	1	1	1

In order that the condition (ii) $f(1)=g(1)=6$ is satisfied, the polynomial *f* (and likewise *g*) contains exactly one of the cyclotomic polynomials $x+1$ and x^2+1, exactly one of the cyclotomic polynomials x^2+x+1 and x^6+x^3+1 and maybe some of the cyclotomic polynomials x^2-x+1, x^4-x^2+1, x^6-x^3+1 and $x^{12}-x^6+1$. Accordingly, we need to investigate the accuracy of the conditions (i) and (iv) in the following 32 cases:

1) $f(x)=(x+1)(x^2+x+1)=x^3+2x^2+2x+1$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$.
The polynomial *f* does not fulfil the condition (i).

2) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)=x^5+x^4+x^3+x^2+x+1$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)=x^{30}+x^{24}+x^{18}+x^{12}+x^6+1$
Both conditions are satisfied. <u>Two solutions</u> appear in this case, due to the symmetry between *f* and *g*.

3) $f(x)=(x+1)(x^2+x+1)(x^4-x^2+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)(x^6-x^3+1)(x^{12}-x^6+1)$.
Because the coefficient of the term x of the polynomial *f* is 2, the condition (i) is not satisfied.

4) $f(x)=(x+1)(x^2+x+1)(x^6-x^3+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)(x^{12}-x^6+1)$.
Because the coefficient of the term x of the polynomial *f* is 2, the condition (i) is not satisfied.

5) $f(x)=(x+1)(x^2+x+1)(x^{12}-x^6+1)$ and
$f(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)$.
Because the coefficient of the term x of the polynomial *f* is 2, the condition (i) is not satisfied.

6) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)=x^9+x^8+x^5+x^4+x+1$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^6-x^3+1)(x^{12}-x^6+1)=x^{26}+x^{24}+x^{14}+x^{12}+x^2+1$.
Both conditions are satisfied. Two solutions appear in this case, due to the symmetry between *f* and *g*.

7) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)(x^6-x^3+1)=x^{11}+x^{10}+x^9+x^2+x+1$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^4-x^2+1)(x^{12}-x^6+1)=x^{24}+x^{21}+x^{18}+x^6+x^3+1$.
Both conditions are satisfied. Two solutions appear in this case, due to the symmetry between *f* and *g*.

8) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)(x^{12}-x^6+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^4-x^2+1)(x^6-x^3+1)=x^{18}+2x^{12}+2x^6+1$.
The polynomial *g* does not fulfil the condition (i).

9) $f(x)=(x+1)(x^2+x+1)(x^4-x^2+1)(x^6-x^3+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)(x^{12}-x^6+1)$.
Because the coefficient of the term x of the polynomial *f* is 2, the condition (i) is not satisfied.

10) $f(x)=(x+1)(x^2+x+1)(x^4-x^2+1)(x^{12}-x^6+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)(x^6-x^3+1)$.
Because the coefficient of the term x of the polynomial *f* is 2, the condition (i) is not satisfied.

11) $f(x)=(x+1)(x^2+x+1)(x^6-x^3+1)(x^{12}-x^6+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)$.
Because the coefficient of the term x of the polynomial *f* is 2, the condition (i) is not satisfied.

12) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^{12}-x^6+1)=x^{20}+x^{18}+x^{17}+x^{15}-x^{11}-x^9+x^5+x^3+x^2+1$.
The polynomial *g* does not fulfil the condition (i).

13) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^{12}-x^6+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^6-x^3+1)$.
Because the coefficient of the term x^6 of the polynomial *f* is –1, the condition (i) is not satisfied.

14) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)(x^6-x^3+1)(x^{12}-x^6+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^4-x^2+1)=x^{12}+x^9+2x^6+x^3+1$.
The polynomial *g* does not fulfil the condition (i).

15) $f(x)=(x+1)(x^2+x+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)(x^2-x+1)$.
Because the coefficient of the term x of the polynomial *g* is –1, the condition (i) is not satisfied.

16) $f(x)=(x+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$ and
$g(x)=(x^2+1)(x^6+x^3+1)$.

Because the coefficient of the term x^3 of the polynomial *f* is –1, the condition (i) is not satisfied.

17) $f(x)=(x^2+1)(x^2+x+1)=x^4+x^3+2x^2+x+1$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$.

The polynomial *f* does not fulfil the condition (i).

18) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)=x^6+2x^4+2x^2+1$ and
$g(x)=(x+1)(x^6+x^3+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$.

The polynomial *f* does not fulfil the condition (i).

19) $f(x)=(x^2+1)(x^2+x+1)(x^4-x^2+1)=x^8+x^7+x^6+x^2+x+1$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)(x^6-x^3+1)(x^{12}-x^6+1)$
$=x^{27}+x^{24}+x^{15}+x^{12}+x^3+1$.

Both conditions are satisfied. Two solutions appear in this case, due to the symmetry between *f* and *g*.

20) $f(x)=(x^2+1)(x^2+x+1)(x^6-x^3+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)(x^{12}-x^6+1)$.

Because the coefficient of the term x^2 of the polynomial *f* is 2, the condition (i) is not satisfied.

21) $f(x)=(x^2+1)(x^2+x+1)(x^{12}-x^6+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)$.

Because the coefficient of the term x^2 of the polynomial *f* is 2, the condition (i) is not satisfied.

22) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)=x^{10}+x^8+x^6+x^4+x^2+1$ and
$g(x)=(x+1)(x^6+x^3+1)(x^6-x^3+1)(x^{12}-x^6+1)=x^{25}+x^{24}+x^{13}+x^{12}+x+1$.

Both conditions are satisfied. Two solutions appear in this case, due to the symmetry between *f* and *g*.

23) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)(x^6-x^3+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^4-x^2+1)(x^{12}-x^6+1)$.

Because the coefficient of the term x^2 of the polynomial *f* is 2, the condition (i) is not satisfied.

24) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)(x^{12}-x^6+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^4-x^2+1)(x^6-x^3+1)$.

Because the coefficient of the term x^2 of the polynomial *f* is 2, the condition (i) is not satisfied.

25) $f(x)=(x^2+1)(x^2+x+1)(x^4-x^2+1)(x^6-x^3+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)(x^{12}-x^6+1)$.

Because the coefficient of the term x^3 of the polynomial *g* is 2, the condition (i) is not satisfied.

26) $f(x)=(x^2+1)(x^2+x+1)(x^4-x^2+1)(x^{12}-x^6+1)=x^{20}+x^{19}+x^{18}+x^2+x+1$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)(x^6-x^3+1)=x^{15}+x^{12}+x^9+x^6+x^3+1$.

Both conditions are satisfied. Two solutions appear in this case, due to the symmetry between *f* and *g*.

27) $f(x)=(x^2+1)(x^2+x+1)(x^6-x^3+1)(x^{12}-x^6+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)(x^4-x^2+1)$.

Because the coefficient of the term x^2 of the polynomial *f* is 2, the condition (i) is not satisfied.

28) $f(x)=(x^2+1)(x^2+x+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^2-x+1)=x^9+2x^6+2x^3+1$.

The polynomial *g* does not fulfil the condition (i).

29) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)(x^6-x^3+1)(x^{12}-x^6+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^4-x^2+1)$.

Because the coefficient of the term x^2 of the polynomial *g* is –1, the condition (i) is not satisfied.

30) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^{12}-x^6+1)$
$=x^{22}+x^{20}+x^{18}+x^4+x^2+1$ and
$g(x)=(x+1)(x^6+x^3+1)(x^6-x^3+1)=x^{13}+x^{12}+x^7+x^6+x+1$.

Both conditions are satisfied. Two solutions appear in this case, due to the symmetry between *f* and *g*.

31) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)$ and
$g(x)=(x+1)(x^6+x^3+1)(x^{12}-x^6+1)=x^{19}+x^{18}+x^{16}+x^{15}-x^{10}-x^9+x^4+x^3+x+1$.

The polynomial *g* does not fulfil the condition (i).

32) $f(x)=(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)(x^6-x^3+1)(x^{12}-x^6+1)$
$g(x)=(x+1)(x^6+x^3+1)$.

Because the coefficient of the term x^3 of the polynomial *f* is –1, the condition (i) is not satisfied.

There exist 14 pairs of solutions in total. Hence, the answer is $14\cdot(6!)^2=7\ 257\ 600$ for $n=6$. These chess boards can be obtained by interchanging the entries of the first row and the first column and / or by changing the places of any two rows or any two columns of the following chess boards (possibly several times):

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1	2	3	7	8	9
4	5	6	10	11	12
13	14	15	19	20	21
16	17	18	22	23	24
25	26	27	31	32	33
28	29	30	34	35	36

1	2	3	10	11	12
4	5	6	13	14	15
7	8	9	16	17	18
19	20	21	28	29	30
22	23	24	31	32	33
25	26	27	34	35	36

1	2	3	19	20	21
4	5	6	22	23	24
7	8	9	25	26	27
10	11	12	28	29	30
13	14	15	31	32	33
16	17	18	34	35	36

1	3	5	7	9	11
2	4	6	8	10	12
13	15	17	19	21	23
14	16	18	20	22	24
25	27	29	31	33	35
26	28	30	32	34	36

1	3	5	19	21	23
2	4	6	20	22	24
7	9	11	25	27	29
8	10	12	26	28	30
13	15	17	31	33	35
14	16	18	32	34	36

1	2	5	6	9	10
3	4	7	8	11	12
13	14	17	18	21	22
15	16	19	20	23	24
25	26	29	30	33	34
27	28	31	32	35	36

Some Further Results

Assuming the first row consists of numbers $1 = a_1 < a_2 < a_3 < ... < a_n$ and the first column consists of numbers $1 = b_1 < b_2 < b_3 < ... < b_n$, we will prove the following equations:

$$\sum_{1 \le i \le n} a_i + \sum_{1 \le j \le n} b_j = \frac{n^3 + 3n}{2}$$

$$n\left(\sum_{1 \le i \le n} a_i^2 + \sum_{1 \le j \le n} b_j^2\right) - \left(\sum_{1 \le i \le n} a_i\right)^2 - \left(\sum_{1 \le j \le n} b_j\right)^2 = \frac{n^6 - n^2}{12}.$$

Note that $\{a_i + b_j - 1 : 1 \le i \le n, 1 \le j \le n\} = \{1, 2, ..., n^2\}$ or equivalently that $\{a_i + b_j : 1 \le i \le n, 1 \le j \le n\} = \{2, 3, ..., n^2 + 1\}$. To deduce the first equation, observe

$$n\left(\sum_{1 \le i \le n} a_i + \sum_{1 \le j \le n} b_j\right) = \sum_{1 \le i \le n, 1 \le j \le n} \left(a_i + b_j\right) = 2 + 3 + ... + \left(n^2 + 1\right) = \frac{\left(n^2 + 1\right)\left(n^2 + 2\right)}{2} - 1 = \frac{n^4 + 3n^2}{2} \Rightarrow$$

$$\sum_{1 \le i \le n} a_i + \sum_{1 \le j \le n} b_j = \frac{n^3 + 3n}{2}.$$

To deduce the second equation, similarly observe

$$\sum_{1 \le i \le n, 1 \le j \le n} \left(a_i + b_j\right)^2 = 2^2 + 3^2 + ... + \left(n^2 + 1\right)^2 = \frac{\left(n^2 + 1\right)\left(n^2 + 2\right)\left(2n^2 + 3\right)}{6} - 1 = \frac{2n^6 + 9n^4 + 13n^2}{6} \Rightarrow$$

$$n\sum_{1 \le i \le n} a_i^2 + n\sum_{1 \le j \le n} b_j^2 + 2\sum_{1 \le i \le n, 1 \le j \le n} a_i b_j = \frac{2n^6 + 9n^4 + 13n^2}{6}.$$

By the first equation, we have

$$2\sum_{1 \le i \le n, 1 \le j \le n} a_i b_j = \left(\sum_{1 \le i \le n} a_i + \sum_{1 \le j \le n} b_j\right)^2 - \left(\sum_{1 \le i \le n} a_i\right)^2 - \left(\sum_{1 \le j \le n} b_j\right)^2 = \left(\frac{n^3 + 3n}{2}\right)^2 - \left(\sum_{1 \le i \le n} a_i\right)^2 - \left(\sum_{1 \le j \le n} b_j\right)^2$$

$$= \frac{n^6 + 6n^4 + 9n^2}{4} - \left(\sum_{1 \le i \le n} a_i\right)^2 - \left(\sum_{1 \le j \le n} b_j\right)^2$$

Inserting this into the above yields

$$n\sum_{1 \le i \le n} a_i^2 + n\sum_{1 \le j \le n} b_j^2 + \frac{n^6 + 6n^4 + 9n^2}{4} - \left(\sum_{1 \le i \le n} a_i\right)^2 - \left(\sum_{1 \le j \le n} b_j\right)^2 = \frac{2n^6 + 9n^4 + 13n^2}{6} \Rightarrow$$

$$n\left(\sum_{1 \le i \le n} a_i^2 + \sum_{1 \le j \le n} b_j^2\right) - \left(\sum_{1 \le i \le n} a_i\right)^2 - \left(\sum_{1 \le j \le n} b_j\right)^2 = \frac{2n^6 + 9n^4 + 13n^2}{6} - \frac{n^6 + 6n^4 + 9n^2}{4} = \frac{n^6 - n^2}{12}$$

as desired. As a result, we will note that the sum of numbers written on any *n* cells any two of which are neither in the same row nor in the same column is equal to $\sum_{1 \le k \le n} \left(a_{i_k} + b_{j_k} - 1\right)$ where $i_1, i_2, ..., i_n$ and $j_1, j_2, ..., j_n$ are permutations of $1, 2, ..., n$ which from the first equation is equal to $\sum_{1 \le i \le n} a_i + \sum_{1 \le j \le n} b_j - n = \frac{n^3 + 3n}{2} - n = \frac{n\left(n^2 + 1\right)}{2}$.

The Case *n* = 10

The boards given above in case *n* = 6 make it reasonable to claim that any two numbers which are symmetric with respect to the centre of the board sum up to n^2+1. Moreover, the sum of any two numbers on the first row which are symmetric with respect to the vertical line passing through the middle of the board is allegedly constant and similarly for the first column. The following computer program written in Python computes the number of chess boards satisfying the required condition and prints them for *n* = 10, taking only the last claim made above for granted among those that remained unproved in this paper:

```
import numpy as np
import time
import pdb

def getAllFirstRows(numbers, count):
  if count == 2:
    return [[1,2]]
  elif count == len(numbers):
    return [numbers]

  combinations = getAllFirstRows(numbers[:-1],
count)
  for new_combination in
getAllFirstRows(numbers[:-1], count-1):
    new_combination.append(numbers[-1])
    combinations.append(new_combination)

  return combinations

def getBoard(rowHalf, n):
  # get the sum of pairs in the row using the last
two numbers
  rowsum = rowHalf[-1] + rowHalf[-2]
  # create the second half of the row using this
um
  rowFirst = np.array(rowHalf[:-1])
  rowSecond = rowHalf[::-1]
  rowSecond = rowsum - np.array(rowSecond[1:])
  row = np.concatenate([rowFirst, rowSecond])
  # if we included a case with too large numbers,
discard
  if row[-1] > n**2 - n: return None

  board = [row]

  check = np.zeros(n**2 + 1)
  check[0] = 1
  check[row] = 1

  for _ in range(n-1):
    i = np.min(np.where(check == 0))

    newRow = row + i - 1
    if newRow[-1] > n**2 or
np.sum(check[newRow]) > 0:
      return None
    board.append(newRow)
    check[newRow] += 1

  return np.array(board)

def numberOfBoards(n):

  # The last number of the row can be at most
n^2 - n since 2 is in the row and
  # the last number of the column must be at
least n+1 (1, 3, 4, ..., n+1)
  numberPool = [i for i in range(1, n**2-n+1)]
  allFirstRows = getAllFirstRows(numberPool,
n//2 + 1)

  boards = []
  for rowHalf in allFirstRows:
    board = getBoard(rowHalf, n)
    if board is not None: boards.append(board)

  return boards

if __name__ == "__main__":
  start = time.time()
  boards = numberOfBoards(10)
  print(len(boards))
  print()

  for board in boards:
    print(board)
    print()
  end = time.time()

  print("this took", end - start, "seconds")
```

Running the above program, we conclude there are $14 \cdot (10!)^2 = 184{,}354{,}652{,}160{,}000$ chess boards for $n = 10$. These chess boards can be obtained by interchanging the entries of the first row and the first column and / or by changing the places of any two rows or any two columns of the following chess boards (possibly several times):

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	5	6	9	10	13	14	17	18
3	4	7	8	11	12	15	16	19	20
21	22	25	26	29	30	33	34	37	38
23	24	27	28	31	32	35	36	39	40
41	42	45	46	49	50	53	54	57	58
43	44	47	48	51	52	55	56	59	60
61	62	65	66	69	70	73	74	77	78
63	64	67	68	71	72	75	76	79	80
81	82	85	86	89	90	93	94	97	98
83	84	87	88	91	92	95	96	99	100

1	2	3	4	5	11	12	13	14	15
6	7	8	9	10	16	17	18	19	20
21	22	23	24	25	31	32	33	34	35
26	27	28	29	30	36	37	38	39	40
41	42	43	44	45	51	52	53	54	55
46	47	48	49	50	56	57	58	59	60
61	62	63	64	65	71	72	73	74	75
66	67	68	69	70	76	77	78	79	80
81	82	83	84	85	91	92	93	94	95
86	87	88	89	90	96	97	98	99	100

1	2	11	12	21	22	31	32	41	42
3	4	13	14	23	24	33	34	43	44
5	6	15	16	25	26	35	36	45	46
7	8	17	18	27	28	37	38	47	48
9	10	19	20	29	30	39	40	49	50
51	52	61	62	71	72	81	82	91	92
53	54	63	64	73	74	83	84	93	94
55	56	65	66	75	76	85	86	95	96
57	58	67	68	77	78	87	88	97	98
59	60	69	70	79	80	89	90	99	100

1	2	3	4	5	26	27	28	29	30
6	7	8	9	10	31	32	33	34	35
11	12	13	14	15	36	37	38	39	40
16	17	18	19	20	41	42	43	44	45
21	22	23	24	25	46	47	48	49	50
51	52	53	54	55	76	77	78	79	80
56	57	58	59	60	81	82	83	84	85
61	62	63	64	65	86	87	88	89	90
66	67	68	69	70	91	92	93	94	95
71	72	73	74	75	96	97	98	99	100

1	2	21	22	41	42	61	62	81	82
3	4	23	24	43	44	63	64	83	84
5	6	25	26	45	46	65	66	85	86
7	8	27	28	47	48	67	68	87	88
9	10	29	30	49	50	69	70	89	90
11	12	31	32	51	52	71	72	91	92
13	14	33	34	53	54	73	74	93	94
15	16	35	36	55	56	75	76	95	96
17	18	37	38	57	58	77	78	97	98
19	20	39	40	59	60	79	80	99	100

1	2	3	4	5	51	52	53	54	55
6	7	8	9	10	56	57	58	59	60
11	12	13	14	15	61	62	63	64	65
16	17	18	19	20	66	67	68	69	70
21	22	23	24	25	71	72	73	74	75
26	27	28	29	30	76	77	78	79	80
31	32	33	34	35	81	82	83	84	85
36	37	38	39	40	86	87	88	89	90
41	42	43	44	45	91	92	93	94	95
46	47	48	49	50	96	97	98	99	100

The execution of the computer program takes approximately 55 seconds and the computation time grows at least exponentially with the size of the board. It is therefore unfortunately not possible to calculate the answer for greater values of *n* using this program.

This answer for *n* = 10 strengthens our belief that the answer of the reduced problem (which requires us to find the number of polynomials satisfying the given conditions) likely depends on the number of prime factors of *n* and their exponents, rather than the values of those prime factors (compare this also to the fact that the answer was the same regardless of what the base prime of *n* was when *n* was power of a prime number). However, both approaches (using modular arithmetics or simple inequalities) become more complicated and none of them works straight in case *n* has at least two different prime factors. As shown above, it is also quite difficult to find the answer in those cases with or without the help of a computer program, even for small values of *n* (Computational methods which rely on an exhaustive search quickly become infeasible, for $n \geq 12$ for example). Hence, the problem still remains open to be solved for such values of *n*.

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