Junior problems

J529. Let a and b be positive real numbers. Prove that for any $x \geq \max(a, b)$

$$(x^2 + a^2)(x^2 + b^2) \geq 8\sqrt{ab}x(x - a)(x - b).$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by the author

We have

$$(x^2 + a^2)(x^2 + b^2) = ((x - a)^2 + 2xa)(x - b)^2 + 2xb) \geq 2(x - a)\sqrt{2xa}2(x - b)\sqrt{2xb} = 8\sqrt{ab}x(x - a)(x - b).$$

Also solved by Polyahedra, Polk State College, FL, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Corneliu Mănescu-Avram, Ploieşti, Romania; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Taes Padhihary, Disha Delphi Public School, India; Ashley Kim, Union County Vocational-Technical School in Scotch Plains, NJ, USA; Arkady Alt, San Jose, CA, USA; Evripides P. Nastou, 6th High School, Nea Smyrni, Greece.

J530. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{\sqrt{2(a^2+b^2)}}+\frac{b+c}{\sqrt{2(b^2+c^2)}}+\frac{c+a}{\sqrt{2(c^2+a^2)}}+\frac{3(a^2+b^2+c^2)}{2(ab+bc+ca)}\geq\frac{9}{2}.$$

Proposed by Marius Stanean, Zalau, România

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
Using Cauchy-Schwarz and AM-GM inequality we have

$$\begin{aligned}
\sum_{\text{cyc}}\frac{a+b}{\sqrt{2(a^2+b^2)}}&=\sum_{\text{cyc}}\frac{(a+b)^2}{(a+b)\sqrt{2(a^2+b^2)}}\\
&\geq\frac{4(a+b+c)^2}{(a+b)\sqrt{2(a^2+b^2)}+(b+c)\sqrt{2(b^2+c^2)}+(c+a)\sqrt{2(c^2+a^2)}}\\
&\geq\frac{4(a+b+c)^2}{\sqrt{[(a+b)^2+(b+c)^2+(c+a)^2](4a^2+4b^2+4c^2)}}\\
&=\frac{2(a+b+c)^2}{\sqrt{2(a^2+b^2+c^2)(a^2+b^2+c^2+ab+bc+ca)}}\\
&\geq\frac{4(a+b+c)^2}{2(a^2+b^2+c^2)+(a^2+b^2+c^2+ab+bc+ca)}\\
&=\frac{4(a+b+c)^2}{3(a^2+b^2+c^2)+ab+bc+ca}.
\end{aligned}$$

It remains to show that

$$\frac{4(a+b+c)^2}{3(a^2+b^2+c^2)+ab+bc+ca}+\frac{3(a^2+b^2+c^2)}{2(ab+bc+ca)}\geq\frac{9}{2}.$$

After setting $a^2+b^2+c^2=x, ab+bc+ca=y$ the inequality becomes

$$\frac{4(x+2y)}{3x+y}+\frac{3x}{2y}\geq\frac{9}{2}.$$

This is equivalent to

$$9x^2-16xy+7y^2\geq 0,$$

$$(x-y)(9x-7y)\geq 0$$

which is true because $x\geq y$. The proof is complete. The equality holds if and only if $a=b=c$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Polyahedra, Polk State College, FL, USA; Taes Padhihary, Disha Delphi Public School, India; Toyesh Prakash Sharma, St. C.F Andrews School, Agra, India; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece.

J531. Solve in real numbers the system of equations

$$\begin{cases} x^5y^5+1 &= y^5 \\ x^9y^9+1 &= y^9. \end{cases}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan

The only solution is $(x,y)=(0,1)$. Let $z=xy$, then we obtain

$$\begin{cases} z^5+1=y^5 \\ z^9+1=y^9. \end{cases}$$

Case 1: If $1<y$, then $1<z<y$ and we obtain

$$1=\frac{z^5}{y^5}+\frac{1}{y^5}>\frac{z^9}{y^9}+\frac{1}{y^9}=1,$$

a contradiction.

Case 2: If $0<y<1$, then $-1<z<0$ and we obtain

$$y^5>y^9=z^9+1>z^5+1=y^5,$$

a contradiction.

Case 3: If $-1<y<0$, then $z<-1$ and we obtain

$$y^9>y^5=z^5+1>z^9+1=y^9,$$

a contradiction.

Case 4: If $y=-1$, then we obtain $\sqrt[5]{-2}=z=\sqrt[9]{-2}$, a contradiction.

Case 5: If $y<-1$, then $z<y<-1$ and we obtain

$$1=\frac{z^9}{y^9}+\frac{1}{y^9}>\frac{z^5}{y^5}+\frac{1}{y^5}=1,$$

a contradiction and we are done.

Also solved by Kyle Dechau, SUNY Brockport, NY, USA; Polyahedra, Polk State College, FL, USA; Taes Padhihary, Disha Delphi Public School, India; Ioannis D. Sfikas, Athens, Greece.

J532. Let a and b be real numbers such that $ab \geq \frac{1}{3}$. Prove that

$$\frac{1}{3a^2+1} + \frac{1}{3b^2+1} \geq \frac{2}{3ab+1}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Taes Padhihary, Disha Delphi Public School, India

Note that

$$\frac{1}{3a^2+1} + \frac{1}{3b^2+1} - \frac{2}{3ab+1} = \frac{3(a-b)^2(3ab-1)}{(3a^2+1)(3b^2+1)(3ab+1)} \geq 0,$$

which is true from the condition.

Also solved by Daniel Văcaru, Pitești, Romania; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Kyle Dechau, SUNY Brockport, NY, USA; Polyahedra, Polk State College, FL, USA ; Nakis Konstantinos Pantelis, Ionidios Model High School, Greece; Toyesh Prakash Sharma, St. C.F Andrews School, Agra, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Maria Billa, Model Middle School of Evaggeliki Athens, Greece; Joel Schlosberg, Bayside, NY, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Srijon Sarkar, Aditya Academy Secondary School, Kolkata, West Bengal, India; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Eleni Bethani, Evangelical Model High School, Nea Smyrni, Athens, Greece; Anderson Torres, São Paulo, Brazil; Arighna Pan, Nabadwip Vidyasagar College, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA; Fallon Lennox, SUNY Brockport, NY, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Evripides P. Nastou, 6th High School, Nea Smyrni, Greece; Ioannis D. Sfikas, Athens, Greece.

J533. Find the range of the expression

$$\frac{a-b}{c}$$

where a, b, c are the side-lengths of a triangle with $\angle A = 90^\circ$ and $c \le b$.

Proposed by Shiva Oswal, Stanford Online High School, USA

Solution by Polyahedra, Polk State College, USA

Since $0^\circ < C \le 45^\circ$,

$$\frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{1-\cos C}{\sin C} = \tan\frac{C}{2},$$

which has range $(0, \sqrt{2}-1]$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Kyle Dechau, SUNY Brockport, NY, USA; Ashley Kim, Union County Vocational-Technical School in Scotch Plains, NJ, USA; Arkady Alt, San Jose, CA, USA; Anderson Torres, São Paulo, Brazil; Arighna Pan, Nabadwip Vidyasagar College, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Joel Schlosberg, Bayside, NY, USA; Jung Hoon Song; Michail Prousalidis, Evangeliki Model High School, Smyrna, Greece; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Evripides P. Nastou, 6th High School, Nea Smyrni, Greece; Ioannis D. Sfikas, Athens, Greece.

J534. Find the greatest prime p such that $3p$ has three digits and divides $31^{31} + 32^{155}$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

First solution by Joel Schlosberg, Bayside, NY, USA

331 is prime, since it is not divisible by 2, 3, 5, 7, 11, 13, or 17 and the product of any two or more larger prime divisors is at least $19^2 = 361 > 331$. Since $2 \mid 332$, $3 \mid 333$, and $3p > 1000$ for $p \geq 334$, 331 is the largest prime such that $3p$ has three digits.

$31^3 - 1 = 90 \cdot 331$, so

$$31^{31} = 31 \cdot (31^3)^{10} \equiv 31 \cdot 1^{10} = 31 \pmod{331}.$$

$32^2 = 31 + 3 \cdot 331$ and $32^3 + 1 = 99 \cdot 331$, so

$$32^{155} = 32^2 \cdot (32^3)^{51} \equiv 31 \cdot (-1)^{51} \equiv -31 \pmod{331}$$

and thus $331 \mid 31^{31} + 32^{155}$.

Second solution by the author

$p = 331$ is the largest prime for which $3p$ has three digits and divides $31^{31} + 32^{155}$. Indeed,

$$31^{31} + 32^{155} = 31^{31} + (32^5)^{31}$$

is divisible by $31 + 32^5 = 32^5 + 32 - 1 = (32^2 - 32 + 1)(32^3 + 32^2 - 1)$ and $32^2 - 32 + 1 = 3 \cdot 331$. We used the identity $a^5 + a - 1 = (a^2 - a + 1)(a^3 + a^2 - 1)$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Polyahedra, Polk State College, USA; Taes Padhihary, Disha Delphi Public School, India; Ashley Kim, Union County Vocational-Technical School in Scotch Plains, NJ, USA; Anderson Torres, São Paulo, Brazil; Arighna Pan, Nabadwip Vidyasagar College, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Sailalitha Kodukula, Archimedean Middle Conservatory, Miami, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam.

Senior problems

S529. Prove that if the number $\overline{abcbca}$ is divisible by 37 then so is

$$(a-b)^2+(b-c)^2+(c-a)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Mircea Becheanu, Montreal, Canada

$abcbca \equiv 0(\text{mod}37) \Leftrightarrow (a+10c+100b)+10^3(c+10b+100a) \equiv 0(\text{mod}37)$

$$\Leftrightarrow (a+10c-11b)+(c+10b-11a) \equiv 0(\text{mod}37) \Leftrightarrow 10a+b-11c \equiv 0(\text{mod}37) \Leftrightarrow 10(a-c)+(b-c) \equiv 0(\text{mod}37)$$

$$\Leftrightarrow b-c \equiv 10(c-a)(\text{mod}37).$$

Let remark that if $x+y+z=0$ then $x^2+y^2+z^2=2(x^2+y^2+xy)$. So, it is enough to show that

$$(b-c)^2+(c-a)^2+(b-c)(c-a) \equiv 0(\text{mod}37).$$

Using the congruence obtained above we have

$$100(c-a)^2+(c-a)^2+10(c-a)^2 \equiv (c-a)^2(100+1+10) \equiv 0(\text{mod}37).$$

Second solution by the author

Because $37|999$, $37|\overline{abcbca}$ means $37|\overline{abc}+\overline{bca}=101a+110b+11c$. Then $111(a+b+c)-(101a+110b+11c)=100c+10a+b=\overline{cab}$ is divisible by 37. Since all cyclic digit permutations of the 3-digit multiples of 37 are themselves divisible by 37, it follows that abc and bca are also divisible by 37. Then, using the properties of determinants,

$$\det\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \det\begin{vmatrix} 100a+10b+c & b & c \\ 100b+10c+a & c & a \\ 100c+10a+b & a & b \end{vmatrix}$$

is divisible by 37. Hence 37 divides

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

and, since it does not divide $a+b+c$ (being bigger) the conclusion follows.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Arighna Pan, Nabadwip Vidyasagar College, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Erfan Amouzad Khalili, Tehran, Iran; Joel Schlosberg, Bayside, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S530. Let ABC be a triangle with area K. Prove that

$$a(s-a)\cos\frac{B-C}{4}+b(s-b)\cos\frac{C-A}{4}+c(s-c)\cos\frac{A-B}{4}\geq 2\sqrt{3}K.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

The inequality is equivalent to

$$a\frac{K}{s}\cot\frac{A}{2}\cos\frac{B-C}{4}+b\frac{K}{s}\cot\frac{B}{2}\cos\frac{C-A}{4}+c\frac{K}{s}\cot\frac{C}{2}\cos\frac{A-B}{4}\geq 2\sqrt{3}K,$$

which can be rewritten as

$$\sin A\cot\frac{A}{2}\cos\frac{B-C}{4}+\sin B\cot\frac{B}{2}\cos\frac{C-A}{4}+\sin C\cot\frac{C}{2}\cos\frac{A-B}{4}\geq\sqrt{3}(\sin A+\sin B+\sin C)$$

and

$$2\cos^2\frac{A}{2}\cos\frac{B-C}{4}+2\cos^2\frac{B}{2}\cos\frac{C-A}{4}+2\cos^2\frac{C}{2}\cos\frac{A-B}{4}\geq\sqrt{3}(\sin A+\sin B+\sin C)$$

Corner transformation $(A,B,C)\to(\pi-2A,\pi-2B,\pi-2C)$ gives us

$$2\sin^2 A\cos\frac{B-C}{2}+2\sin^2 B\cos\frac{C-A}{2}+2\sin^2 C\cos\frac{A-B}{2}\geq\sqrt{3}(\sin 2A+\sin 2B+\sin 2C),$$

$$2\sin A(\sin B+\sin C)\sin\frac{A}{2}+2\sin B(\sin C+\sin A)\sin\frac{B}{2}+2\sin C(\sin A+\sin B)\sin\frac{C}{2}\geq\sqrt{3}(\sin 2A+\sin 2B+\sin 2C),$$

$$\sin A(\sin B+\sin C)\sin\frac{A}{2}+\sin B(\sin C+\sin A)\sin\frac{B}{2}+\sin C(\sin A+\sin B)\sin\frac{C}{2}\geq 2\sqrt{3}\sin A\sin B\sin C\ (1)$$

Now, inscribed circle substitution $a=y+z, b=z+x, c=x+y (x,y,z\in\mathbb{R}_+)$ gives us

$\sin A=\dfrac{2\sqrt{xyz(x+y+z)}}{(z+x)(x+y)}$, similarly for $\sin B$ and $\sin C$

Next, $\sin\dfrac{A}{2}=\sqrt{\dfrac{yz}{(z+x)(x+y)}}$ and the permutations. Thus, (1) is equivalent to

$$\frac{y^2+z^2+2(xy+yz+zx)}{\sqrt{x(z+x)(x+y)}}+\frac{z^2+x^2+2(xy+yz+zx)}{\sqrt{y(x+y)(y+z)}}+\frac{x^2+y^2+2(xy+yz+zx)}{\sqrt{z(y+z)(z+x)}}\geq 4\sqrt{3(x+y+z)}\ (2)$$

Now, set $x+y+z=1$. Then (2) can be transformed into

$$\frac{(1-x)^2+2x(1-x)}{\sqrt{x^3+x^2(1-x)+xyz}}+\frac{(1-y)^2+2y(1-y)}{\sqrt{y^3+y^2(1-y)+xyz}}+\frac{(1-z)^2+2z(1-z)}{\sqrt{z^3+z^2(1-z)+xyz}}\geq 4\sqrt{3}$$

since $yz\leq\left(\dfrac{y+z}{2}\right)^2=\dfrac{1}{4}(1-x)^2$.

Finally, we need to prove the following inequality

$$\frac{(1-x)^2+2x(1-x)}{\sqrt{x^3+x^2(1-x)+\frac{1}{4}x(1-x)^2}}+\frac{(1-y)^2+2y(1-y)}{\sqrt{y^3+y^2(1-y)+\frac{1}{4}y(1-y)^2}}+$$
$$+\frac{(1-z)^2+2z(1-z)}{\sqrt{z^3+z^2(1-z)+\frac{1}{4}z(1-z)^2}}\geq 4\sqrt{3},$$

which is equivalent to

$$\frac{1-x}{\sqrt{x}}+\frac{1-y}{\sqrt{y}}+\frac{1-z}{\sqrt{z}}\geq 2\sqrt{3} \tag{3}$$

since $\dfrac{1-x}{\sqrt{x}}-\dfrac{4-6x}{\sqrt{3}}=\dfrac{2\sqrt{3}}{\sqrt{x}}\left(\sqrt{x}-\dfrac{1}{\sqrt{3}}\right)^2\left(\sqrt{x}+\dfrac{\sqrt{3}}{2}\right)\geq 0$ Therefore, $\dfrac{1-x}{\sqrt{x}}\geq\dfrac{4-6x}{\sqrt{3}}$. In the same way, two more formulae can be obtained, it is clear that the superposition of three formulas is valid.

Note: It is a kind of strengthening of the famous Finsler - Hadwiger inequality (1937)

$$a(s-a)+b(s-b)+c(s-c)\geq 2\sqrt{3}K$$

$$a^2+b^2+c^2\geq 4\sqrt{3}K+(a-b)^2+(b-c)^2+(c-a)^2$$

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Telemachus Baltsavias, Keramies Junior High School, Kefalonia, Greece.

S531. Let x, y, z be real numbers such that $-1 \le x, y, z \le 1$ and $x+y+z+xyz=0$. Prove that

$$x+y+z+\frac{72}{9+xy+yz+zx} \ge 8.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

If $x=-1$ then $(y-1)(z-1)=0$ so let $y=1$ and the inequality becomes $z \ge -1$, true. Similar if $x=1$ then $(y+1)(z+1)=0$ so let $y=-1$ and the inequality becomes $z \ge -1$. Let $x, y, z \in (-1,1)$ and denote $\frac{b}{a}=\frac{1-x}{1+x}$, $\frac{c}{b}=\frac{1-y}{1+y}$, $\frac{a}{c}=\frac{1-z}{1+z}$, where $a, b, c > 0$. It follows that

$$x=\frac{a-b}{a+b},\ y=\frac{b-c}{b+c},\ z=\frac{c-a}{c+a},$$

and the inequality becomes

$$\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{9(a+b)(b+c)(c+a)}{(a+b+c)(ab+bc+ca)} \ge 8,$$

or

$$\frac{9(a+b)(b+c)(c+a)}{(a+b+c)(ab+bc+ca)}-8-\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)} \ge .0$$

Note that the inequality is cyclic, so without loss of generality, we may assume that b is between a and c. We have two cases:

Case 1. $a \ge b \ge c$, clearly true since

$$\frac{9(a+b)(b+c)(c+a)}{(a+b+c)(ab+bc+ca)}-8 \ge 0 \Longleftrightarrow \sum_{cyc} a(b-c)^2 \ge 0.$$

Case 2. $c \ge b \ge a$, then we can rewrite the inequality as

$$\frac{2c(a-b)^2+(a+b)(c-a)(c-b)}{(a+b+c)(ab+bc+ca)}-\frac{(b-a)(c-a)(c-b)}{(a+b)(b+c)(c+a)} \ge 0,$$

which is true because

$$\frac{a+b}{(a+b+c)(ab+bc+ca)} \ge \frac{b-a}{(a+b)(b+c)(c+a)},$$

that is, using the identity $(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc$,

$$2a(a+b+c)(ab+bc+ca) \ge abc(a+b).$$

Equality holds if $a=b=c$ yields $x=y=z=0$ or $x=-1, y=1, z=-1$ and its cyclic permutations.

Also solved by Kyle Dechau, SUNY Brockport, NY, USA; Arighna Pan, Nabadwip Vidyasagar College, India.

S532. Let a, b, c be positive real numbers such that $abc = 1$. Prove that for any $0 \le t \le \min\{a, b, c\}$,

$$(2a^2 - 6at + 7t^2)(2b^2 - 6bt + 7t^2)(2c^2 - 6ct + 7t^2) \ge 108t^5(a - t)(b - t)(c - t).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Because $0 \le t \le a$, it is natural to consider the expression $t^3 + (a - t)^3$ and get

$$2a^2 - 6at + 7t^2 = \frac{1}{a}[2t^3 + 2(a - t)^3 + at^2] \ge \frac{1}{a}3t(a - t)\sqrt[3]{4at^2}$$

by the AM-GM inequality. Writing the two analogous inequalities and multiplying out yields the conclusion.

Second solution by Marius Stanean, Zalau, România

Let $a = tx$, $b = ty$, $c = tz$, then $x, y, z \ge 1$ and $xyz = \frac{1}{t^3}$, $t \in (0, 1)$. The inequality becomes

$$(2x^2 - 6x + 7)(2y^2 - 6y + 7)(2z^2 - 6z + 7) \ge 108t^2(x - 1)(y - 1)(z - 1).$$

If one of the numbers x, y, z is equal to 1 the inequality is clearly true, so consider $x, y, z > 1$. Then, we have

$$\begin{aligned}\frac{2x^2 - 6x + 7}{x - 1} &= 2(x - 1) + \frac{3}{x - 1} - 2\\ &\ge x - 1 + \frac{2}{x - 1} \ge 3\sqrt[3]{\frac{1}{x - 1}}.\end{aligned}$$

We only need to prove that

$$\frac{1}{\sqrt[3]{(x - 1)(y - 1)(z - 1)}} \ge 4t^2,$$

or equivalently

$$x^2y^2z^2 \ge 64(x - 1)(y - 1)(z - 1),$$

true by multiplying the following inequalities

$$x^2 = (x - 1 + 1)^2 \ge 4(x - 1),$$

$$y^2 = (y - 1 + 1)^2 \ge 4(y - 1),$$

$$z^2 = (z - 1 + 1)^2 \ge 4(z - 1).$$

Equality holds when $x = y = z = 2$ which means $t = \frac{1}{2}$ and $a = b = c = 1$.

Also solved by Arkady Alt, San Jose, CA, USA.

S533. For every positive integer we denote by $\tau(a)$ the number of positive divisors of a and by $d_1(a) < d_2(a) < \cdots < d_{\tau(a)}(a)$ the increasing sequence of these divisors. Let $(x_1, x_2, \ldots, x_n)$ be a sequence of positive integers. We say that the refinement of this sequence is the sequence

$$d_1(x_1), d_2(x_1), \ldots, d_{\tau(x_1)}(x_1), d_1(x_2), \ldots, d_{\tau(x_2)}(x_2), \ldots, d_1(x_n), \ldots, d_{\tau(x_n)}(x_n).$$

For example, the refinement of the sequence $(1, 4, 7, 10)$ is the sequence $(1, 1, 2, 4, 1, 7, 1, 2, 5, 10)$. Let p be a prime number. Starting with the sequence $(1, p, p^2, \ldots, p^p)$, we refine this sequence 2020 times in a row. Evaluate how many times each prime power p^i appears in the last sequence.

Proposed by Besfort Shala, University of Primorska, Koper, Slovenia

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let p be a prime number, and consider the sequence $(1, p, p^2, \ldots, p^p)$. For each integer i, $0 \le i \le p$, denote by $c_i^{(k)}$ the number of times the prime power p^i appears in the kth refinement of $(1, p, p^2, \ldots, p^p)$. Because the divisors of the prime power p^i are $1, p, p^2, \ldots, p^i$, it follows that

$$c_i^{(k+1)} = \sum_{j=i}^{p} c_j^{(k)}$$

for each integer i, $0 \le i \le p$. Now,

$$c_i^{(0)} = 1 = \binom{p-i}{0},$$

so

$$c_i^{(1)} = \sum_{j=i}^{p} c_j^{(0)} = \sum_{j=i}^{p} 1 = p + 1 - i = \binom{p+1-i}{1}.$$

For some non-negative integer k, suppose

$$c_i^{(k)} = \binom{p+k-i}{k}.$$

Then

$$c_i^{(k+1)} = \sum_{j=i}^{p} \binom{p+k-j}{k} = \binom{p+k+1-i}{k+1}$$

by the hockey stick identity. Thus,

$$c_i^{(k)} = \binom{p+k-i}{k}$$

for all non-negative integers k by induction. Finally,

$$c_i^{(2020)} = \binom{p+2020-i}{2020}.$$

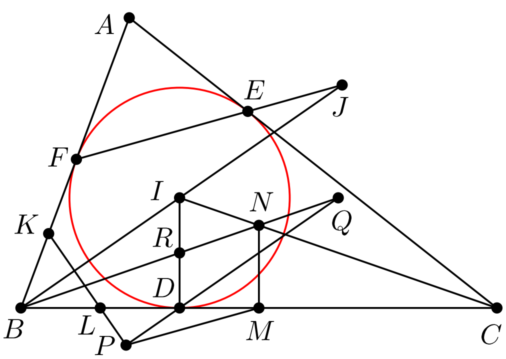
Also solved by Kyle Dechau, SUNY Brockport, NY, USA; Maiteyo Bhattacharjee, IACS, Kolkata, India.

S534. Let ABC be a triangle with incircle tangent to sides BC, CA, AB at points D, E, F, respectively. Suppose that K, L, M are midpoints of BF, BD, BC, respectively. The line which passes through D and is parallel to the bisector of the angle CBA intersects KL at P. Prove that $PM \parallel EF$ if and only if $BC = 3BD$.

Proposed by Dominik Burek, Krakow, Poland

Solution by Li Zhou, Polk State College, FL, USA

Let I be the incenter of $\triangle ABC$. Suppose that BI intersects EF at J, CI intersects the perpendicular bisector of BC at N, and BN intersects PD at Q and ID at R. Since $\angle FJB = C/2 = \angle DBN$, we see that $PM \parallel EF$ if and only if $\triangle PDM \sim \triangle BDQ$.



If $\triangle PDM \sim \triangle BDQ$, then

$$\frac{DM}{DQ} = \frac{DP}{DB} = \frac{1}{2}\cos\frac{B}{2},$$

so MN bisects DQ. Therefore, $NQ = NR = NI$, thus $QI \perp ID$. Hence, $BD = IQ = 2DM$, that is, $BC = 3BD$.

Conversely, if $BD = 2DM$, then Q must be the reflection of I across MN, so MN bisects DQ. Hence, $DM/DQ = DP/DB$, thus $\triangle PDM \sim \triangle BDQ$.

Also solved by Corneliu Mănescu-Avram and Nicuşor Zlota, Romania; Ehsan Heidari, Atomic Energy High School, Tehran, Iran.

Undergraduate problems

U529. Let $P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + 1$ be a polynomial whose roots $x_1, x_2, \ldots, x_n$ are real and positive. Prove that for any positive real number t,

$$(t^2 - tx_1 + x_1^2)(t^2 - tx_2 + x_2^2)\cdots(t^2 - tx_n + x_n^2) \geq 2^n t^{\frac{n}{2}} |P(t)|.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Besfort Shala, University of Primorska, Koper, Slovenia

Firstly, we show that n must be even. For if n was odd, the product of the positive real roots of $p(x)$ would be -1, an absurdity. Next, note that $t^2 - tx_i + x_i = (t - x_i)^2 + tx_i$ which by the AM-GM inequality (which we may use as $t, x_i \in \mathbb{R}^+$), is greater than or equal to $2|t - x_i|\sqrt{tx_i}$. Multiplying these inequalities (note that both sides of all inequalities are non-negative) for $i = 1, 2, \ldots, n$, and using $\prod_{i=1}^n x_i = 1$, we obtain

$$\prod_{i=1}^n (t^2 - tx_i + x_i^2) \geq 2^n t^{\frac{n}{2}} \prod_{i=1}^n |(t - x_i)| \sqrt{\prod_{i=1}^n x_i} = 2^n t^{\frac{n}{2}} |p(t)|,$$

as desired.

Let us also consider when does equality hold. Equality holds if and only if

$$(t - x_i)^2 = tx_i \iff t^2 - 3tx_i + x_i^2 = 0 \iff t = \frac{(3 \pm \sqrt{5})x_i}{2}$$

for all $i = 1, 2, \ldots, n$. For the polynomial $p(x)$, this means that for all distinct values of i, j we either have $x_i = x_j$ or $(3 \pm \sqrt{5})x_i = (3 \mp \sqrt{5})x_j$. For example, equality holds for $t = \frac{3 \pm \sqrt{5}}{2}$ when $p(x) = (x - 1)^n$ for even values of n.

Also solved by Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Ibrahim Suleiman, New York University, Abu Dhabi; Arkady Alt, San Jose, CA, USA; David Son, Stevens Institute of Technology, NJ, USA.

U530. Evaluate

$$\lim_{n\to\infty} \frac{\sin 1 + (\sin \frac{1}{2})^2 + \cdots + (\sin \frac{1}{n})^n}{\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}}$$

Proposed by Florin Rotaru, Focşani, Romania

Solution by the author

We shall use generalized Stolz-Cesaro Lemma. We have:

$$\lim_{n\to\infty} \frac{\sin 1 + \left(\sin \frac{1}{2}\right)^2 + \cdots + \left(\sin \frac{1}{n}\right)^n}{\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}} =$$

$$= \lim_{n\to\infty} \frac{\sin^n \frac{1}{n}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{\sin^n \frac{1}{n}}{\frac{1}{n^n}} \cdot \lim_{n\to\infty} \frac{n!}{n^n}.$$

Since

$$\frac{n!}{n^n} < \frac{1}{n}$$

we have

$$\lim \frac{n!}{n^n} = 0.$$

So, the required limit is 0.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Toyesh Prakash Sharma, St. C.F Andrews School, Agra, India.

U531. Evaluate

$$\int_0^1 \frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}+\sqrt[3]{1-x}}dx.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Arkady Alt, San Jose, CA, USA

Let $I := \int_0^1 \frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}+\sqrt[3]{1-x}}dx$ and $t := \frac{\sqrt[3]{1-x}}{\sqrt[3]{1+x}}$.

Then $t^3 = \frac{1-x}{1+x} \iff x = \frac{1-t^3}{1+t^3}$, $dx = -\frac{6t^2}{(t^3+1)^2}dt$ and

$$I+1 = \int_0^1 \left(\frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}+\sqrt[3]{1-x}}+1\right)dx = 2\int_0^1 \frac{\sqrt[3]{1+x}}{\sqrt[3]{1+x}+\sqrt[3]{1-x}}dx = 2\int_0^1 \frac{dx}{1+\frac{\sqrt[3]{1-x}}{\sqrt[3]{1+x}}} =$$

$$12\int_0^1 \frac{1}{1+t}\cdot\frac{t^2}{(t^3+1)^2}dt = 12\int_0^1 \frac{t^2}{(1+t)^3(t^2-t+1)^2}dt =$$

$$12\int_0^1 \left(\frac{1}{9(t+1)^3}-\frac{2}{27(t+1)}+\frac{t+1}{9(t^2-t+1)^2}+\frac{2t-4}{27(t^2-t+1)}\right)dt.$$

Since

$$\frac{t^2}{(1+t)^3(t^2-t+1)^2} = \frac{1}{9(t+1)^3}-\frac{2}{27(t+1)}+\frac{t+1}{9(t^2-t+1)^2}+\frac{2(t-2)}{27(t^2-t+1)}$$

and

$$\int_0^1 \frac{dt}{(t+1)^3} = \frac{3}{8}, \int_0^1 \frac{dt}{t+1} = \ln 2,\ \int_0^1 \frac{t+1}{(t^2-t+1)^2}dt = \frac{2}{9}\sqrt{3}\pi+1, \int_0^1 \frac{t-2}{t^2-t+1}dt = -\frac{\sqrt{3}}{3}\pi$$

then

$$I+1 = 12\left(\frac{1}{9}\cdot\frac{3}{8}-\frac{2}{27}\cdot\ln 2+\frac{1}{9}\left(\frac{2}{9}\sqrt{3}\pi+1\right)+\frac{2}{27}\cdot\left(-\frac{\sqrt{3}}{3}\pi\right)\right) = \frac{11}{6}-\frac{8}{9}\ln 2 \iff$$

$$I = \frac{5}{6}-\frac{8}{9}\ln 2.$$

Also solved by Arighna Pan, Nabadwip Vidyasagar College, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ioannis D. Sfikas, Athens, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Gabriel Mititelu, Western Sydney University, Sydney, Australia; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U532. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt[n]{1^2 3^2 \cdots (2n+1)^2}}{1+3+\cdots+(2n+1)}.$$

Proposed by the author Mircea Becheanu, Montreal, Canada

Solution by the author

We denote

$$x_n = \frac{\sqrt[n]{1^2 3^2 \ldots (2n+1)^2}}{1+3+\cdots+(2n+1)}$$

Observe that $1+3+5+\cdots+(2n+1) = (n+1)^2$. By algebraic computations we obtain

$$x_n = \sqrt[n]{\frac{((2n+1)!)^2}{2^{2n}(n+1)^{2n}(n!)^2}}$$

Let denote

$$a_n = \frac{((2n+1)!)^2}{2^{2n}(n+1)^{2n}(n!)^2}$$

We remind the following well-known result: given a sequence $(a_n)_{n\geq 1}$ of positive real numbers such that the limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and it is finite, then the limit

$$\lim_{n\to\infty} \sqrt[n]{a_n}$$

exists and the two limits are equal. In our case we have:

$$\frac{a_{n+1}}{a_n} = \frac{((2n+3)!)^2}{2^{2n+2}(n+2)^{2n+2}((n+1)!)^2} . \frac{2^{2n}(n+1)^{2n}(n!)^2}{((2n+1)!)^2} =$$

$$\frac{1}{4} . \frac{(2n+2)^2(2n+3)^2}{(n+1)^2(n+2)^2} . \left(\frac{n+1}{n+2}\right)^2 = \left(\frac{2n+3}{n+2}\right)^2 \left(\frac{n+1}{n+2}\right)^{2n}$$

We have

$$\lim_{n\to\infty} \left(\frac{2n+3}{n+2}\right)^2 = 4, \text{and}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^{2n} = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^{2n+4} . \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^4 =$$

$$\lim_{n\to\infty} \left[\left(1 - \frac{1}{n+2}\right)^{n+2}\right]^2 = \frac{1}{e^2}$$

Therefore,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} x_n = \frac{4}{e^2}$$

Also solved by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Toyesh Prakash Sharma, St. C.F Andrews School, Agra, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ioannis D. Sfikas, Athens, Greece; Besfort Shala, University of Primorska, Koper, Slovenia; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Henry Ricardo, Westchester Area Math Circle; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sebastian Foulger, Oxford University, Oxford, UK; Arkady Alt, San Jose, CA, USA.

U533. Evaluate

$$\int_1^2 (1+\ln x)x^x dx.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Henry Ricardo, Westchester Area Math Circle

Making the substitution $u = x\ln x$, we see that

$$\int_1^2 (1+\ln x)x^x dx = \int_0^{2\ln 2} e^u\, du = e^{2\ln 2} - e^0 = 3.$$

Also solved by Taes Padhihary, Disha Delphi Public School, India; Toyesh Prakash Sharma, St. C.F Andrews School, Agra, India; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Arighna Pan, Nabadwip Vidyasagar College, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Besfort Shala, University of Primorska, Koper, Slovenia; Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Keramies Junior High School, Kefalonia, Greece; Daniel Văcaru, Pitești, Romania; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; David Son, Stevens Institute of Technology, NJ, USA.

U534. Find all functions $f : \mathbb{R}- \to \mathbb{R}$ such that

$$f(x)f(x+y) \geq f(x)^2 + xy$$

for all $x, y, \in \mathbb{R}$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Assume that $f(0) = 0$, choose $x \geq y \geq 0$ and plug $(y, x-y)$:

$$f(x)f(y) = f(y)^2 + x(x-y) > 0.$$

Hence, the sign of $f(x)$ is fixed for all $x > 0$. Now, set $x \leq y \leq 0$ and find that the sign of $f(x)$ is also fixed for all $x < 0$.Plugging $y = -x$ yields to $x^2 \geq f(x)^2$ and $\left(\frac{x}{2}, \frac{x}{2}\right)$ results into $f\left(\frac{x}{2}\right) f(x) \geq f\left(\frac{x}{2}\right)^2 + \frac{x^2}{4}$. Hence, $f(x)^2 - x^2 \geq 2\left(\frac{x}{2} - f(x)\right)^2 \geq 0$. Hence, $f(x)^2 \geq x^2$. That is, $f(x)^2 = x^2$. Since $f(x)$ doesn't change its sign in negative or positive rays, we find that

$$f(x) = x, -x, |x|, -|x|$$

are all solutions. Now, assume that $f(0) = a \neq 0$. It's easy to find that is $f(x)$ is a solution, then so is $-f(x)$. Thus, WLOG, assume that $a > 0$. Setting $x = 0$ gives us $af(y) \geq a^2$. Hence, $f(y) \geq f(0) = a > 0$. Next, we can find that $f(x) \geq |x|$. Plugging $(x, y-x)$ resuts into $f(y) - f(x) \geq \frac{x(y-x)}{f(x)}$. Plugging $(y, x-y)$ results into $\frac{x(y-x)}{f(y)} \geq f(y) - f(x)$.

Therefore,

$$|y-x| \geq \frac{|y||y-x|}{f(y)} \geq \frac{y(y-x)}{f(y)} \geq f(y) - f(x) \geq \frac{x(y-x)}{f(x)}.$$

Then, $f(x)$ would be continuous. Further, for all $y > x$, $\frac{y}{f(y)} \geq \frac{f(y)-f(x)}{y-x} \geq \frac{x}{f(x)}$. Hence the funtion is differentiable. Further, $f'(x) = \frac{x}{f(x)}$. This implies that $\frac{d}{dx} f(x)^2 = 2x$. Hence,

$$f(x)^2 = x^2 + a \text{ and } f(x) = \pm\sqrt{x^2 + a}.$$

Also solved by David Son, Stevens Institute of Technology, NJ, USA; Ioannis D. Sfikas, Athens, Greece.

O529. Let a and b be odd positive integers. Prove that for any positive integer n there is a positive integer m such that 2^n divides at least one of the numbers $a^m b^2 - 1$ and $b^m a^2 - 1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Noting that $a^2, b^2 \equiv 1 \pmod 8$ and looking for m of the form $m = 2m_1$, we reduce the following: if $a, b \equiv 1 \pmod 8$,then for any $n \geq 1$ there is $m \geq 1$ such that 2^n divides $a^m b - 1$ or $b^m a - 1$. Fix n in the sequel.

We first claim that we can find integers $c, d \geq 0$ such that $a \equiv 9^c \pmod 2^n$ and $b \equiv 9^d \pmod 2^n$. Indeed, it suffices to prove that for any $x \equiv 1 \pmod 8$ there is $y \geq 0$ such that $x \equiv 9^y \pmod 2^n$. We may assume that $n > 3$. Let d be the order of 9 modulo 2^n Then d divides 2^{n-1}, so $d = 2^k$ for some k.

Now, $9^{2^k} - 1 = 8 \cdot (9+1)(9^2+1)...(9^{2^{k-1}}+1)$ and each factor except for the first one is of the form $4l + 2$, thus 2^n divides $9^{2^k} - 1$ if and only if $k \geq n-3$, thus the order of 9 modulo 2^n is 2^{n-3}. In particular the 2^{n-3} numbers$9^0, 9^1, ..., 9^{2^{n-3}} - 1$ are pairwise distinct modulo 2^n and all congruent to 1 modulo 8. It follows that they cover all residue classes mod 2^n of the form $8l + 1$ and the claim is proved.

We see now that it suffices to find $m \geq 1$ such that 2^{n-3} divides one of $mc + d$ and $md + c$. Write $c = 2^{u_1} v_1, d = 2^{u_2} v_2$ with $u_1, u_2 \geq 0$ and v_1, v_2 odd. By symmetry we may assume that $u_1 \geq u_2$. It suffices to ensure that $2^{u_1 - u_2} v_1 + m v_2$ is divisible by 2^{n-3}, which is certainly possible since v_2 is odd, thus prime to 2^{n-3}.

Also solved by Pratik Donga, Junagadh, India.

O530. Let a, b, c be positve real numbers such that $abc = 1$. Prove that

$$\sqrt{4a^2 - a + 1} + \sqrt{4b^2 - b + 1} + \sqrt{4c^2 - c + 1} \geq 2(a + b + c).$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

We rewrite the inequality as

$$\sum_{\text{cyc}} (\sqrt{4a^2 - a + 1} - 2a) \geq 0,$$

or

$$\sum_{\text{cyc}} \frac{1 - a}{\sqrt{4a^2 - a + 1} + 2a} \geq 0.$$

Now we will show that

$$\frac{1 - a}{\sqrt{4a^2 - a + 1} + 2a} \geq \frac{1}{4} \left(\frac{3}{a^2 + a + 1} - 1 \right).$$

Indeed, this is equivalent to

$$(1 - a) \left(\frac{4}{\sqrt{4a^2 - a + 1} + 2a} - \frac{2 + a}{a^2 + a + 1} \right) \geq 0,$$

$$(1 - a) \left[4(a^2 + a + 1) - (a + 2)(\sqrt{4a^2 - a + 1} + 2a) \right] \geq 0,$$

$$(1 - a) \left[(2a^2 + 4) - (a + 2)\sqrt{4a^2 - a + 1}) \right] \geq 0,$$

$$(1 - a) \left[(2a^2 + 4)^2 - (a + 2)^2 (4a^2 - a + 1) \right] \geq 0,$$

$$(1 - a)^2 (4 + 4a + 5a^2) \geq 0$$

which is clearly true. Similarly we have

$$\frac{1 - b}{\sqrt{4b^2 - b + 1} + 2b} \geq \frac{1}{4} \left(\frac{3}{b^2 + b + 1} - 1 \right),$$

$$\frac{1 - c}{\sqrt{4c^2 - c + 1} + 2c} \geq \frac{1}{4} \left(\frac{3}{c^2 + c + 1} - 1 \right).$$

Adding them up we obtain

$$\sum_{\text{cyc}} \frac{1 - a}{\sqrt{4a^2 - a + 1} + 2a} \geq \frac{1}{4} \left(\sum_{\text{cyc}} \frac{3}{a^2 + a + 1} - 3 \right).$$

To finish the proof we only need to show that

$$\sum_{\text{cyc}} \frac{1}{a^2 + a + 1} \geq 1.$$

After clearing the denominators and some computations, the inequality reduces to

$$a^2 + b^2 + c^2 + a + b + c + 2 \geq (a + 1)(b + 1)(c + 1)$$

or equivalently

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

But this is one of the most elementary inequalities, so we are done.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O531. Let ABC be an acute triangle and A_1, B_1, C_1 be the middle points of the sides BC, AC respectively AB. Denote: • A' and A'', the projections of point A_1 on the side AB, respectively AC , • B' and B'', the projections of B_1 on BC, respectively AB, • C' and C'', the projections of C_1 on AC, respectively BC, • X, the intersection point of the tangents to $\triangle A_1A'A''$ circumcircle in A' and A'', • Y the intersection point of the tangents to $\triangle B_1B'B''$ circumcircle in B' and B'', • Z the intersection point of the tangents to $\triangle C_1C'C''$ circumcircle in C' and C''. Show that A_1X, B_1Y and CZ are concurrent lines.

Proposed by Mihaela Berindeanu, teacher, Bucharest, România

First solution by the author

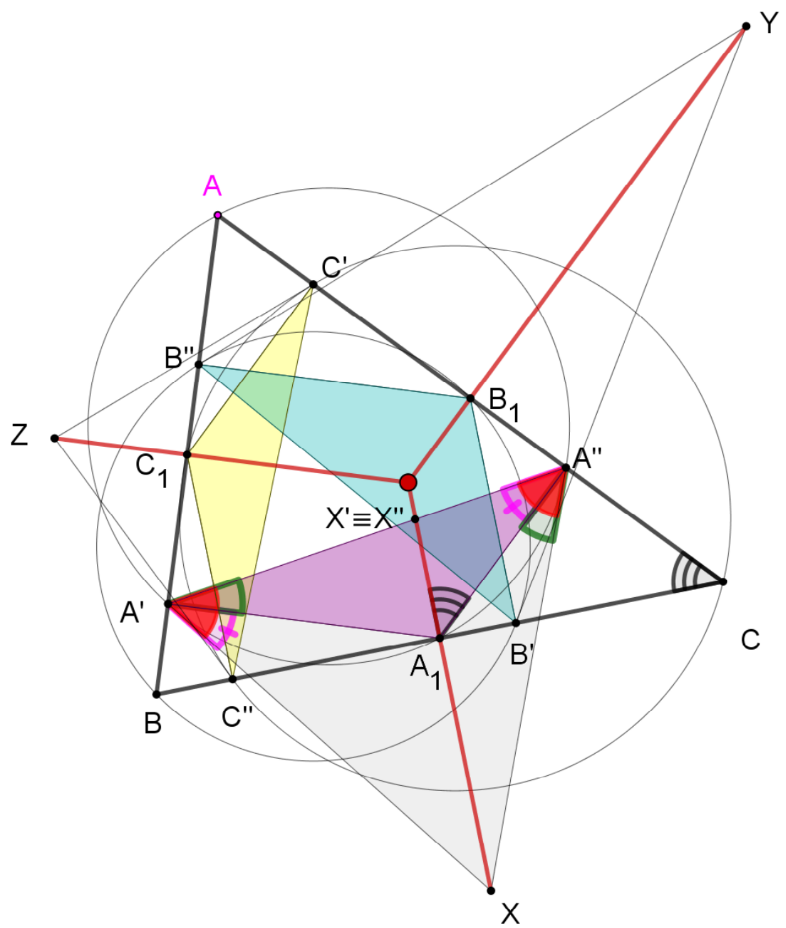


Figura 1:

Denote $XA_1 \cap A'A'' = \{X'\}$

- Show that XX' is a symmedian in $\triangle A'A''A_1$. In other words, we will prove that $\dfrac{X'A'}{X'A''} = \left(\dfrac{A_1A'}{A_1A''}\right)^2$

  XA', XA'' = tangents to $A_1A''A'$ circumcircle from an exterior point $\Rightarrow \triangle A'XA''$ = isosceles triangle $\Rightarrow XA'' = XA'$ and $\measuredangle XA''A' = \measuredangle XA'A''$.
  From XA'' = tangent line $\Rightarrow \measuredangle\,(XA''A_1) = \measuredangle\,(A_1A'A'')$
  From XA' = tangent line $\Rightarrow \measuredangle\,(XA'A_1) = \measuredangle\,(A_1A''A')$

$$\frac{X'A''}{X'A'} = \frac{\sigma(XX'A'')}{\sigma(XX'A')} = \frac{XA'' \cdot XX' \cdot \sin(X'XA'') \cdot \frac{1}{2}}{XA' \cdot XX' \cdot \sin(X'XA') \cdot \frac{1}{2}} = \frac{\sin(X'XA'')}{\sin(X'XA')}$$

According to the Sine Rule, in $\triangle XA_1A''$, respectively in $\triangle XA_1A'$:

$$\frac{\sin(X'XA'')}{A_1A''} = \frac{\sin(A_1A''X)}{XA_1} = \frac{\sin(A_1A'A'')}{XA_1} \tag{1}$$

$$\frac{\sin(X'XA')}{A_1A'} = \frac{\sin(A_1A'X)}{XA_1} = \frac{\sin(A_1A''A')}{XA_1} \tag{2}$$

$(1)/(2) \Rightarrow \dfrac{\sin(X'XA'')}{\sin(X'XA')} = \dfrac{A_1A'' \cdot \sin(A_1A'A'')}{A_1A' \cdot \sin(A_1A''A')} = \left(\dfrac{A_1A''}{A_1A'}\right)^2 \Rightarrow A_1X'$ is a symmedian in $\triangle A'A_1A''$

- If $X'' \in A'A''$ and $A_1X'' \perp BC$, show that $X' \equiv X''$
  $\measuredangle(A'A_1X'') = \measuredangle(A_1BA') = 90^\circ - \measuredangle(A'A_1B)$
  $\measuredangle(A''A_1X'') = \measuredangle(A''CA_1) = 90^\circ - \measuredangle(A''A_1C)$

$$\frac{X''A''}{X''A'} = \frac{\sigma(A_1A''X'')}{\sigma(A_1A'X'')} = \frac{A_1A''\sin(X''A_1A'') \cdot A_1X'' \cdot \frac{1}{2}}{A_1A'\sin(X''A_1A') \cdot A_1X'' \cdot \frac{1}{2}} = \frac{A_1A''}{A_1A'} \cdot \frac{\sin(A_1CA'')}{\sin(A_1BA')}$$

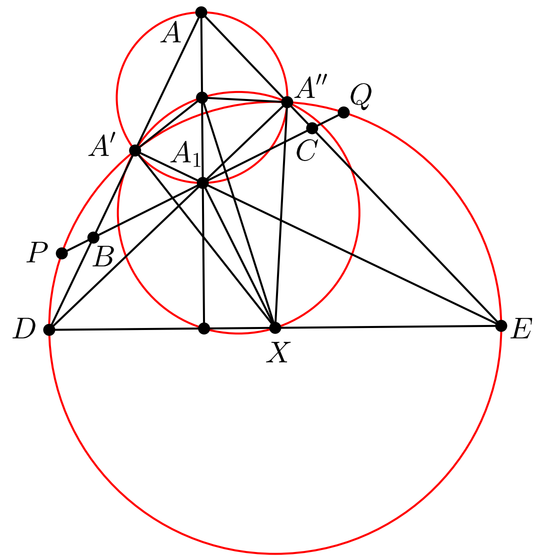
$\triangle A_1A''C$ is a right triangle $\Rightarrow \sin(A_1CA'') = \dfrac{A_1A''}{A_1C}$

$\sin(A_1BA') = \dfrac{A_1A'}{A_1B}$ and $A_1C = A_1B \Rightarrow \dfrac{\sin(A_1CA'')}{\sin(A_1BA')} = \dfrac{A_1A''}{A_1A'} \Rightarrow \dfrac{X''A''}{X''A'} = \left(\dfrac{A_1A''}{A_1A'}\right)^2 \Rightarrow X' \equiv X''$

So, XA_1 is the mediator of BC.

Analogously, YB_1, ZC_1 are mediators $\Rightarrow XA_1, YB_1, ZC_1$ are concurrent lines.

Second solution by Li Zhou, Polk State College, FL, USA



Suppose that AB intersects A_1A'' at D and AC intersects A_1A' at E. It is easy to see that A_1 is the orthocenter of $\triangle ADE$, and A', A'' lie on the circle Ω with diameter DE and center X. Extend A_1B and A_1C to intersect Ω at P and Q, respectively. By Haruki's lemma, $PB \cdot A_1Q/BA_1 = PA_1 \cdot CQ/A_1C$. Writing $PB = PA_1 - BA_1$ and $CQ = A_1Q - A_1C$ and using the condition $BA_1 = A_1C$, we get $PA_1 = A_1Q$. Therefore, XA_1 is the perpendicular bisector of BC, and thus XA_1, YB_1, ZC_1 concur at the circumcenter of $\triangle ABC$.

O532. Points D and E lie on sides AB and BC of triangle ABC, respectively. Assume the incenter I of triangle ABC lies inside quadrilateral $ADEC$ and $BD > AD$. Let$r$ be the inradius of triangle ABC and suppose the distance x from point I to line DE satisfies

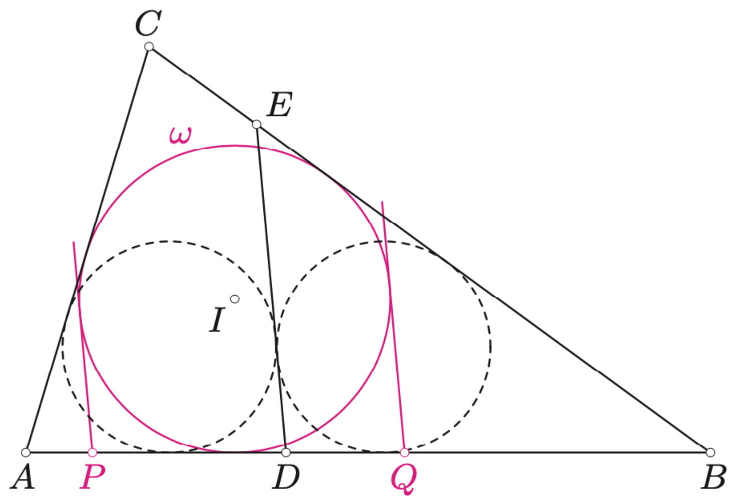
$$\frac{x}{r} = \frac{BD - AD}{AB}.$$

Prove that the inradius of triangle BDE is equal to the radius of a circle tangent to segment AD and to rays AC, DE.

Proposed by Waldemar Pompe, Warsaw, Poland

First solution by the author

Denote by r_1 the radius of the circle tangent to segment AD and to rays AC, DE, and by r_2 the inradius of triangle BDE.



Let ω be the incircle of triangle ABC. Draw two tangent lines to ω parallel to DE, which intersect line AB at points P and Q, as shown in the figure above. Then, by homotheties with centers A and B,

$$\frac{r_1}{r} = \frac{AD}{AQ} \quad \frac{r_2}{r} = \frac{BD}{BP}.$$

Therefore, in order to prove that $r_1 = r_2$, we need to verify that

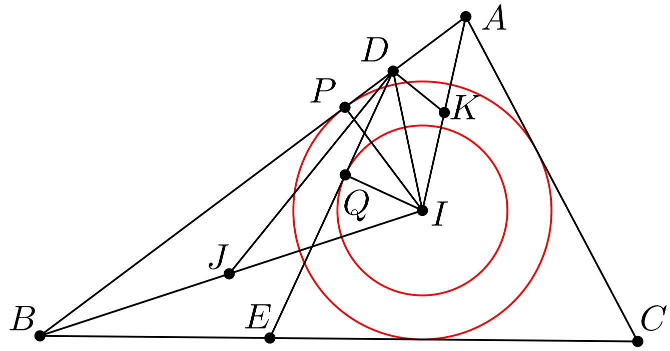
$$\frac{AD}{AQ} = \frac{BD}{BP}.$$

However, $\frac{AD}{BD} = \frac{r - x}{r + x} = \frac{DQ}{DP}$, which gives

$$\frac{AD}{BD} = \frac{AD + DQ}{BD + DP} = \frac{AQ}{BP}.$$

This is the desired equality.

Second solution by Li Zhou, Polk State College, FL, USA



Let J be the incenter of $\triangle BDE$ and K be the point where the bisector of $\angle ADE$ intersects AI. Note that $DJ \perp DK$. By the ratio lemma,

$$\frac{BJ}{JI} = \frac{BD \sin \angle BDJ}{ID \sin \angle JDI}, \quad \frac{AK}{KI} = \frac{AD \sin \angle KDA}{DI \sin \angle IDK} = \frac{AD \cos \angle BDJ}{DI \cos \angle JDI}.$$

Let P and Q be the orthogonal projections of I onto AB and DE, respectively. From $(BD - AD)/(BD + AD) = x/r$ and the sum-to-product formulas, we get

$$\frac{BD}{AD} = \frac{r + x}{r - x} = \frac{\sin \angle PDI + \sin \angle QDI}{\sin \angle PDI - \sin \angle QDI} = \frac{\tan \angle JDI}{\tan \angle BDJ}.$$

Therefore, $BJ/JI = AK/KI$, that is, $JK \parallel BA$, from which the claim follows.

O533. Let $a, b, c, d > 0$. Prove that

$$\frac{bcd}{a^2} + \frac{acd}{b^2} + \frac{abd}{c^2} + \frac{abc}{d^2} \geq 2\sqrt{a^2 + b^2 + c^2 + d^2}$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

Let $a = \dfrac{1}{x}, b = \dfrac{1}{y}, c = \dfrac{1}{z}, d = \dfrac{1}{u}$. The inequality is equivalent to

$$x^3 + y^3 + z^3 + u^3 \geq 2\sqrt{x^2y^2z^2 + y^2z^2u^2 + z^2u^2x^2 + u^2x^2y^2}$$
$$\Leftrightarrow (x^3 + y^3 + z^3 + u^3)^2 \geq 4(x^2y^2z^2 + y^2z^2u^2 + z^2u^2x^2 + u^2x^2y^2)$$

By the power average inequality

$$\left(\frac{x^3 + y^3 + z^3 + u^3}{4}\right)^{\frac{1}{3}} \geq \left(\frac{x^2 + y^2 + z^2 + u^2}{4}\right)^{\frac{1}{2}}$$

Therefore, we need to prove

$$(x^2 + y^2 + z^2 + u^2)^3 \geq 16(x^2y^2z^2 + y^2z^2u^2 + z^2u^2x^2 + u^2x^2y^2).$$

Now, let $p = x^2, q = y^2, r = z^2, t = u^2$ The inequlity rewrites as

$$(p + q + r + t)^3 \geq 16(pqr + qrt + rtp + tpq)$$
$$\Leftrightarrow \sum_{cyc} p^3 - \sum_{cyc} pqr + 3\sum_{cyc}(p^2q + q^2r + r^2p - 3pqr) \geq 0$$

By applying the three-dimensional mean value inequality, we conclude the result.

Also solved by Taes Padhihary, Disha Delphi Public School, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania.

O534. Let n be a positive integer. An arithmetic sequence is called n-small if it consists of integers and if its common difference has at most n positive divisors. Prove that there is $c > 0$ such that for any $N \geq 1$ the set $\{1^n, 2^n, 3^n, ...\}$ shares at most $cN^{1/n}$ elements with any n-small arithmetic sequence of length N.

Proposed by Titu Andreescu, USA and Gabriel Dospinescu, France

Solution by the authors

Let $e(k)$ be the number of positive divisors of k. We will prove a stronger result by induction on n (and one really needs this stronger claim to make the induction work!): for any $n \geq 1$ there is $c_n > 0$ such that for all $N \geq 1$, al lpolynomials $f \in \mathbb{Z}[X]$ of degree n and all integers a, d with $d > 0$, there are at most $c_n N^{1/n} e(d)^{n-1}$ numbers $x \in \mathbb{Z}$ such that $f(x) \in \{a, a+d, ..., a+(N-1)d\}$

This is obvious if $n = 1$, so assume it holds for n and let us prove it for $n+1$. Say f has degree $n+1$, fix an n-small sequence $a, a+d, ..., a+(N-1)d$ and consider the set A of $x \in \mathbb{Z}$ such that $f(x) \in \{a, a+d, ..., a+(N-1)d\}$

We may assume that A is nonempty. Fix $x_0 \in A$ and choose, if possible, a different element $x \in A$. Define the polynomial $g(X) = \dfrac{f(X) - f(x_0)}{X - x_0}$, it has integer coefficients and degree n. By $X - x_0$ assumption we can write $f(x_0) = a + ud$ and $f(x) = a + vd$ with $0 \leq u, v < N$. Thus $(x - x_0)g(x) = f(x) - f(x_0) = d(v - u)$. Let $d_1 = \gcd(d, x - x_0)$, so that d_1 divides d and $g(x) = \dfrac{d}{d_1}s$. Since $d_1 \mid x - x_0, r := \dfrac{v-u}{s}$ is an integer. If $|r| \leq N^{\frac{1}{n+1}}$, we have at most $2N^{\frac{1}{n+1}}$ possibilities for $x = x_0 + r_1 r$. Assume that this inequality fails, then using the obvious inequality $|u - v| < N$ we obtain $|s| < N^{\frac{n}{n+1}}$ and so $g(x) \in \{\frac{d}{d_1} \cdot s \mid |s| < N^{\frac{n}{n+1}}\}$. By the inductive hypothesis this happens for at most

$$c_n e(d/d_1)^{n-1} \left(2N^{\frac{n}{n+1}}\right)^{\frac{1}{n}} \leq 2c_n e(d)^{n-1} N^{\frac{1}{n+1}}$$

values of x. Finally, since d_1 is a positive divisor of d, the total number of possibilities for x is bounded by $e(d)\left(2N^{\frac{1}{n+1}} + 2c_n e(d)^{n-1} N^{\frac{1}{n+1}}\right)$ and the result follows with $c_{n+1} = 2c_n + 3$ (to take into account x_0).

Remark: By far the most delicate part in this problem is to realize that a more general result holds. Proving this more general statement is relatively simple, even though one can spend a lot of time with other strategies which fail.