Triangle BIC

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1 Orthocenter of BIC

![Orthocenter of BIC](image)

**Theorem 1** $\triangle ABC$ is a triangle with $I$ incenter and $H'$ is the orthocenter of $\triangle BIC$. The polar of $H'$ with respect to $\odot(I)$ is the A-midline in $\triangle ABC$.

**Proof.** Let $DEF$ be the contact triangle and $B'$ be the midpoint of $AC$, so we need to show that $B'$ lies on the polar of $H'$. Dealing with the polar of $B'$ is much better than working with polar of $H'$ and this motivates us to use La Hire.

It suffices to show that $H'$ lies on the polar of $B'$. Now let $BK$ be the tangent to $\odot(I)$ other than $BE$, so $KE$ is the polar of $B'$. It is known that $B, E', K$ are collinear where $E'$ is the reflection of $E$ in $I$. This means that it is enough to show that $\angle H'KB = 90^\circ = \angle HDB$ or just $\angle BH'D = \angle BKD$. Now,

$$\angle BH'D = \angle ICD = \angle IED = \angle BKD$$

which completes the proof.

**Corollary 1 (India TST 2014)** $H'E \perp IB'$.

The proof of this is left to the readers (as it is mentioned in the above proof itself). Let’s try this problem now:

Before moving we need to define certain points. $M$ is midpoint of $EF$, $N$ is the midpoint of the segment joining $D$ and the feet of perpendicular from $D$ on $EF$. $A', B', C'$ are the midpoints of the sides $BC, AC, AB$. The line $EF$ meets $A'B'$ at $T$.

**Problem (Iran TST 2009)** Prove that $H', M, N$ are collinear.

**Solution 1.** First realise that $C, T, H'$ are collinear. Now define $W = EF \cap BH'$. This problem is a beast just because the key triangle to focus on is $\triangle DWT$. Note that since $IT$ is a perpendicular bisector of $DF$,
we have $\angle WTI = \angle ITD$. Hence it follows that $I$ is the incenter of $DWT$. Since $H'T \perp IT, H'W \perp IW$, we see that $H'$ is the $D$-excenter of this triangle. Now using the fact that midpoint of $D$-altitude, the $D$-intouch point and the $D$-excenter are collinear, we're done! ■

There is another proof to this problem which has nothing to do with the incircle, but is definitely too good to be left out. Also, this is easier to motivate!

**Solution 2.** First of all, we rephrase the problem to get rid of the incircle and deal just with the contact triangle:

Let $\triangle ABC$ be a triangle with circumcenter $O$ and orthocenter $H$. $D$ and $E$ are intersection of the tangents to $\odot(ABC)$ at $A, B$ and $A, C$ respectively. $H'$ is the orthocenter of $\triangle DEO$. Prove that $H'$ lies on the $A$-Schwatt line of $\triangle ABC$.

Recall the property of the Schwatt Line - it is the locus of the centers of rectangles inscribed in the triangle. The stubbornness to use this fact motivates the whole solution. Therefore it suffices to show that $H'$ is the circumcenter of $\triangle AFG$ where $F, G$ are on $AB, AC$ such that $FG \parallel BC$ and $H'$ is equidistant to $FG, BC$. (why?)

The similarity motivates us to work on lengths. We have $AO \cdot AH' = AD \cdot AE$ as $H'$ is the orthocenter. Let $J$ be the foot of the $A-$ altitude. Now,

$$\frac{AO}{AD} = \frac{CJ}{AJ} \implies \frac{AO}{AH} = \frac{AO^2}{AD \cdot AE} = \frac{BJ \cdot CJ}{AJ^2}.$$  

Let $L$ be the reflection of $H$ in $BC$, so $L$ lies on $\odot(ABC)$. Therefore

$$\frac{LO}{AH} = \frac{AO}{AH} = \frac{BJ \cdot CJ}{AJ^2} = \frac{AH \cdot JA}{AJ^2} \implies \frac{JL}{LO} = \frac{AJ}{AH}.$$  

Now $\angle OLI = \angle JAH' \implies \triangle OLI \sim \triangle H'JA$. Combining this result with $JH' = H'K$ gives $\angle OJA = \angle LJI = \angle H'KJ$ where $K = AJ \cap FG$, so $JO \parallel KH'$ and this implies that $H'$ is the circumcenter of $\triangle AFG$ which completes the proof! ■

**Comments:** Indeed, this is the infamous Iran TST 2009/9 that gave a lemma the name ‘Iran Lemma’.
Exercise 1(a). Show that $H'$ is the reflection of $I_a$ in the midpoint of $BC$, denoted as $M$, where $I_a$ is the $A$-excenter.

If the previous point is denoted $I_A$, show that $I$ is the reflection of the orthocenter of $BIA_C$ about $M$.

Finally, can you say anything about the reflection of orthocenter of $AIA_C$ about the midpoint of $AC$?

Exercise 1(b). $AD$ is parallel to $H'A'$ where $A'$ is the midpoint of $BC$.

2 Circle BIC

Exercise 1. Prove that $A_1A_2$ is the $D-$midline of $\triangle DEF$. (Hint: Invert!)

Exercise 2. (a) Prove that $A_1, A_2$ are isogonal conjugates. (Hint: Incenter-Excenter Lemma)

Exercise 2. (b) Show that there are exactly six points on $(I)$ whose isogonal conjugates also lie on $(I)$.

Exercise 3. Prove that if $M$ is the midpoint of arc $BAC$, then $BM$ and $CM$ are tangent to $\odot(BIC)$.

Problem (2017 Belarus TST 3.1) Let $I$ be the incenter of a non-isosceles triangle $ABC$. The line $AI$ intersects $\odot(ABC)$ at $A$ and $D$. Let $M$ be the midpoint of the arc $BAC$. The line through $I$ perpendicular to $AD$ intersects $BC$ at $F$. The line $MI$ intersects the circle $BIC$ at $N$. Prove that the line $FN$ is tangent to $\odot(BIC)$.

Solution. From Exercise 3, $MB, MC$ is tangent to $\odot(BIC)$ which gives that $BICN$ is a harmonic quadrilateral. Clearly $FI$ is tangent to $\odot(BIC)$ so $FN$ must also be tangent and this completes the proof.
Example Problems

Problem (Russia 2013/11.8) Let $\triangle ABC$ be a triangle with incenter $I$. $\odot(BIC)$ meets $(I)$ again at $X$ and $Y$, and the two common tangents of $\triangle BIC$ and $(I)$ meet at point $Z$. Prove that $\odot(ABC)$ and $\odot(XYZ)$ are tangent to each other. Further, the tangency point $T$, lies on $\odot(AI)$.

The circle $\odot(BIC)$ immediately motivates the idea to invert about the incircle. And in this case, the inverted configuration is rather neat. We’ll denote the image of $W$ under inversion about $(I)$ as $W^*$.

(a) Redefine $T$ as $\odot(AI) \cap \odot(ABC)$ and show that $T^*$ is the feet from $D$ on $EF$.

(b) Show that $Z'$ lies on the perpendicular bisector of $XY$.

(c) Show that $\odot(T^*XY)$ is tangent to $\odot(A^*B^*C^*)$.

(d) Let $N$ be the midpoint of arc $XNY$. Show that it suffices to prove that $Z'$ is the reflection of $N$ in $XY$.

(e) Show that $NX^2 = NI \cdot NZ'$.

(f) Conclude by computing the power of the midpoint of $XY$.

Problem (USA TSTST 2016/6) Let $ABC$ be a triangle with incenter $I$, and whose incircle is tangent to $BC$, $CA$, $AB$ at $D$, $E$, $F$, respectively. Let $K$ be the foot of the altitude from $D$ to $EF$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points $C_1$ and $C_2$, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points $B_1$ and $B_2$. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint $M$ of $DK$.

(a) Show that midpoint of $EF$ lies on the radical axis.

(b) Show that it suffices to prove that $G$ lies on the radical axis, where $G$ is the Gergonne point.

(c) Define $A_1, A_2$ similarly. Let $A_b, A_c$ be points on $AB, AC$ such that $A_bA_c||EF$ and $G, A_b, A_c$ are collinear. Show that $A_b, A_c$ lies on $\odot(AA_1A_2)$.

(d) Conclude by finding the power of $G$.

Comments: The hardest part of this problem was the motivation to introduce any of the Gergonne point or the orthocenter of $\triangle BIC$. The rest of the problem follows fairly quickly thereafter.
4 Problems

Problem 4.1. Let $ABC$ be a triangle. Prove that the poles of the sides of the medial triangle of $\triangle ABC$ with respect to the incircle of $\triangle ABC$ all lie on the Feuerbach hyperbola of $\triangle ABC$.

Problem 4.2. Let $I$ be the incenter of a scalene $\triangle ABC$. Prove that if $D, E$ are two points on rays $BA, CA$, satisfying $BD = CA, AB = CE$ then line $DE$ pass through the orthocenter of $\triangle BIC$.

Problem 4.3 (IMO 2009 Shortlist G3). Let $ABC$ be a triangle. The incircle of $ABC$ touches the sides $AB$ and $AC$ at the points $Z$ and $Y$, respectively. Let $G$ be the point where the lines $BY$ and $CZ$ meet, and let $R$ and $S$ be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

Problem 4.4 (2017 Belarus Team Selection Test 3.1). Let $I$ be the incenter of a non-isosceles triangle $ABC$. The line $AI$ intersects the circumcircle of the triangle $ABC$ at $A$ and $D$. Let $M$ be the middle point of the arc $BAC$. The line through the point $I$ perpendicular to $AD$ intersects $BC$ at $F$. The line $MI$ intersects the circle $BIC$ at $N$. Prove that the line $FN$ is tangent to the circle $BIC$. 

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Here is the solution to the second walkthrough problem:

**Problem (USA TSTST 2016/6)** Let $ABC$ be a triangle with incenter $I$, and whose incircle is tangent to $BC$, $CA$, $AB$ at $D$, $E$, $F$, respectively. Let $K$ be the foot of the altitude from $D$ to $EF$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points $C_1$ and $C_2$, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points $B_1$ and $B_2$. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint $M$ of $DK$.

**Solution.** Denote by $Ge$ the Gergonne point of $\triangle ABC$, the contact triangle by $\triangle DEF$, $\odot(AIB) \cap \odot(BT_1'T_2)$ by $Z$, and the midpoint of $EF$ by $N$. I hope the diagram provided helps with the illustration.

**Lemma 1** $Ge, M, N$ are collinear.

**Proof.** Well-known. It’s the $D$-Schwatt line of $\triangle DEF$. □

**Claim 1** $N$ lies on the radical axis of $\odot(BB_1B_2)$ and $\odot(CC_1C_2)$.

**Proof.** Incircle inversion sends $A, B, C$ to the midpoints of $EF, FD, DE$ respectively. Hence $B_1B_2$ and $C_1C_2$ are just the midlines of $\triangle DEF$. Thus, $N$ lies on both of them. Now we’re done by PoP. □
Claim 2 Ge lies on the radical axis of \( \odot(BB_1B_2) \) and \( \odot(CC_1C_2) \).

The proof is rather involved and left as an exercise to the bold readers.

Lemma 2 The three lines through the symmedian point of a triangle parallel to each side cut the sides at 6 concyclic points. Also, the intersections closer to any vertex are antiparallel to the corresponding side.

Proof. Theorem 10 of this PDF by Cosmin Pohoata.

Step 1 Points \( E, T_1, T'_2, V, D \) are concyclic.

Proof. \( D, E, T'_1T'_2 \) are trivially concyclic (antiparallel). Now \( \angle DT'_1T_1 = \angle DU_2T_2 = \angle DEF = \angle DET_1 \).

Step 2 \( Z \) lies on \( \odot(BB_1B_2) \).

Proof. \( \angle BZT_2 = \angle VZT'_2 = \angle VDT'_2 = \angle FDB = BT'_1T'_2 \).

Step 3 \( N \) lies on the radical axis of \( \odot(DEF) \) and \( \odot(BT'_1T'_2) \).

Proof. Now note that the homothety from \( B \) sending \( GeV \) to \( EF \) sends \( V \) to \( N \) as the reflection of \( Ge \) about \( AB \), which proves that \( B, V, N \) are collinear. Now, since \( BF \parallel VT_1 \) and \( VT_1ZE \) is cyclic, we find \( BFZE \) is cyclic too. Hence, \( BN \cap NZ = NF \cap NE \), and we are done.

Step 4 \( T'_1, T'_2 \) lie on \( \odot(BB_1B_2) \).

Proof. Repeat Step 3 for the midpoint of \( DE \) to find that \( \odot(DEF), \odot(BT'_1T'_2), \odot(BB_1B_2) \) are coaxial.

Since \( T'_1 \) is the reflection of \( Ge \) about \( T_1 \), we find \( GeT'_1 \cdot GeT'_2 = 4GeT_1 \cdot GeT_2 \). Apply the same reasoning for \( \odot(CC_1C_2) \) to finish the problem.
References

1. Euclidean Geometry in Mathematical Olympiads by Evan Chen.
   https://web.evanchen.cc/geombook.html

2. The Art of Problem Solving website.
   https://artofproblemsolving.com/

3. Let’s Talk About Symmedians! by Sammy Luo and Cosmin Pohoata