Triangle BIC

Shuborno Das* Aditya Khurmi† Aatman Supkar‡
Arindam Bhattacharyya§

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*Contact at shuborno2107@gmail.com
†Contact at adityak1135@gmail.com
‡Contact at aatmansupkar@gmail.com
§Contact at arindampartha2002@gmail.com

✠1 Orthocenter of BIC

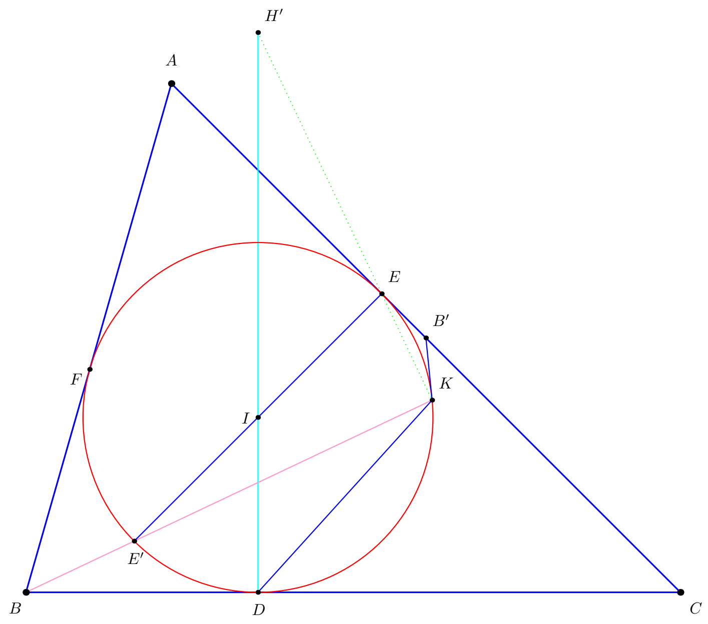


Figure 1: Orthocenter of BIC

Theorem 1 $\triangle ABC$ is a triangle with I incenter and H' is the orthocenter of $\triangle BIC$. The polar of H' with respect to $\odot(I)$ is the A-midline in $\triangle ABC$.

Proof. Let DEF be the contact triangle and B' be the midpoint of AC, so we need to show that B' lies on the polar of H'. Dealing with the polar of B' is much better than working with polar of H' and this motivates us to use La Hire.

It suffices to show that H' lies on the polar of B'. Now let BK be the tangent to $\odot(I)$ other than BE, so KE is the polar of B'. It is known that B, E', K are collinear where E' is the reflection of E in I. This means that it is enough to show that $\angle H'KB = 90^\circ = \angle HDB$ or just $\angle BH'D = \angle BKD$. Now,

$$\angle BH'D = \angle ICD = \angle IED = \angle BKD$$

which completes the proof.

□

Corollary 1 (India TST 2014) $H'E \perp IB'$.

The proof of this is left to the readers (as it is mentioned in the above proof itself). Let's try this problem now:

Before moving we need to define certain points. M is midpoint of EF, N is the midpoint of the segment joining D and the feet of perpendicular from D on EF. A', B', C' are the midpoints of the sides BC, AC, AB. The line EF meets $A'B'$ at T.

Problem (Iran TST 2009) Prove that H', M, N are collinear.

Solution 1. First realise that C, T, H' are collinear. Now define $W = EF \cap BH'$. This problem is a beast just because the key triangle to focus on is $\triangle DWT$. Note that since IT is a perpendicular bisector of DF, we have $\angle WTI = \angle ITD$. Hence it follows that I is the incenter of DWT. Since $H'T \perp IT, H'W \perp IW$, we see that H' is the D-excenter of this triangle. Now using the fact that midpoint of D-altitude, the D-intouch point and the D-excenter are collinear, we're done! ■

There is another proof to this problem which has nothing to do with the incircle, but is definitely too good to be left out. Also, this is easier to motivate!

Solution 2. First of all, we rephrase the problem to get rid of the incircle and deal just with the contact triangle:

Let $\triangle ABC$ be a triangle with circumcenter O and orthocenter H. D and E are intersection of the tangents to $\odot(ABC)$ at A, B and A, C respectively. H' is the orthocenter of $\triangle DEO$. Prove that H' lies on the A-Schwatt line of $\triangle ABC$.

Recall the property of the Schwatt Line - it is the locus of the centers of rectangles inscribed in the triangle. The stubbornness to use this fact motivates the whole solution. Therefore it suffices to show that H' is the circumcenter of $\triangle AFG$ where F, G are on AB, AC such that $FG||BC$ and H' is equidistant to FG, BC. (why?)

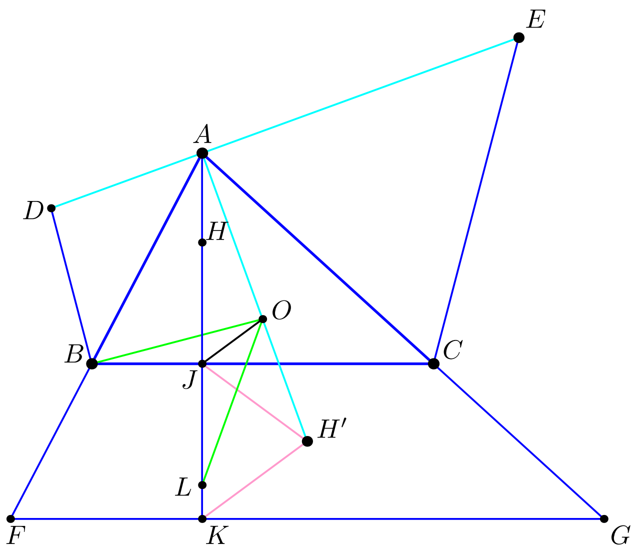


Figure 2: Iran TST 2009

The similarity motivates us to work on lengths. We have $AO \cdot AH' = AD \cdot AE$ as H' is the orthocenter. Let J be the foot of the $A-$ altitude. Now,

$$\frac{AO}{AD} = \frac{CJ}{AJ} \implies \frac{AO}{AH} = \frac{AO^2}{AD \cdot AE} = \frac{BJ \cdot CJ}{AJ^2}.$$

Let L be the reflection of H in BC, so L lies on $\odot(ABC)$. Therefore

$$\frac{LO}{AH} = \frac{AO}{AH} = \frac{BJ \cdot CJ}{AJ^2} = \frac{JH \cdot JA}{AJ^2} = \frac{JL}{AJ} \implies \frac{JL}{LO} = \frac{AJ}{AH}.$$

Now $\angle OLJ = \angle JAH' \implies \triangle OJL \sim \triangle H'JA$. Combining this result with $JH' = H'K$ gives $\angle OJA = \angle LJH' = \angle H'KJ$ where $K = AJ \cap FG$, so $JO||KH'$ and this implies that H' is the circumcenter of $\triangle AFG$ which completes the proof! ■

Comments: Indeed, this is the infamous Iran TST 2009/9 that gave a lemma the name 'Iran Lemma'.

Exercise 1(a). Show that H' is the reflection of I_a in the midpoint of BC, denoted as M, where I_a is the $A-$ excenter.

If the previous point is denoted I_A, show that I is the reflection of the orthocenter of BI_AC about M. Finally, can you say anything about the reflection of orthcenter of AI_AC about the midpoint of AC?

Exercise 1(b). AD is parallel to $H'A'$ where A' is the midpoint of BC.

✠2 Circle BIC

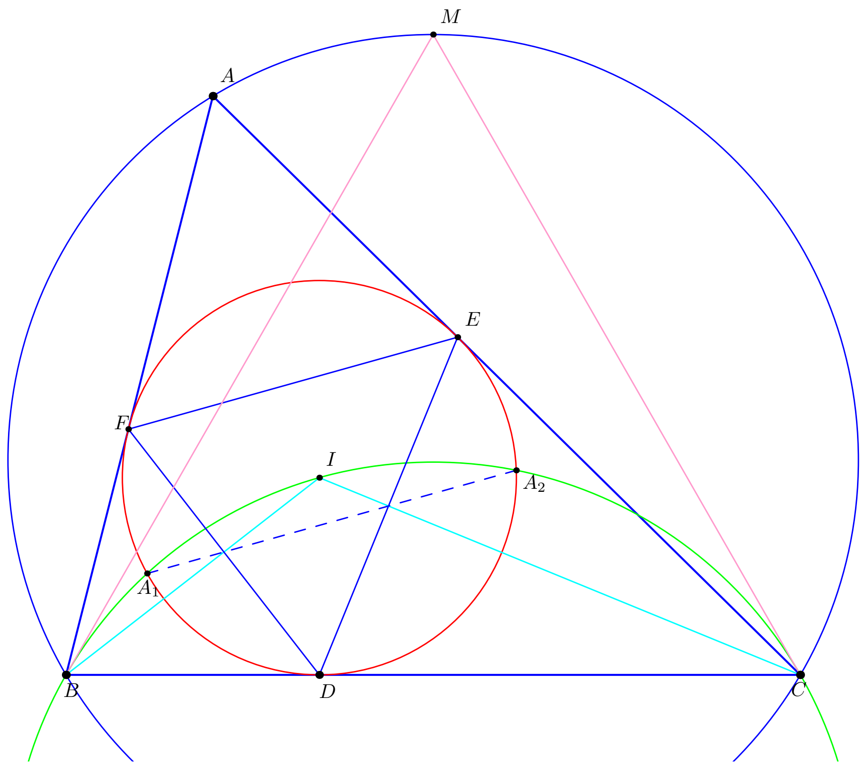


Figure 3: Circle BIC

Exercise 1. Prove that A_1A_2 is the $D-$ midline of $\triangle DEF$. (Hint: Invert!)

Exercise 2. (a) Prove that A_1, A_2 are isogonal conjugates. (Hint: Incenter-Excenter Lemma)

Exercise 2. (b) Show that there are exactly six points on (I) whose isogonal conjugates also lie on (I).

Exercise 3. Prove that if M is the midpoint of arc BAC, then BM and CM are tangent to $\odot(BIC)$.

Problem (2017 Belarus TST 3.1) Let I be the incenter of a non-isosceles triangle ABC. The line AI intersects $\odot(ABC)$ at A and D. Let M be the midpoint of the arc BAC. The line through I perpendicular to AD intersects BC at F. The line MI intersects the circle BIC at N.
Prove that the line FN is tangent to $\odot(BIC)$.

Solution. From Exercise 3, MB, MC is tangent to $\odot(BIC)$ which gives that $BICN$ is a harmonic quadrilateral. Clearly FI is tangent to $\odot(BIC)$ so FN must also be tangent and this completes the proof. ■

✠3 Example Problems

Problem (Russia 2013/11.8) Let $\triangle ABC$ be a triangle with incenter I. $\odot(BIC)$ meets (I) again at X and Y , and the two common tangents of $\triangle BIC$ and (I) meet at point Z. Prove that $\odot(ABC)$ and $\odot(XYZ)$ are tangent to each other. Further, the tangency point T, lies on $\odot(AI)$.

The circle $\odot(BIC)$ immediately motivates the idea to invert about the incircle. And in this case, the inverted configuration is rather neat. We'll denote the image of W under inversion about (I) as W^*.

(a) Redefine T as $\odot(AI) \cap \odot(ABC)$ and show that T^* is the feet from D on EF.

(b) Show that Z' lies on the perpendicular bisector of XY.

(c) Show that $\odot(T^*XY)$ is tangent to $\odot(A^*B^*C^*)$.

(d) Let N be the midpoint of arc XNY. Show that it suffices to prove that Z' is the reflection of N in XY.

(e) Show that $NX^2 = NI \cdot NZ'$.

(f) Conclude by computing the power of the midpoint of XY.

Problem (USA TSTST 2016/6) Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

(a) Show that midpoint of EF lies on the radical axis.

(b) Show that it suffices to prove that G lies on the radical axis, where G is the Gergonne point.

(c) Define A_1, A_2 similarly. Let A_b, A_c be points on AB, AC such that $A_bA_c || EF$ and G, A_b, A_c are collinear. Show that A_b, A_c lies on $\odot(AA_1A_2)$.

(d) Conclude by finding the power of G.

Comments: The hardest part of this problem was the motivation to introduce any of the Gergonne point or the orthocenter of $\triangle BIC$. The rest of the problem follows fairly quickly thereafter.

4 Problems

Problem 4.1. Let ABC be a triangle. Prove that the poles of the sides of the medial triangle of $\triangle ABC$ with respect to the incircle of $\triangle ABC$ all lie on the Feuerbach hyperbola of $\triangle ABC$.

Problem 4.2. Let I be the incenter of a scalene $\triangle ABC$. Prove that if D, E are two points on rays $\overrightarrow{BA}, \overrightarrow{CA}$, satisfying $BD = CA, AB = CE$ then line DE pass through the orthocenter of $\triangle BIC$.

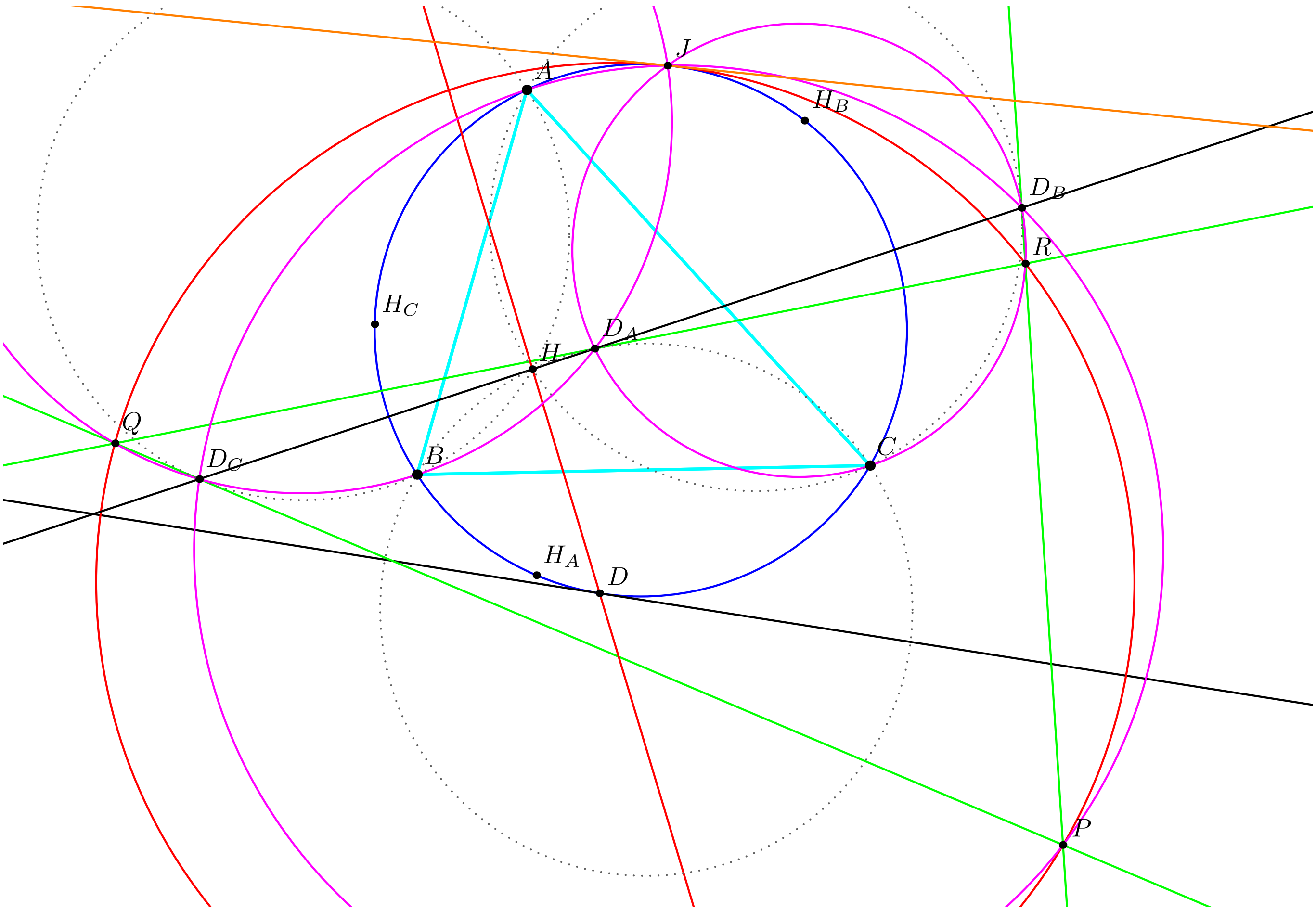
Problem 4.3 (IMO 2009 Shortlist G3)**.** Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

Problem 4.4 (2017 Belarus Team Selection Test 3.1)**.** Let I be the incenter of a non-isosceles triangle ABC. The line AI intersects the circumcircle of the triangle ABC at A and D. Let M be the middle point of the arc BAC. The line through the point I perpendicular to AD intersects BC at F. The line MI intersects the circle BIC at N. Prove that the line FN is tangent to the circle BIC.

✠5 Solutions to Example Problems

Here is the solution to the second walkthrough problem:

Problem (USA TSTST 2016/6) Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.



Solution. Denote by Ge the Gergonne point of $\triangle ABC$, the contact triangle by $\triangle DEF$, $\odot(AIB) \cap \odot(BT_1'T_2')$ by Z, and the midpoint of EF by N. I hope the diagram provided helps with the illustration.

Lemma 1 Ge, M, N are collinear.

Proof. Well-known. It's the D-Schwatt line of $\triangle DEF$. □

Claim 1 N lies on the radical axis of $\odot(BB_1B_1)$ and $\odot(CC_1C_2)$.

Proof. Incircle inversion sends A, B, C to the midpoints of EF, FD, DE respectively. Hence B_1B_2 and C_1C_2 are just the midlines of $\triangle DEF$. Thus, N lies on both of them. Now we're done by PoP. □

Claim 2 Ge lies on the radical axis of $\odot(BB_1B_2)$ and $\odot(CC_1C_2)$.

The proof is rather involved and left as an exercise to the bold readers.

Lemma 2 The three lines through the symmedian point of a triangle parallel to each side cut the sides at 6 concyclic points. Also, the intesections closer to any vertex are antiparallel to the corresponding side.

Proof. Theorem 10 of this PDF by Cosmin Pohoata. □

Step 1 Points E, T_1, T_2', V, D are concyclic.

Proof. $D, E, T_1'T_2'$ are trivially concyclic (antiparallel). Now $\measuredangle DT_2'T_1 = \measuredangle DU_2T_2 = \measuredangle DEF = \measuredangle DET_1$. □

Step 2 Z lies on $\odot(BB_1B_2)$.

Proof. $\measuredangle BZT_2' = \measuredangle VZT_2' = \measuredangle VDT_2' = \measuredangle FDB = BT_1'T_2'$. □

Step 3 N lies on the radical axis of $\odot(DEF)$ and $\odot(BT_1'T_2')$.

Proof. Now note that the homothety from B sending GeV to EF sends V to N as the reflection of Ge about V lies on AB, which proves that B, V, N are collinear. Now, since $BF||VT_1$ and VT_1ZE is cyclic, we find $BFZE$ is cyclic too. Hence, $BN \cap NZ = NF \cap NE$, and we are done. □

Step 4 T_1', T_2' lie on $\odot(BB_1B_2)$.

Proof. Repeat Step 3 for the midpoint of DE to find that $\odot(DEF), \odot(BT_1'T_2'), \odot(BB_1B_2)$ are coaxial. □

Since T_1' is the reflection of Ge about T_1, we find $GeT_1' \cdot GeT_2' = 4GeT_1 \cdot GeT_2$. Apply the same reasoning for $\odot(CC_1C_2)$ to finish the problem.

✠6 References

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