ANTI-STEINER POINT REVISITED

Nguyen Duc Toan
Le Quy Don High School For The Gifted, Da Nang, Vietnam

Abstract

Many students, although familiar with problems involving the Steiner line, may not be aware of Anti-Steiner point applications. In this article, we introduce some properties and problems related to the Anti-Steiner point of a line with respect to a triangle. This is suitable for high school students, particularly those involved in mathematical Olympiads.

Note. In this article, we label that \( \odot(O) \) is the circle with center \( O \), \( \odot(XYZ) \) is the circumcircle of triangle \( XYZ \), \( (d, d') \) is the directed angle of lines \( d \) and \( d' \) modulo \( \pi \), and \( AB \) is the algebraic length of segment \( AB \).

1 Anti-Steiner point definition

First, the Steiner’s theorem about the Steiner line is commonly known and used in olympiad mathematics. The theorem is illustrated below.

Theorem 1 (Steiner). Let \( ABC \) be a triangle with orthocenter \( H \). \( D \) is a point on the circumcircle of triangle \( ABC \). Then, the reflections of \( D \) in three edges \( BC, CA, AB \) and point \( H \) lie on a line \( l \). We call that \( l \) is the Steiner line of point \( D \) with respect to triangle \( ABC \).

Now, let’s discuss the converse of the Steiner’s theorem regarding the Steiner line.

Theorem 2 (Collings). Let \( ABC \) be the triangle with orthocenter \( H \). Given a line \( l \) passing through \( H \). Denote that \( a', b', c' \) are the reflections of \( l \) in the edges \( BC, CA, AB \), respectively. Then, the lines \( a', b', c' \) are concurrent at a point \( T \) on the circumcircle of triangle \( ABC \).

Proof. First, we have a lemma.

Lemma 1.1. Let \( ABC \) be the triangle with altitudes \( AD, BE, CF \) and orthocenter \( H \). The line \( AD \) meets the circumcircle of triangle \( ABC \) at the second point \( H_a \). Then, \( H_a \) is the reflection of \( H \) in line \( BC \).
Proof. We can easily see that $A, C, D, F$ lie on the circle with diameter $BC$. Then,

$$(CH, CD) \equiv (CF, CD) \equiv (AF, AD) \equiv (AB, AH_a) \equiv (CB, CH_a) \equiv (CD, CH_a)$$

Similarly, we have $(BH, BD) \equiv (BD, BH_a)$. Then, $H_a$ is the reflection of $H$ in line $BC$.

Back to our main proof,

Let $D, E, F$ be the intersections of $l$ and three edges $BC, CA, AB$, respectively. According to the Lemma 1.1., we have $H_a, H_b, H_c$ are the reflections of $H$ in $BC, CA, AB$, respectively. Hence $H_aD, H_bE, H_cF$ are the reflections of $l$ in lines $BC, CA, AB$, respectively. Therefore,

$$(EH_b, FH_c) \equiv (EH_b, CA) + (AC, AB) + (AB, FH_c)$$

$$\equiv (Ca, l) + (AC, AB) + (l, AB)$$

$$\equiv 2(AC, AB)$$

$$\equiv (H_aH_b, H_aH_c)$$

Then, we have the intersection of lines $EH_b$ and $FH_c$ lying on $\odot(ABC)$. Similarly, the intersection of lines $EH_b$ and $DH_a$ lies on $\odot(ABC)$. Therefore, $H_aD, H_bE, H_cF$ are concurrent at a point $T$ on $\odot(ABC)$.

Note. We call that $T$ is the Anti-Steiner point of line $l$ with respect to triangle $ABC$. Moreover, given a point $K$ lying on line $l$. We can also call that $T$ is the Anti-Steiner point of point $K$ with respect to triangle $ABC$.

Next, I will introduce to you an extension of the Anti-Steiner point definition in a complete quadrilateral from a popular theorem.

Theorem 3. Given a convex quadrilateral $ABCD$ that no side is parallel to another side. $AB$ meets $CD$ at $E$. $AD$ meets $BC$ at $F$. Let $M$ be the Miquel point of the complete quadrilateral $AC.BD.EF$. Then, the orthocenters of four triangles $BCE, CDF, ADE, ABF$ lie on line $l$.
Note. We call that line $l$ is the Steiner line of the complete quadrilateral $AC.BD.EF$. Also, we can see that $M$ is Anti-Steiner point of line $l$ with respect to triangles $BCE, CDF, ADE, ABF$. So, we call that $M$ is the Anti-Steiner point of the complete quadrilateral $AC.BD.EF$.

2 Concurrency related problems

Theorem 4. Given triangle $ABC$ and an arbitrary point $P$ on the plane. $P_a, P_b, P_c$ are the reflections of $P$ in $BC, CA, AB$, respectively. Lines $AP, BP, CP$ meet the circumcircle of $\triangle ABC$ at $D, E, F$, respectively. Then, the circumcircles of triangles $AP_bP_c$, $BP_aP_c$, $CP_aP_b$, $PP_aD$, $PP_bD$, $PP_cE$, $PP_eF$ pass through the Anti-Steiner point $S$ of $P$ with respect to triangle $ABC$.

Solution. Denote that $H$ is the orthocenter of triangle $ABC$. $AH, BH, CH$ meets $\odot(ABC)$ at the second points $H_a, H_b, H_c$, respectively. According to the prove of Collings's above, we have $P_aH_a, P_bH_b, P_cH_c$ are concurrent at the Anti-Steiner point $S$ of $P$ with respect to triangle $ABC$. We will prove that $\odot(AP_bP_c)$ and $\odot(PP_aD)$ pass through $S$. The other cases can be proved similarly.
By directed angle chasing, we have,

\[(SP_b, SP_c) \equiv (H_b P_b, H_c P_c)\]
\[\equiv (H_b P_c, AC) + (AC, AB) + (AB, H_c P_c)\]
\[\equiv (AC, HP) + (AC, AB) + (HP, AB)\]
\[\equiv 2(AC, AB)\]
\[\equiv (AP_b, AP_c)\]

Hence, \(S, A, P_b, P_c\) are concyclic. Besides,

\[(SD, SP_a) \equiv (SD, SH_a)\]
\[\equiv (AD, AH_a)\]
\[\equiv (PD, PP_a)\]

Hence, \(S, D, P_a, P\) are concyclic. Therefore, we have \(\odot(AP_b P_c)\) and \(\odot(PP_a D)\) pass through \(S\). Similarly, the other circumcircles pass through \(S\).

Comment. This is a very popular and useful theorem used to solve the problem related to the Anti-Steiner point. A part of this theorem is the geometric problem from the China TST 2016. Now we can see some other applications of this theorem into some Olympiad problems.

**Problem 2.1 (EGMO 2017 - Problem 6).** Let \(ABC\) be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid \(G\) and the circumcentre \(O\) of \(ABC\) in its sides \(BC, CA, AB\) are denoted by \(G_1, G_2, G_3\) and \(O_1, O_2, O_3\), respectively. Show that the circumcircles of triangles \(G_1 G_2 C, G_1 G_3 B, G_2 G_3 A, O_1 O_2 C, O_1 O_3 B, O_2 O_3 A\) and \(ABC\) have a common point.

**Solution.** Denote \(T\) be the Anti-Steiner point of the Euler line of triangle \(ABC\) with respect to the triangle. According to Theorem 4., we can easily have \(\odot(G_1 G_2 C), \odot(G_1 G_3 B), \odot(G_2 G_3 A), \odot(O_1 O_2 C), \odot(O_1 O_3 B), \odot(O_2 O_3 A)\) pass through \(T\) lying on \(\odot(ABC)\).

**Problem 2.2.** Denote triangle \(ABC\) and an arbitrary point \(P\) (\(P\) does not lie on \(\odot(ABC)\)). \(PA, PB, PC\) meet \(\odot(ABC)\) at the second points \(D, E, F\) respectively. The perpendicular bisectors of segments \(PD, PE, PF\) meet each other creating triangle \(A'B'C'\). Prove that \(AA', BB', CC'\) are concurrent at a point on \(\odot(ABC)\), and \(PT\) passes through the Anti-Steiner point of \(P\).
Solution. Let $K$ be the Anti-Steiner point of $P$ with respect to triangle $ABC$. $PT$ meets $\circ(ABC)$ at the second point $T$. We will show that $A, A', T$ are collinear. Then, we can have $AA', BB', CC'$ are concurrent at $T$. Indeed, let $M$ be the projection of $P$ to $BC$, $X$ be the reflection of $P$ in $BC$. According to the Theorem 4., we have $P, X, D, K$ are concyclic. Moreover, we can easily see that $A'$ is the center of $\circ(EPF)$. Let $P_a$ be the antipode of $P$ in $\circ(PEF)$. Let $\Phi$ be the inversion with center $P$ and power $PA \cdot PD$. We have,

$$\Phi : P \leftrightarrow P, T \leftrightarrow K, A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F, \circ(PEF) \leftrightarrow BC$$

Since $\angle(PM, BC) = \angle(PP_a, \circ(PEF)) = 90^\circ$. Hence,

$$\Phi : PM \leftrightarrow PP_a, M \leftrightarrow P_a$$

Since, $A', M$ are the midpoints of segments $PP_a$ and $PX$. Hence,

$$\Phi : A' \leftrightarrow X, \circ(PXKD) \leftrightarrow \overline{A'TA}$$

Therefore, we have $A', T, A$ are collinear. Similarly, we have $B', B, T$ are collinear, and $C', C, T$ are collinear. Then, $AA', BB', CC'$ are concurrent at $T$.

3 Two circles tangent to each other at the Anti-Steiner point

In some problems about two circles tangent to each other, the tangent point is the Anti-Steiner point or a point related to it. In many cases, defining the tangent point may help us to find the way to solve the problem and explore many other properties.

Problem 3.1 (Le Phuc Lu). Let $ABC$ be a triangle. Suppose that perpendicular bisector of $AB$ cuts $AC$ at $A_1$, perpendicular bisector of $AC$ cuts $AB$ at $A_2$. Similarly define $B_1, B_2, C_1, C_2$. Three lines $A_1A_2, B_1B_2, C_1C_2$ cut each other creating a triangle $DEF$. Prove that the circumcircle of triangle $DEF$ is tangent to the circumcircle of triangle $ABC$. 

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Solution. First, we have a converse Steiner’s Theorem.

**Lemma 3.1.** Let $ABC$ be a triangle with orthocenter $H$ and circumcenter $O$. $K$ is a point on plane such that the reflections of $K$ in $BC, CA, AB$ are collinear with $H$. Then, $K$ lies on the circumcenter of triangle $ABC$.

Back to our main problem.

Let $E$ be the intersection of $A_bA_c$ and $BC$. Let $K$ be the Miquel point of complete quadrilateral $AE.BA_c.A_bC$. We can easily see that $O$ is the orthocenter of triangle $AA_bA_c$, hence, $OH$ is the Steiner line of the complete quadrilateral $AE.BA_c.A_bC$. Let $K_1, K_2$ be the reflections of $K$ in $AB$ and $AC$, respectively, $H_1$ be the orthocenter of triangle $EA_cC$. Then, we have $O, K_1, K_2, H_1$ are collinear. Moreover, $O$ and $H_1$ are the orthocenters of triangles $C_bC_c$ and $EA_cC$, respectively. According to the **Lemma 3.1.**, $K \in \odot(CaCcB)$ via $K \in \odot(CEA_c)$. Hence, $K$ is the Miquel point of the complete quadrilateral $C_aE.C_bA_c.B'C$. Hence,

$$(KE, KB') \equiv (C_bE, C_bB') \equiv (C_bB, C_bC_a) \equiv (AB, AC)$$

Given a point $A_1$ on $\odot(O)$ such that $AA_1 \parallel BC$. Denote $B_1, C_1$ similarly. We have,

$$(KE, KB_1) \equiv (KE, KC) + (KC, KB_1)$$
$$\equiv (A_cE, A_cC) + (A_cC, AB_1)$$
$$\equiv (CB, CA) + (BA, BC)$$
$$\equiv (AB, AC)$$

Then, we have $(KE, KB_1) \equiv (KE, KB')$. Therefore, $K, B, B_1$ are collinear. Similarly, $K, A_1, A'$ are collinear, and $K, C_1, C'$ are collinear. We can easily see that $A_1B_1 \parallel A'B', A_1C_1 \parallel A'C'$, and $B_1C_1 \parallel B'C'$. Hence, $K$ is the center of the homothety $\Phi$ mapping $\triangle A_1B_1C_1$ to $\triangle A'B'C'$. Besides, $K$ lie on both circles $\odot(A_1B_1C_1)$ and $\odot(A'B'C')$. Therefore, $\odot(ABC)$ is tangent to $\odot(A'B'C')$ at the Anti-Steiner point $K$. 

\[\Box\]
Problem 3.2 (Dao Thanh Oai). Denote triangle $ABC$ and an arbitrary point $P$. The line through $P$ and perpendicular to $BC$ meets $AB, AC$ at $A_c, A_b$, respectively. The line through $P$ and perpendicular to $CA$ meets $BC, BA$ at $B_a, B_c$, respectively. The line through $P$ and perpendicular to $AB$ meets $CA, CB$ at $C_b, C_a$, respectively. $\odot(PC_bA_b) \cap \odot(PA_cB_c) = \{D; P\}$, $\odot(PA_cB_c) \cap \odot(PB_aC_a) = \{E; P\}$, $\odot(PB_aC_a) \cap \odot(PA_bC_b) = \{F; P\}$. Prove that $\odot(DEF)$ and $\odot(ABC)$ are tangent to each other at the Anti-Steiner point of $P$.

Solution. Let $S$ be the Anti-Steiner point of $P$ with respect to triangle $ABC$. We will prove that $\odot(DEF)$ passes through point $S$. Indeed, let $P_a, P_b, P_c$ be the reflections of $P$ in $BC, CA, AB$, respectively. According to the Lemma 3.1., since the reflections of $S$ in $AB, AC$ are collinear with orthocenter $P$ of triangle $AB, C_b, S \in \odot(AB, C_b)$. According to the Lemma 1.1., we have $P_b, P_c$ also lie on the $\odot(AB, C_b)$. Moreover, we have,

$$(DB_c, DC_b) \equiv (DB_c, DP) + (DP, DC_b)$$

$$\equiv (A_cB_c, A_cP) + (A_bP, A_bC_b)$$

$$\equiv (AB, AC)$$

$$\equiv (AB_c, AC_b)$$

Hence, $D \in (AB_cC_b)$. Therefore, six points $S, A, P_a, P_c, C_b, B_c$ are concyclic. Similarly, we have $S, B, P_a, P_c, C_a, A_c$ are concyclic, and $S, B, P_a, P_c, A_b, B_a$ are concyclic. Therefore,

$$(DE, DF) \equiv (DE, DP) + (DP, DF)$$

$$\equiv (A_cE, A_cP) + (A_bP, A_bF)$$

$$\equiv (A_cE, A_bF)$$

$$\equiv (A_cE, A_cB) + (A_cB, A_bC) + (A_bC, A_bF)$$

$$\equiv (SE, SB) + (AB, AC) + (SC, SF)$$

$$\equiv (SE, SB) + (SB, SC) + (SC, SF)$$

$$\equiv (SE, SF)$$
Hence, $S, D, E, F$ are concyclic. Denote that $St$ is the tangent at $S$ of $\odot(DEF)$. We will show that $St$ is tangent to $\odot(ABC)$. Indeed,

$$(St, SA) \equiv (St, SD) + (SD, SA)$$

$$\equiv (ES, DE) + (C_b D, C_b A)$$

$$\equiv (ES, E A_c) + (E A_c, ED) + (C_b D, C_b A_b)$$

$$\equiv (BS, B A_c) + (P A_c, PD) + (PD, P A_b)$$

$$\equiv (BS, BA)$$

Hence, $St$ is tangent to $\odot(ABC)$. Therefore, $\odot(DEF)$ is tangent to $\odot(ABC)$ at the Anti-Steiner point $S$.

**Problem 3.3 (Nguyen Van Linh).** Let $ABC$ is a triangle with circumcenter $O$ and orthocenter $H$. $OH$ meets $BC, CA, AB$ at $X, Y, Z$ respectively. The lines passing through $X, Y, Z$ and perpendicular to $OA, OB, OC$ respectively meet each other that create triangle $A'B'C'$. Prove that $\odot(A'B'C')$ is tangent to $\odot(O)$.

**Solution.** Let $N$ be the Anti-Steiner point of the line $OH$ with respect to triangle $ABC$. Denote that $L$ is the reflection of $N$ in $OH$. Hence, $L \in \odot(O)$. We have $A$ is the excenter of triangle $NYZ$. Hence, $\angle YNZ = 180^\circ - 2 \angle BAC = \angle ZA'Y$. Since, $N$ and $L$ are symmetric with respect to $OH$, $\angle YLZ = \angle YNZ$. Hence, $\angle YLZ = \angle ZA'Y$ or $L \in \odot(A'YZ)$. Lines $AH, BH, CH$ meet $\odot(O)$ at second points $D, E, F$, respectively. Let $A'', B'', C''$ be the antipodes of $D, E, F$ in $\odot(O)$, respectively. We can easily see that $EF \perp AO$, hence $B''C'' \parallel EF \parallel B'C'$. Similarly, we can have that there is a homothety $\Phi$ mapping $\triangle A'B'C'$ to $\triangle A''B''C''$. We will show that $A'A'', B'B'', C'C''$ are concurrent at point $L$.

Indeed, we have $E, Z, N$ are collinear, hence $\angle OEZ = \angle ONZ = \angle OLZ$. Hence, $E, Z, O, L$ are concyclic. We can see that $A'A'', L$ are collinear if and only if $\angle A'LD = 90^\circ$, which equivalent to $\angle A'YZ + \angle ZLD = 90^\circ$. Note that $YA' \perp OB$ then $\angle A'YZ + \angle ZLD = 90^\circ \iff \angle ZOB = \angle ZLD \iff \angle AYO - \angle ABO = \angle ZLO + \angle OLD = \angle NEC'' + 90^\circ - \angle DAL = \frac{1}{2} \angle NOC'' + \angle ACB - \frac{1}{2} \angle COL$ (1). Let $S$ be the reflection of $C$ in $OH$. We have $\angle NOC = \angle LOS$, hence $\angle CHO = \angle NEC = \angle LNS$. Besides, $LN \perp OH$ and $NS \perp CH$. Hence, $EC, EO$ are isogonal conjugate lines in $\angle NES$, or
\[ \angle \text{NOC}'' = \angle \text{COS}. \] Hence, (1) \( \iff \angle \text{ACB} - \angle \text{NEC} = \frac{1}{2} \angle \text{NOC}'' + \angle \text{ACB} - \frac{1}{2} \angle \text{COL}. \] This is equivalent to \( \angle \text{COL} = \angle \text{NOC}'' + \angle \text{NOC} = \angle \text{COS} + \angle \text{SOL}, \) which is correct. Therefore, we have \( A'A'', B'B'', C'C'' \) are concurrent at point \( L. \) Hence, \( L \) is the center of the homothety \( \Phi. \) Then, \( \circ(A'B'C') \) is tangent to \( \circ(A''B''C'') \) at point \( L. \)

**Problem 3.4 (IMO Shortlist 2018 - G5).** Let \( ABC \) be a triangle with circumcircle \( \Omega \) and incentre \( I. \) A line \( \ell \) intersects the lines \( AI, BI, \) and \( CI \) at points \( D, E, \) and \( F, \) respectively, distinct from the points \( A, B, C, \) and \( I. \) The perpendicular bisectors \( x, y, \) and \( z \) of the segments \( AD, BE, \) and \( CF, \) respectively determine a triangle \( \Theta. \) Show that the circumcircle of the triangle \( \Theta \) is tangent to \( \Omega. \)

![Diagram of triangle ABC with incentre I and perpendicular bisectors](image)

**Solution.** Denote that \( X, Y, Z \) are the midpoints of the minor arcs \( BC, CA, AB \) of \( \circ(ABC). \) We can easily see that \( XB = XC = XI, ZI = ZB = ZA, YI = YC = YA. \) Then \( I \) is the orthocenter of triangle \( XYZ. \) Line \( d \) passing through \( I \) and parallel to \( DE \) meets \( XZ, XY \) at \( U, T \) respectively. Let \( K \) be the Anti-Steiner point of \( UT \) with respect to triangle \( XYZ. \) We can easily see that \( B, U, K \) are collinear and \( C, T, K \) are collinear. \( BK \) meets \( DE \) at \( V. \) Then,

\[ (EV, EB) \equiv (IU, IB) \equiv (BI, BU) \equiv (BE, BV) \]

Then \( V \) lies on the perpendicular bisector of segment \( BE. \) Similarly, denote \( W \) be the intersection of \( CK \) and \( DE, \) then \( W \) lies on the perpendicular bisector of segment \( CF. \) Let \( Z', X' \) are the intersections of the perpendicular bisector of segment \( EB \) with \( KZ \) and \( KX \) respectively. We have,

\[ \frac{KY}{KY'} = \frac{KT}{KW} = \frac{KU}{KV} = \frac{KZ}{KZ'} = \frac{KX}{KX'} \Rightarrow Z'Y' \parallel ZY, X'Y' \parallel XY \]

Hence \( X'Y'Z' \) is the triangle \( \Theta. \) Let \( \Phi \) is the homothety with center \( K \) and ratio \( \frac{KY'}{KY}, \) we have

\[ \Phi : X \mapsto X', Y \mapsto Y', Z \mapsto Z', \circ(XYZ) \mapsto \circ(X'Y'Z') \]

Moreover, \( K \) lies on both \( \circ(XYZ) \) and \( \circ(X'Y'Z'). \) Therefore, \( \circ(XYZ) \) is tangent to \( \circ(X'Y'Z') \) at \( K. \)
The orthopole of a line through the circumcenter with respect to the original triangle

First, I will introduce the definition of the orthopole.

**Problem 4.1.** Given triangle $ABC$ and line $l$. Let $X,Y,Z$ be the projections of $A,B,C$ to $l$, respectively. Prove that, the lines passing through $X,Y,Z$ and perpendicular to $BC,CA,AB$ respectively are concurrent at a point $S$.

**Solution.** First, we have two lemmas.

**Lemma 4.1.** Given the line $AB$ and a constant $k$. Denote that $O$ is the midpoint of segment $AB$, $H$ is a point on segment $AB$ such that $\frac{OH}{2AB} = k$. Then, the set of the points $M$ such that $MA^2 - MB^2 = k$ is the line $\triangle$ passing through $H$ and perpendicular to $AB$.

![Diagram](image)

**Proof.** We have,

$$MA^2 - MB^2 = (M'M^2 + M'A^2) - (M'M^2 + M'B^2)$$
$$= M'A^2 - M'B^2$$
$$= (M'A - M'B)(M'A + M'B)$$
$$= BA(M'O + OA + M'O + OB)$$
$$= BA2M'O$$
$$= 2OM'.AB$$

Therefore, we have

$$MA^2 - MB^2 = k \iff OM' = \frac{k}{2AB}$$
$$\iff OM' = OH$$
$$\iff M' \equiv H$$
$$\iff M \in \triangle$$

**Lemma 4.2 (Four-point Lemma).** Given two lines $AB$ and $CD$. Then, $AB \perp CD$ if and only if $AC^2 - AD^2 = BC^2 - BD^2$.

![Diagram](image)
Proof. First, assume that $AB \perp CD$.
Let $H$ be the intersection of $AB$ and $CD$. According to the Pythagorean Theorem, we have,

$$AC^2 - AD^2 = (HA^2 + HC^2) - (HA^2 + HD^2)$$
$$= (HB^2 + HC^2) - (HB^2 + HD^2)$$
$$= BC^2 - BD^2$$

Second, assume that $AC^2 - AD^2 = BC^2 - BD^2$.
Let $H, H'$ be the projections of $A, B$ to $CD$. According to the Pythagorean Theorem, we have,

$$HC^2 - HD^2 = (HA^2 + HC^2) - (HA^2 + HD^2)$$
$$= AC^2 - AD^2$$
$$= BC^2 - BD^2$$
$$= (H'B^2 + H'C^2) - (H'B^2 + H'D^2)$$
$$= H'C^2 - H'D^2$$

Then, according to the Lemma 4.1., we have $AB \perp CD$.

Back to our main problem

Let $S$ be the intersection of the line passing through $Y, Z$ and perpendicular to $CA, AB$, respectively. We will show that $SX \perp BC$. Indeed, according to the Four-point Lemma and the Pythagoras’s theorem, we have,

$$SB^2 - SC^2 = (SB^2 - SA^2) - (SC^2 - SA^2)$$
$$= (ZB^2 - ZA^2) - (YC^2 - YA^2)$$
$$= ((BY^2 + YZ^2) - (ZX^2 + XA^2)) - ((YC^2 + ZC^2) - (YX^2 + XA^2))$$
$$= (BY^2 + XY^2) - (ZC^2 + ZX^2)$$
$$= (XB^2 - XC^2)$$

According to the Four-point Lemma, we have $SX \perp BC$.

Note. The concurrent point $S$ in the problem is called the orthopole of the line $l$ with respect to the triangle $ABC$.

Next, I will introduce three popular theorems related to the orthopole from Fontene.
Theorem 5 (The first Fontene’s Theorem). Given triangle $ABC$ and an arbitrary point $P$ on the plane. Let $A_1, B_1, C_1$ be the midpoints of segments $BC, CA, AB$, respectively. Let $\triangle A_2B_2C_2$ be the Pedal triangle of $P$ with respect to triangle $ABC$. Let $X, Y, Z$ be the intersections of three pairs of lines $(B_1C_1, B_2C_2), (C_1A_1, C_2A_2), (A_1B_1, A_2B_2)$, respectively. Then, $A_2X, B_2Y, C_2Z$ are concurrent at one common point of two circles $\odot(A_1B_1C_1)$ and $\odot(A_2B_2C_2)$.

Theorem 6 (The second Fontene’s Theorem). Given an arbitrary point $P$ lying on a fixed line $d$ passing through the circumcenter $O$ of triangle $ABC$. Then, the Pedal triangle of $P$ with respect to triangle $ABC$ always meets the Nine-point circle of triangle $ABC$ at a fixed point.

Theorem 7 (The third Fontene’s Theorem). Let $P, Q$ be the two isogonal conjugate points in triangle $ABC$. Let $O$ be the circumcenter of the triangle. Then, the Pedal triangle of $P$ with respect to triangle $ABC$ is tangent to the Nine-point circle of triangle $ABC$ if and only if $PQ$ passes through center $O$.

We will prove those theorems in the same proof.

Proof. Let $F, L$ be the projections of $A$ to $l$ and $PA_2$, respectively. Then, $A, C_1, B_1, F, O$ lie on the circle with diameter $AO$, and $A, L, C_2, B_2, P, F$ lie on the circle with diameter $AP$. Hence, $F$ is the Miquel point of the complete quadrilateral from four lines $AB, AC, C_1B_1, C_2B_2$, then, $X, C_1, C_2, F$ are concyclic. We have

$$\angle XFC_2 = \angle AC_1X = \angle ALX = \angle LXC_1$$

Hence $L, X, F$ are collinear. Note that the circle with diameter $AO$ is symmetric to the Nine-point circle ($\odot(A_1B_1C_1)$) of triangle $ABC$ with respect to $B_1C_1$. Since, $L$ is the reflection of $A_2$ in $B_1C_1$, the reflection $T$ of $F$ in $B_1C_1$ is the second intersection of $A_2X$ and $\odot(A_1B_1C_1)$. Since $l$ passes through the orthocenter $O$ of triangle $A_1B_1C_1$, then $T$ be the Anti-Steiner point of $P$ with respect to triangle $A_2B_2C_2$. Similarly, we have $B_2Y, C_2Z$ also pass through the Anti-Steiner point $T$. We proved the first Fontene’s Theorem.

Note that the location of $T$ on $\odot(A_1B_1C_1)$ does not depend on the location of $P$ on line $l$, it just depends on the location of $l$ passing through $O$. Therefore, we can prove the second Fontene’s Theorem.

Now, according to a popular property of the Pedal triangle of two isogonal conjugate points, we have the center of $\odot(A_2B_2C_2)$ is the midpoint of segment $PQ$. Hence, if $S$ is the second intersection
of $\odot(A_1B_1C_1)$ and $\odot(A_2B_2C_2)$, $S$ is the Anti-Steiner of $Q$ with respect to triangle $A_1B_1C_1$. Hence, $T \equiv S$ if and only if $OP \equiv OQ$ or $O, P, Q$ are collinear. We proved the third Fontene’s Theorem. 

Comment.

- From the above proof, we notice that the orthopole of a line passing through the circumcircle of a triangle is the Anti-Steiner point of that line with respect to the median triangle (creating from three midpoints of three sides).

- Also, from the proof, if $P \equiv Q$ at the incenter of triangle $ABC$, the point $T$ coincides with the Feuerbach point $F_e$ of the triangle $ABC$, which is the tangent point of the incircle and the Nine-point circle of triangle $ABC$. Then, we have the next theorem.

**Theorem 8.** The Feuerbach point is the Anti-Steiner point of the line passing the incenter and the circumcenter with respected to the tangent triangle.

Comment. It is a beautiful property that can be applied to solve some other problems related to the Anti-Steiner point with respect to the tangent triangle.

**Problem 4.2.** In $\triangle ABC$ let $N$ be the Nagel point, $O$ the circumcenter and $T$ the Anti-Steiner point of $N$ with respect to $\triangle ABC$. Prove that $T$ lies on $ON$.

**Solution.** First, we have a popular lemma

**Lemma 4.3.** Let $ABC$ be a triangle with incenter $I$, centroid $G$, and the Nagel point $N$. Then, 

$$\overrightarrow{GI} = -\frac{1}{2} \overrightarrow{GN}$$

**Proof.** Indeed, denote that $D, E, F$ is the tangents points of the incircle with $BC, CA, AB$, respectively. Denote that $I_a, I_b, I_c$ is the excenter of triangle $ABC$ with respect to $A, B, C$, respectively. $\odot(I_a)$ is tangent to $BC$ at $A_1$, similarly define $B_1, C_1$. Let $X, Y, Z$ be the antipodes of $D, E, F$ in the incircle, respectively. Let $A', B', C'$ be the midpoints of segments $BC, CA, AB$, respectively. We have $A'$ is the center of the homothety $\Phi$ mapping $\odot(I)$ to $\odot(I_a)$. Then,

$$\Phi : \odot(I) \mapsto \odot(I_a), I \mapsto I_a, IX \mapsto I_aA_1, X \mapsto A_1$$

Then, we have $A, X, A_1$ are collinear. Moreover, we can easily find that $BD = CA_1 = \frac{BA + BC - AC}{2}$. Hence, $A'$ is the midpoint of segment $DA_1$. Hence, $IA_1$ is the midline of triangle $XDA_1$. Then, $IA' \parallel AA_1$. Similarly, we have $IB' \parallel BB_1, IC' \parallel CC_1$. Let $\Theta$ be the homothety with center $G$ and ratio $-\frac{1}{2}$. We have,

$$\Theta : A \mapsto A', B \mapsto B', C \mapsto C', \triangle ABC \mapsto \triangle A'B'C'$$

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Since $IA' \parallel AA_1, IB' \parallel BB_1, IC' \parallel CC_1$, hence,

$$\Theta : AA_1 \mapsto A'I, BB_1 \mapsto B'I, CC_1 \mapsto C'I$$

Since, $A'I, B'I, C'I$ are concurrent at $I$ and $AA_1, BB_1, CC_1$ are concurrent at $N$, we have

$$\Theta : N \mapsto I \iff \overrightarrow{GI} = -\frac{1}{2} \overrightarrow{GN}$$

Back to our main problem,

![Diagram](image)

Let $G$ be the centroid of triangle $ABC$. Let $\Phi$ is the homothety with center $G$ and ratio -2, we have,

$$\Phi : A \mapsto A', B \mapsto B', C \mapsto C', I \mapsto N$$

Therefore, $N$ is the incenter of triangle $A'B'C'$. According to the Theorem 8., we have $T$ is the Feuerbach point of triangle $A'B'C'$. Moreover, we have $O$ is the center of the Nine-point circle of triangle $A'B'C'$, and the Nine-point circle is tangent to the incircle at the Feuerbach point $T$. Therefore, $T$ lies on the line passing through two centers $O$ and $N$ of those circles.

Now, we will see some other problems related to the orthopole of a line passing through the circumcenter and the application of Fontene's Theorems.

**Problem 4.3.** Let $ABC$ be the triangle with circumcenter $O$. A line $l$ passing through $O$ meets $BC, CA, AB$ at $D, E, F$ respectively. Prove that three circles with diameters $AD, BE, CF$ are concurrent at two points: one point is the orthopole of $l$ with respect to triangle $ABC$, and the other is the reflection of the Miquel point of complete quadrilateral $AD.BE.CF$. 
Solution. Let $M$ be the Miquel point of the complete quadrilateral creating from four lines $AB, AC, BC, l$ and $K$ be the reflection of $M$ in $l$. We have,

\[ \angle MKA = \angle MBF = \angle MDF = \angle KDF \]

Hence, $\angle DKA = 90^\circ$ or $K$ lies on the circle with diameter $AD$. Similarly, we have $K$ also lies on the circles with diameters $BE$ and $CF$, respectively. Let $L$ be the orthopole of $l$ with respect to triangle $ABC$. According to the Fontene’s Theorem, we have $L$ is the intersection of the Pedal circles of points $D, E, F$ with respect to triangle $ABC$ and the Nine-point circle of the triangle. Therefore, we can conclude that three circles with diameters $AD, BE, CF$ are concurrent at two points: one point is the orthopole of $l$ with respect to triangle $ABC$, and the other is the reflection of the Miquel point of complete quadrilateral $AD.BE.CF$.

Problem 4.4 (Taiwan TST 2013 - Round 2). Let $ABCD$ be a cyclic quadrilateral with circumcircle $\odot(O)$. Let $l$ be a fixed line passing through $O$ and $P$ be a point varies on $l$. Let $\Omega_1, \Omega_2$ be the pedal circles of $P$ with respect to $\triangle ABC, \triangle DBC$, respectively. Find the locus of $T \equiv \Omega_1 \cap \Omega_2$ (different from the projection of $P$ on $BC$).

Solution. (Telv Cohl)
First, we have a lemma.
Lemma 4.4. Given a fixed circle $\odot(O)$ and a fixed point $P$. Let $B, C$ be two fixed points on $\odot(O)$ and $A$ be a point varies on $\odot(O)$. Let $T$ be the orthopole of $OP$ with respect to $\triangle ABC$ and $\triangle XYZ$ be the pedal triangle of $P$ with respect to $\triangle ABC$. Then $\angle XZT$ is fixed when $A$ varies on $\odot(O)$.

Proof. Let $D, E, F$ be the midpoint of $BC, CA, AB$, respectively. Let $Z'$ be the reflection of $Z$ in $DE$ and $R \equiv DE \cap XY$. Let $T'$ be the reflection of $T$ in $DE$ (i.e. the projection of $C$ on $OP$). From Fontene’s Theorem we get $Z, T, R$ are collinear. Since $C, P, X, Y, T', Z' \in \odot(CP)$ and $C, O, D, E, T' \in \odot(CO)$, so $\angle PZT = \angle Z'RZ = \angle Z'T'P = \angle T'CP = \text{constant}$, hence combine $\angle XZP = \angle CBP = \text{constant} \implies \angle XZT$ is fixed when $A$ varies on $\odot(O)$.

Back to our main problem,

Let $H_{AB}$ be the projection of $P$ on $AB$ (define $H_{BC}, H_{BD}$, similarly). Let $T_A, T_D$ be the orthopoles of $\ell$ with respect to $\triangle DBC, \triangle ABC$, respectively. According to Fontene’s Theorem, we have $T_A \in \Omega_2, T_D \in \Omega_1$. According to the Lemma 4.4., we have $\angle H_{BC}H_{AB}T_D = \angle H_{BC}H_{BD}T_A$, so $\angle T_DTH_{BC} + \angle H_{BC}TT_A = 180^\circ$. Therefore, $T_A, T, T_D$ are collinear. Hence the locus of $T$ is a line passing through $T_A$ and $T_D$ when $P$ varies on $\ell$.

5 Other Anti-Steiner point related problems

In this section, I will introduce some other beautiful problems and properties related to the Anti-Steiner point.

Problem 5.1 (DeuX Mathematics Olympiad 2020 Shortlist G5 (Level II Problem 3)). Given a triangle $ABC$ with circumcenter $O$ and orthocenter $H$. Line $OH$ meets $AB, AC$ at $E, F$ respectively. Define $S$ as the circumcenter of $AEF$. The circumcircle of $AEF$ meets the circumcircle of $ABC$ again at $J$, $J \neq A$. Line $OH$ meets circumcircle of $JSO$ again at $D$, $D \neq O$ and circumcircle of $JSO$ meets circumcircle of $ABC$ again at $K$, $K \neq J$. Define $M$ as the intersection of $JK$ and $OH$ and $DK$ meets circumcircle of $ABC$ at points $K, G$. Prove that circumcircle of $GHM$ and circumcircle of $ABC$ are tangent to each other.

Solution. $EF \cap BC = L$. $I, O'$ are the reflections of $H$ and $O$ in $BC$. $A'$ is the antipode of $A$ in circle $(O)$. We will use six claims to solve this problem.

Claim 1. $K$ is the Anti-Steiner point of $OH$ with respect to triangle $ABC$. 


Indeed, Let $K'$ is the Anti-Steiner point of $OH$. $AT$ is the altitude of triangle $AEF$. We will prove that $AT, AK'$ are isogonal conjugate in $\angle BAC$. First, we can see that $L, K, I, O'$ are collinear. Then, $\angle O'K'A' = \angle IAA' = \angle O'OA'$. Hence, $K', O', O, A'$ are concyclic. Hence, $\angle AHT = \angle IO'O = \angle OAA'$. Then, $\angle HAT = 90^\circ - \angle AHT = 90^\circ - \angle AA'K' = \angle K'AA'$. Hence, $AT, AK'$ are isogonal conjugate in $\angle BAC$. Moreover, since $A, O$ are isogonal conjugate in $\angle BAC$, $AT, AK'$ are isogonal conjugate in $\angle BAC$. Besides, since $AS, AT$ are isogonal conjugate in $\angle BAC$, $A, S, K'$ are collinear. Besides, we can easily see that $\triangle JSE \sim \triangle JOB$. Hence, $\triangle JSO \sim \triangle JEB$. Hence, $\angle K'AA' = \angle JBE = \angle JOS$. Then, $J, S, O, K'$ are concyclic. Hence, $K' \equiv K$ and $K$ is the Anti-Steiner point of line $OH$.

Claim 2. $R$ is a point on $\odot(O)$ such that $KR \perp BC$. Then $AR \parallel OH$.

This is a popular property of Simpson line.

Claim 3. $R, J, L$ are collinear.

Indeed, we can easily see that $J$ is the Miquel point of the complete quadrilateral $(EC, FB, AL)$. $RJ \cap OH = L'$. We have, $\angle JLE = \angle JRA = \angle JBE$. Hence, $J, B, E, L'$ are concyclic. Then, $L' \equiv L$. Hence, $R, J, L$ are collinear.

Claim 4. Let $N$ be the antipode of $K$. Then, $LA' \cap NH = G$

Indeed, since $AR \parallel OH$ and $AR \perp RA'$, we have $RA' \perp OH$. Then, $A'$ is the reflection of $R$ in $OH$. Besides, since $O$ is the midpoint of arc $JK$ of $\odot(JOK)$, $DO$ is the bisector of $\angle JDK$. Hence, $J$ is the reflection of $G$ in $OH$. Then, $L, G, A'$ are concyclic. Moreover, applying the Pascal’s Theorem to the set of 6 points $\left(\begin{array}{ccc} G & A & K \\ I & N & A' \end{array}\right)$, we have $G, H, N$ are collinear.

Claim 5. $\odot(GMN) \cap OH = \{M; P\}$. Then, $PN$ is tangent to cirle $\odot(O)$.

Indeed, since $\angle ODK = \angle OKJ = \angle OKJ, \angle OMK = \angle OKG$. Then, we have $\angle GNP = \angle GMH = \angle J MH = 180^\circ - \angle OMK = \angle OKG$. Hence, $PN$ is tangent to $\odot(O)$.

Claim 6. $\odot(MGH)$ is tangent to cirle $\odot(O)$.

Indeed, let $\Phi$ is the inversion with center $H$ and power $HA$. Then, we have $\Phi : H \leftrightarrow H, A \leftrightarrow I, G \leftrightarrow N, M \leftrightarrow P, \odot(O) \leftrightarrow \odot(O), \odot(MGH) \leftrightarrow PN$.

Since, $PN$ is tangent to $\odot(O)$, $\odot(MGH)$ is tangent to $\odot(O)$.

From Claim 1., we have a corollary:
Corollary 1. Let $ABC$ be a triangle with orthocenter $H$. Let $l$ be a line passing through $H$. Let $K$ be the Anti-Steiner point of $l$ with respect to triangle $ABC$. Then, line $AK$ and the line passing through $A$ perpendicular to $l$ are two isogonal conjugate lines in $\angle BAC$.

Problem 5.2. Let $\Gamma$ be the circumcircle of a triangle $ABC$ with Euler line $l$. Let $DEF$ be the orthic triangle and $H$ the orthocenter. $X = l \cap EF, M = AX \cap \Gamma, J = l \cap MC$ and $G = CH \cap MD$. Prove that $HJMG$ are concyclic.

Solution. I will redefine the problem. Let $\Gamma$ be the circumcircle of a triangle $ABC$ with Euler line $l$. Let $DEF$ be the orthic triangle and $H$ is the orthocenter. $X = l \cap EF, M = AX \cap \Gamma, J = l \cap MC$ and $G = CH \cap MD$. We have $T$ is the Anti-Steiner point of line $l$ with respect to $\triangle ABC$. $TD \cap \Gamma = \{T, M\}, AM \cap l = X$. We will prove that $X, E, F$ are collinear. Indeed, we have

$$(MX, MD) \equiv (MA, MT) \equiv (RA, RT) \equiv (RA, RP) \equiv (HP, HD) \equiv (HX, HD)$$

Therefore $X, M, F, B$ are concyclic, hence,

$$(XM, XF) \equiv (BM, BF) \equiv (BM, BA)$$

Similarly, we have $X, M, E, C$ are concyclic, hence,

$$(XM, XE) \equiv (CM, CA) \equiv (BM, BA) \equiv (XM, XF)$$

Therefore, $X, E, F$ are collinear. Then

$$(HG, HJ) \equiv (HC, HP) \equiv (RP, RC) \equiv (RT, RC) \equiv (MT, MC) \equiv (MG, MJ)$$

Therefore, $H, G, J, M$ are concyclic.

Problem 5.3 (“Quan Hinh” Topic - April 2019 - Nguyen Duc Toan). Triangle $ABC$ is scalene. Let two points $D$ and $E$ lie on rays $BC$ and $CB$ respectively such that $BD = BA$ and $CE = CA$. The circumcircle of triangle $ABC$ meets the circumcircle of triangle $ADE$ at $A$ and $L$. Let $F$ be the midpoint of arc $AB$ which not contain $C$ of the circumcircle of $ABC$. Let $K$ lie on $CF$ such that $\angle FDA = \angle KDE$. $KL \cap AE = J$. Prove that $J$ lies on the Euler line of triangle $ADE$.

Solution. First, we have a lemma.

Lemma 5.1. Let $ABC$ be a triangle with circumcenter $O$. $O'$ is the reflection of $O$ in $BC$. Let $J$ be the circumcircle of triangle $BOC$. Then, $AJ, AO'$ are isogonal conjugate lines in angle $\angle BAC$. 

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Proof. Let $D$ be the intersection of two tangents from $B$ and $C$ of $\odot(O)$. Denote that $M$ is the midpoint of segment $BC$. Since, $AD$ is the symmedian, $AD$ and $AM$ are isogonal conjugate lines in angle $\angle BAC$. Moreover, we have

$$OM.OD = OB^2 = OA^2 = OJ.OO'$$

Hence, $\triangle OAM \sim \triangle ODA$ and $\triangle OJA \sim \triangle OAO'$. Hence, $\angle OAM = \angle ODA$ and $\angle OAJ = \angle OO'A$. Then, $\angle DAO' = \angle JAM$. Hence, $AJ, AO'$ are two isogonal conjugate lines in angle $\angle BAC$.

Back to our main problem,

Denote that $O$ and $H$ are the circumcenter and orthocenter of triangle $AED$, respectively. $OC$ meets $AD$ at $S$; $OB$ meets $AE$ at $I$. We can easily get that $C, O, L$ lie on the perpendicular bisector of segment $AE$, and $O$ is the incenter of triangle $ABC$. Then, $F$ is the center of $\odot(AOB)$. Besides, we have

$$\angle AEB = 90^\circ - \frac{\angle ACB}{2} = 180^\circ - (90^\circ - \frac{\angle ACB}{2}) = \angle 180^\circ - \angle AOB$$
Hence, $A, E, O, B$ are concyclic and $F$ is the center of $⊙(AOE)$. Similarly, we have $O, E, R, S, D$ are concyclic. According to the Lemma 5.1., we have that $K$ is the reflection of $O$ in $AE$.

Now, we need to prove that $L$ is the Anti-Steiner point of $OH$ with respect to triangle $AED$. Indeed, let $L'$ be the Anti-Steiner point of $OH$ with respect to triangle $AED$. We can easily see that $O$ is the orthcenter of triangle $AIS$. Hence, $L'$ is the Anti-Steiner point of $IS$ with respect to triangle $AIS$. Hence, $L' \in ⊙(AIS)$. IS meets $ED$ at $R$. Hence, $L'$ is the Miquel point of the complete quadrilateral $AR.ES.ID$. Let $T$ and $U$ be the midpoints of segments $AE$ and $AD$, respectively. Hence, $$(L'I, L'A) \equiv (SI, SA) \equiv (SI, SU) \equiv (TI, TU) \equiv (EI, ER) \equiv (L'I, L'R)$$

Hence, $L', R, A$ are collinear. We can easily prove that $OB.OI = OE^2 = OS.OC$. Hence, $I, B, S, C$ are concyclic. Then, $RB.OC = RS.RI = RL'.RA$. Hence, $A, B, C, L'$ are concyclic. Therefore, $L' \equiv L$ or $L$ is the Anti-Steiner point of $OH$ with respect to triangle $ADE$.

Hence, line $LK$ is the reflection of line $OH$ in $AE$. Hence, $J$ lies on the Euler line of triangle $ADE$.

Problem 5.4 (Nguyen Van Linh). Let $ABC$ be a triangle with circumcenter $O$ and orthocenter $H$. A line $d$ passing through $H$ meets $BC, CA, AB$ at $D, E, F$ respectively. The line through $F$ and perpendicular to $OB$ meets the line through $E$ and perpendicular to $OC$ at $X$. Similarly define $Y, Z$. Prove that $X, Y, Z$ lie on a line tangent to $⊙(O)$.

Solution. Let $T$ be the Anti-Steiner point of line $d$ with respect to triangle $ABC$. $CH$ meets $⊙(O)$ at the second point $J$. Then, $J, H, C$ are collinear. Since $A$ is the excenter of triangle $TEF$, $∠FTE = 180° - 2∠BAC = 180° - ∠BOC = ∠EXF$. Hence, $E, T, X, F$ are concyclic. Let $K$ be the intersection of $EX$ and $TC$. We have $∠EKC = 90° - ∠OCT = ∠TJC = ∠EHC$. Hence, $E, H, K, C$ are concyclic. Then, $∠TCJ = ∠XEF = ∠XTJ$. Hence, $XT$ is tangent to $⊙(O)$. Similarly, we have $X, Y, Z$ lie on the tangent at $T$ of $⊙(O)$. 

Problem 5.5. Let $(I)$ be the incircle of triangle $ABC$ and $D, E, F$ be the contacts triangle. Let $F_e$ be the Feuerbach point of triangle $ABC$. Let $K$ be the orthocenter of triangle $DEF$. Let $S$ be the Anti Steiner point of $I$ with respect to triangle $ABC$. Prove that $KF_e \parallel IS$.

Solution. First, we have two lemmas.
Lemma 5.2 (Ha Huy Khoi). Let $ABC$ be the triangle with circumcenter $O$, incenter $I$, and orthocenter $H$. $K$ is the Anti-Steiner point of $HI$ with respect to $\triangle ABC$. $K$ meets $OI$ at the second point $L$. $AL$ meets $OI$ at $J$. Then, $AL$ is symmetric to $OI$ with respect to the perpendicular bisector of segment $AI$.

Proof. Indeed, $AH$ meets $\odot(O)$ at the second point $E$. $I'$ is the reflection of $I$ in $BC$. Then, $EI'$ passes through $K$. $AI$ meets $\odot(O)$ at the second point $D$. Then,

$$(II', ID) \equiv (AE, AD) \equiv (KI', KD)$$

Hence, $I, I', D, K$ are concyclic. Let $D'$ be the reflection of $D$ in $BC$, $M$ be the midpoint of segment $BC$, $DN$ be the diameter of $\odot(O)$. We have: $DI^2 = DB^2 = DM.DN = DD'.DO \Rightarrow \triangle DIO \sim \triangle DD'I$. Therefore,

$$(IA, IO) \equiv (IO, ID) \equiv (D'D, D'I) \equiv (KD, KI) \equiv (AI, AL)$$

Hence,

$$(AI, AL) \equiv (IA, IO) \equiv (IJ, IA)$$

Hence, we have $JA = JI$. Hence, $AL$ is symmetric to $OI$ with respect to the perpendicular bisector of segment $AI$. \qed

Lemma 5.3 (Nguyen Duc Toan). Let $ABC$ be the triangle with circumcenter $O$ and orthocenter $H$. $T$ is the Anti-Steiner point of $OH$ with respect to triangle $ABC$. Three altitudes of triangle $ABC$ meet $\odot(O)$ at the second points $D, E, F$, respectively. $TH$ meets $\odot(O)$ at the second point $K$. Then, $K$ is the Anti-Steiner of $H$ with respect to triangle $DEF$. 

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Proof. (Ha Huy Khoi)

Let $K'$ be the Anti-Steiner point of $H$ with respect to triangle $DEF$. $K'H$ meets $(O)$ at the second point $T'$. Applying Lemma 5.2. into the $\triangle DEF$ with its incenter $H$, we have $DT'$ is symmetric to $OH$ with respect to the perpendicular bisector of segment $DH$. Then, $DT'$ is the reflection of $OH$ in $BC$. Similarly, we have $ET', FT'$ are the reflections of $OH$ in $CA, AB$, respectively. Hence, $T'$ is the Anti-Steiner point of $OH$ with respect to $\triangle ABC$. Therefore, $T' \equiv T, K' \equiv K$. Hence, $K$ is the Anti-Steiner point of $H$ with respect to triangle $DEF$.

Back to our main problem.

![Diagram of geometric configuration involving points A, B, C, D, E, F, and T, with circles and lines indicating the relationships described in the proof.]

Let $O, H$ be the circumcenter and the orthocenter of triangle $ABC$, $X, Y, Z$ be the second intersections of $AI, BI, CI$ and $\odot(O)$, respectively. Let $M, N, P$ be the midpoint of segments $EF, FD, DE$ respectively. Let $\Theta$ be the inversion with center $I$ and power $IE^2$. We have

$$\Theta : A \leftrightarrow M, B \leftrightarrow N, C \leftrightarrow P, \odot(ABC) \leftrightarrow \odot(MNP)$$

Hence we have, $I, O$ and the center of $\odot(MNP)$ are collinear. Hence, $O$ lies on the Euler line of triangle $DEF$. Hence, $O, I, K$ are collinear. Moreover, according to Fontene’s theorem, we have $F_e$ is the Anti-Steiner point of $OK$. Let $T$ be the Anti-Steiner point of $H$ with respect to triangle $XYZ$. According to Lemma 5.3., we have $T, I, S$ are collinear. Besides, we can easily get $EF \parallel YZ, FD \parallel XZ, DE \parallel XY$. Then, there is a homothety $\Phi$ that

$$\Phi : D \mapsto X, E \mapsto Y, F \mapsto Z, \triangle DEF \mapsto \triangle XYZ, \odot(DEF) \mapsto \odot(XYZ), K \mapsto I, F_e \mapsto T$$

Therefore, $KF_e \parallel IT$ or $KF_e \parallel IS$.

To end this article, I will introduce some practice problems related to the Anti-Steiner point.
Problem 6.1. Let \( ABC \) be triangle with orthocenter \( H \). \( D, E \) lies on \( \odot(ABC) \) such that \( DE \) pass through \( H \). \( S \) is an Anti-Steiner point of \( DE \). \( N \) lies on \( BC \) such that \( ON \) is perpendicular to \( SE \). \( EN \) cuts \( \odot(ABC) \) at the second point \( F \). Prove that \( A, O, F \) are collinear.

Problem 6.2 (Nguyen Duc Toan). Let \( ABC \) be the triangle with circumcenter \( O \), orthocenter \( H \), and the center of Nine-point circle \( N \). Let \( P \) be the Anti-Steiner point of \( OH \) with respect to the triangle. Prove that \( A, N, P \) are collinear if and only if \( AP \perp OH \).

Problem 6.3 (IMOC 2019 - G5). Given a scalene triangle \( \triangle ABC \) with orthocenter \( H \) and circumcenter \( O \). The exterior angle bisector of \( \angle BAC \) intersects circumcircle of \( \triangle ABC \) at \( N \neq A \). Let \( D \) be another intersection of \( HN \) and the circumcircle of \( \triangle ABC \). The line passing through \( O \), which is parallel to \( AN \), intersects \( AB, AC \) at \( E, F \), respectively. Prove that \( DH \) bisects the angle \( \angle EDF \).

Problem 6.4. Let \( ABC \) be a triangle with orthocentre \( H \) and circumcentre \( O \). The circle through \( A \) and \( B \) touching \( AC \) meets the circle through \( A \) and \( C \) touching \( AB \) at \( X \neq A \). Define \( X_B, X_C \) similarly. Prove that the four circles \( \odot(AX_BX_C), \odot(BX_CX_A), \odot(CX_AX_B), \odot(ABC) \) meet at the Anti-Steiner point of \( OH \) in \( \triangle ABC \).

Problem 6.5 (Nguyen Duc Toan). Let \( ABC \) be a triangle with the incircle \( I \). \( D, E, F \) are the tangent points of the incircle to \( BC, CA \) and \( AB \) respectively. \( ID \) intersects \( AB \) and \( AC \) at \( A_b \) and \( A_C \) respectively. Similarly, we define points \( B_a, B_c, C_a, C_b \). Assume that 3 lines \( A_aB_c, C_cB_a, A_bC_b \) are pairwise intersect create triangle \( A'B'C' \). Prove that the circumcircle of triangle \( A'B'C' \) is tangent to the circumcircle of triangle \( ABC \).

Problem 6.6 (Luis Gonzalez). An arbitrary line \( \ell \) through the circumcenter \( O \) of \( \triangle ABC \) cuts \( AC, AB \) at \( Y, Z \), respectively. The circle with diameter \( YZ \) cuts \( AC, AB \) again at \( M, N \), respectively. Show that \( MN \) passes through the orthopole of \( \ell \) with respect to \( \triangle ABC \).

Problem 6.7. Let \( ABC \) be a triangle with circumcenter \( O \) and \( \angle B > 90^\circ \). A line \( l \) passes through \( O \) cuts \( CA, AB \) at \( E, F \) such that \( BE \perp CF \) at \( D \). Let \( S \) be the orthopole of \( l \) with respect to \( \triangle ABC \). Perpendicular lines from \( D \) to \( SB, SC \) cut \( SC, SB \) at \( M, N \). Draw parallelogram \( DMKN \). Prove that \( S \) is the midpoint of \( AK \).

Problem 6.8. Suppose the triangle \( \triangle ABC \) has circumcenter \( O \) and orthocenter \( H \). Parallel lines \( \alpha, \beta, \gamma \) are drawn through the vertices \( A, B, C \). Let \( \alpha', \beta', \gamma' \) be the reflections of \( \alpha, \beta, \gamma \) over \( BC, CA, AB \). Then these reflections are concurrent if and only if \( \alpha, \beta, \gamma \) are parallel to \( OH \) lines of \( \triangle ABC \). In this case, their point of concurrency \( P \) is the reflection of \( O \) over the Euler Reflection Point (the Anti-Steiner Point of the Euler Line).

Problem 6.9 (Tev Cohl). Let \( I, H \) be the incenter, orthocenter of \( \triangle ABC \), respectively. Let \( \triangle DEF \) be the intouch triangle of \( \triangle ABC \) and \( T \) be the orthocenter of \( \triangle DEF \). Let \( Fe \) be the Feuerbach point of \( \triangle ABC \) and \( S \) be the Anti-steiner point of \( TF \) with respect to \( \triangle DEF \). Prove that \( IH \perp SF \).

Problem 6.10 (Nguyen Duc Toan). Let \( ABC \) is a triangle with circumcircle \( \odot(O) \) and Anti-median triangle \( A'B'C' \) (\( A \) is the midpoint of segment \( B'C' \), and similar to \( B, C \)). Let \( A_1B_1C_1 \) is the triangle created from three tangents of circle \( \odot(O) \) at \( A, B, C \).

1. Prove that \( B'B_1, C'C_1, BC \) are concurrent at point \( X \). Similarly denote \( Y, Z \).

2. Prove that the orthopole of the Euler line of triangle \( ABC \) with respect to triangle \( XYZ \) lies on the circumcircle of triangle \( A'B'C' \).
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