ANTI-STEINER POINT REVISITED

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Abstract

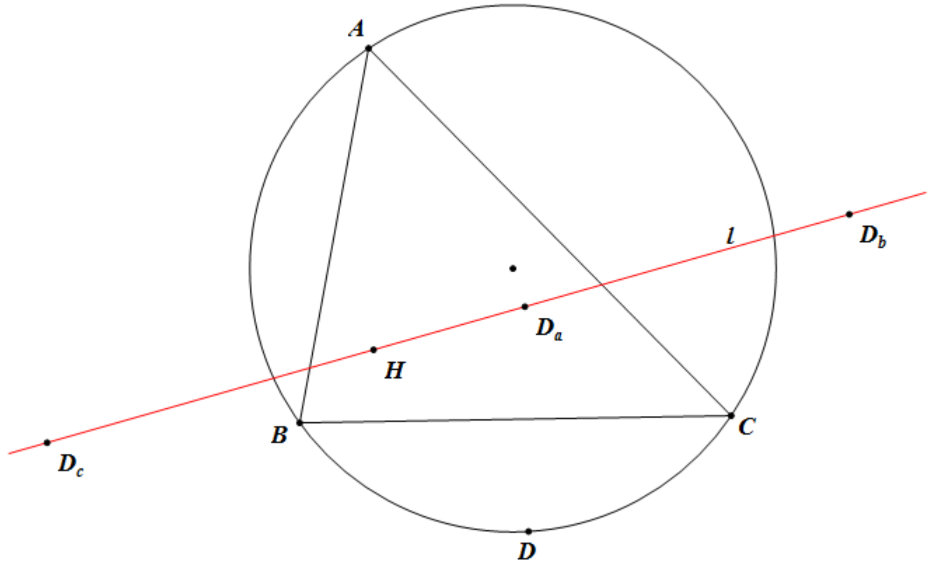
Many students, although familiar with problems involving the Steiner line, may not be aware of Anti-Steiner point applications. In this article, we introduce some properties and problems related to the Anti-Steiner point of a line with respect to a triangle. This is suitable for high school students, particularly those involved in mathematical Olympiads.

Note. *In this article, we label that $\odot(O)$ is the circle with center O, $\odot(XYZ)$ is the circumcircle of triangle XYZ, (d, d') is the directed angle of lines d and d' modulo π, and $\overline{AB}$ is the algebraic length of segment AB.*

1 Anti-Steiner point definition

First, the Steiner's theorem about the Steiner line is commonly known and used in olympiad mathematics. The theorem is illustrated below.

Theorem 1 (Steiner). *Let ABC be a triangle with orthocenter H. D is a point on the circumcircle of triangle ABC. Then, the reflections of D in three edges BC, CA, AB and point H lie on a line l. We call that l is the Steiner line of point D with respect to triangle ABC.*



Now, let's discuss the converse of the Steiner's theorem regarding the Steiner line.

Theorem 2 (Collings). *Let ABC be the triangle with orthocenter H. Given a line l passing through H. Denote that a', b', c' are the reflections of l in the edges BC, CA, AB, respectively. Then, the lines a', b', c' are concurrent at a point T on the circumcircle of triangle ABC.*

Proof. First, we have a lemma.

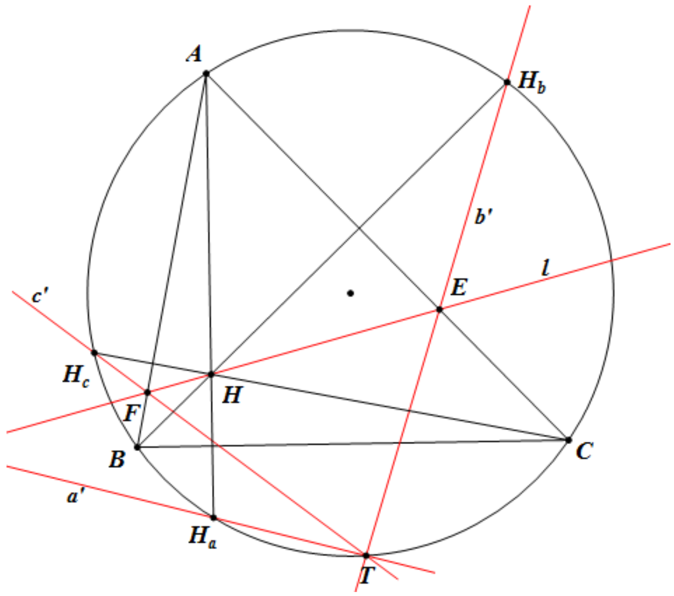
Lemma 1.1. *Let ABC be the triangle with altitudes AD, BE, CF and orthocenter H. The line AD meets the circumcircle of triangle ABC at the second point H_a. Then, H_a is the reflection of H in line BC.*

Proof. We can easily see that A, C, D, F lie on the circle with diameter BC. Then,

$$(CH, CD) \equiv (CF, CD) \equiv (AF, AD) \equiv (AB, AH_a) \equiv (CB, CH_a) \equiv (CD, CH_a)$$

Similarly, we have $(BH, BD) \equiv (BD, BH_a)$. Then, H_a is the reflection of H in line BC. □

Back to our main proof,



The lines AH, BH, CH meet the circumcircle of triangle ABC at the second points H_a, H_b, H_c. Let D, E, F be the intersections of l and three edges BC, CA, AB, respectively. According to the **Lemma 1.1.**, we have H_a, H_b, H_c are the reflections of H in BC, CA, AB, respectively. Hence H_aD, H_bE, H_cF are the reflections of l in lines BC, CA, AB, respectively. Therefore,

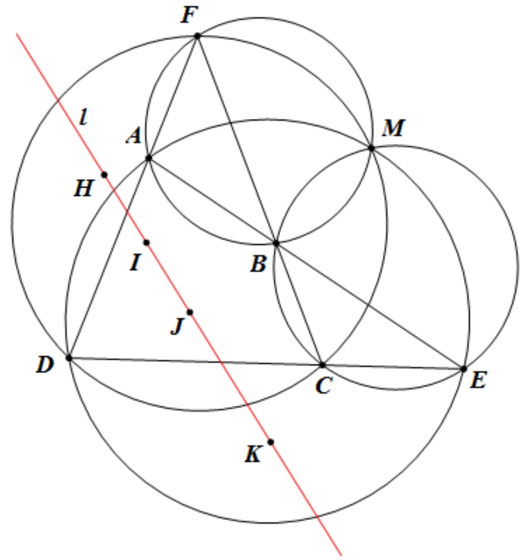
$$\begin{aligned}(EH_b, FH_c) &\equiv (EH_b, CA) + (AC, AB) + (AB, FH_c)\\ &\equiv (Ca, l) + (AC, AB) + (l, AB)\\ &\equiv 2(AC, AB)\\ &\equiv (H_aH_b, H_aH_c)\end{aligned}$$

Then, we have the intersection of lines EH_b and FH_c lying on $\odot(ABC)$. Similarly, the intersection of lines EH_b and DH_a lies on $\odot(ABC)$. Therefore, H_aD, H_bE, H_cF are concurrent at a point T on $\odot(ABC)$. □

Note. *We call that T is the Anti-Steiner point of line l with respect to triangle ABC. Moreover, given a point K lying on line l. We can also call that T is the Anti-Steiner point of point K with respect to triangle ABC.*

Next, I will introduce to you an extension of the Anti-Steiner point definition in a complete quadrilateral from a popular theorem.

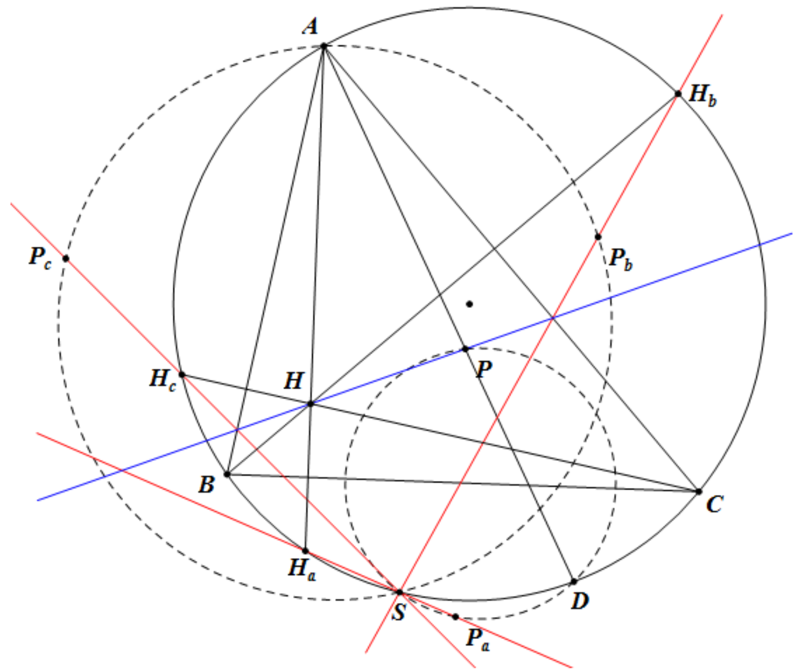
Theorem 3. *Given a convex quadrilateral $ABCD$ that no side is parallel to another side. AB meets CD at E. AD meets BC at F. Let M be the Miquel point of the complete quadrilateral $AC.BD.EF$. Then, the orthocenters of four triangles BCE, CDF, ADE, ABF lie on line l.*



Note. *We call that line l is the Steiner line of the complete quadrilateral $AC.BD.EF$. Also, we can see that M is Anti-Steiner point of line l with respect to triangles BCE, CDF, ADE, ABF. So, we call that M is the Anti-Steiner point of the complete quadrilateral $AC.BD.EF$.*

2 Concurrency related problems

Theorem 4. *Given triangle ABC and an arbitrary point P on the plane. P_a, P_b, P_c are the reflections of P in BC, CA, AB, respectively. Lines AP, BP, CP meet the circumcircle of $\triangle ABC$ at D, E, F, respectively. Then, the circumcircles of triangles $AP_bP_c, BP_cP_a, CP_aP_b, PP_aD, PP_bD, PP_bE, PP_cF$ pass through the Anti-Steiner point S of P with respect to triangle ABC.*



Solution. Denote that H is the orthocenter of triangle ABC. AH, BH, CH meets $\odot(ABC)$ at the second points H_a, H_b, H_c, respectively. According to the prove of Collings's above, we have P_aH_a, P_bH_b, P_cH_c are concurrent at the Anti-Steiner point S of P with respect to triangle ABC. We will prove that $\odot(AP_bP_c)$ and $\odot(PP_aD)$ pass through S. The other cases can be proved similarly.

By directed angle chasing, we have,

$$\begin{aligned}(SP_b, SP_c) &\equiv (H_bP_b, H_cP_c)\\ &\equiv (H_bP_c, AC) + (AC, AB) + (AB, H_cP_c)\\ &\equiv (AC, HP) + (AC, AB) + (HP, AB)\\ &\equiv 2(AC, AB)\\ &\equiv (AP_b, AP_c)\end{aligned}$$

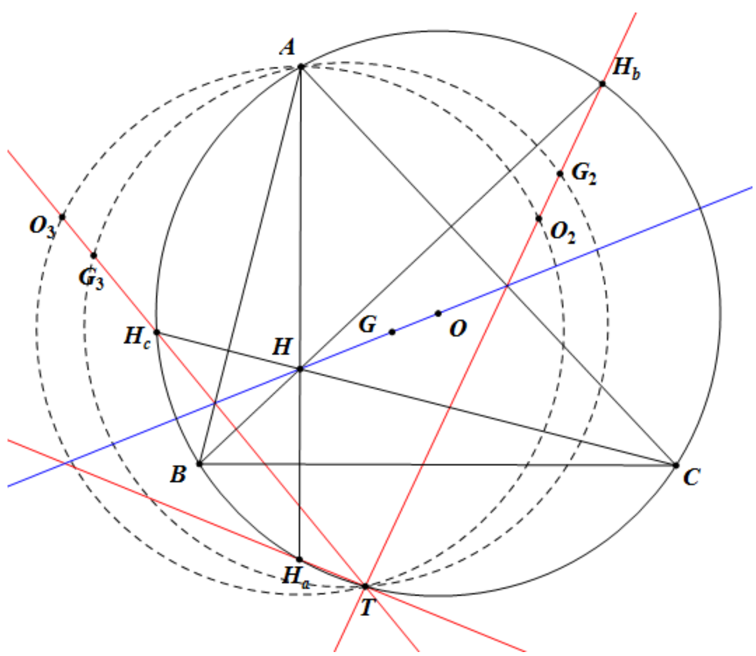
Hence, S, A, P_b, P_c are concylic. Besides,

$$\begin{aligned}(SD, SP_a) &\equiv (SD, SH_a)\\ &\equiv (AD, AH_a)\\ &\equiv (PD, PP_a)\end{aligned}$$

Hence, S, D, P_a, P are concyclic. Therefore, we have $\odot(AP_bP_c)$ and $\odot(PP_aD)$ pass through S. Similarly, the other circumcircles pass through S. □

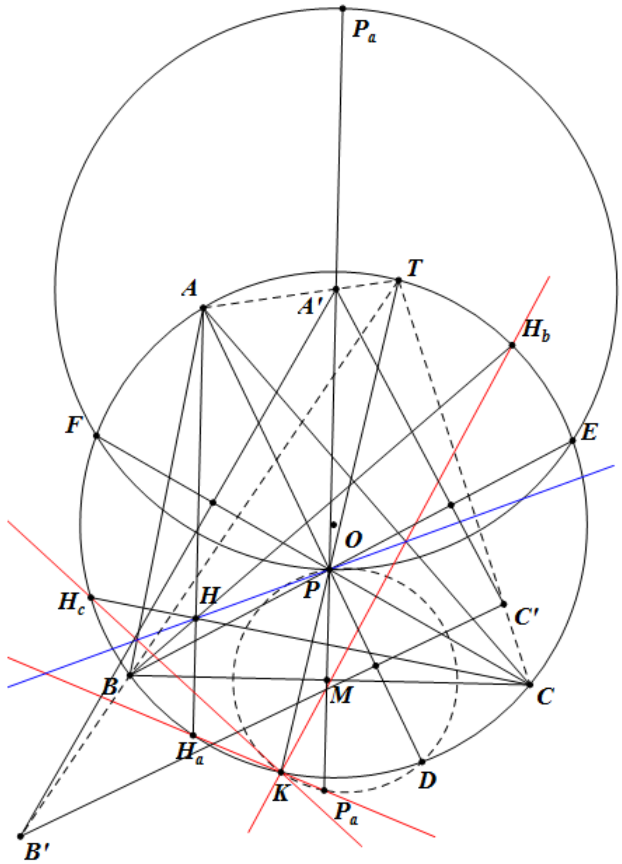
Comment. *This is a very popular and useful theorem used to solve the problem related to the Anti-Steiner point. A part of this theorem is the geometric problem from the* ***China TST 2016****. Now we can see some other applications of this theorem into some Olympiad problems.*

Problem 2.1 (EGMO 2017 - Problem 6). *Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcentre O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3 and O_1, O_2, O_3, respectively. Show that the circumcircles of triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.*



Solution. Denote T be the Anti-Steiner point of the Euler line of triangle ABC with respect to the triangle. According to **Theorem 4.**, we can easily have $\odot(G_1G_2C)$, $\odot(G_1G_3B)$, $\odot(G_2G_3A)$, $\odot(O_1O_2C)$, $\odot(O_1O_3B)$, $\odot(O_2O_3A)$ pass through T lying on $\odot(ABC)$. □

Problem 2.2. *Denote triangle ABC and an arbitrary point P (P does not lie on $\odot(ABC)$). PA, PB, PC meet $\odot(ABC)$ at the second points D, E, F respectively. The perpendicular bisectors of segments PD, PE, PF meet each other creating triangle $A'B'C'$. Prove that AA', BB', CC' are concurrent at a point on $\odot(ABC)$, and PT passes through the Anti-Steiner point of P.*



Solution. Let K be the Anti-Steiner point of P with respect to triangle ABC. PT meets $\odot(ABC)$ at the second point T. We will show that A, A', T are collinear. Then, we can have AA', BB', CC' are concurrent at T. Indeed, let M be the projection of P to BC, X be the reflection of P in BC. According to the **Theorem 4.**, we have P, X, D, K are concyclic. Moreover, we can easily see that A' is the center of $\odot(EPF)$. Let P_a be the antipode of P in $\odot(PEF)$. Let Φ be the inversion with center P and power $\overline{PA}.\overline{PD}$. We have,

$$\Phi : P \leftrightarrow P, T \leftrightarrow K, A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F, \odot(PEF) \leftrightarrow BC$$

Since $\angle(PM, BC) = \angle(PP_a, \odot(PEF)) = 90^o$. Hence,

$$\Phi : PM \leftrightarrow PP_a, M \leftrightarrow P_a$$

Since, A', M are the midpoints of segments PP_a and PX. Hence,

$$\Phi : A' \leftrightarrow X, \odot(PXKD) \leftrightarrow \overline{A', T, A}$$

Therefore, we have A', T, A are collinear. Similarly, we have B', B, T are collinear, and C', C, T are collinear. Then, AA', BB', CC' are concurrent at T. $\square$

3 Two circles tangent to each other at the Anti-Steiner point

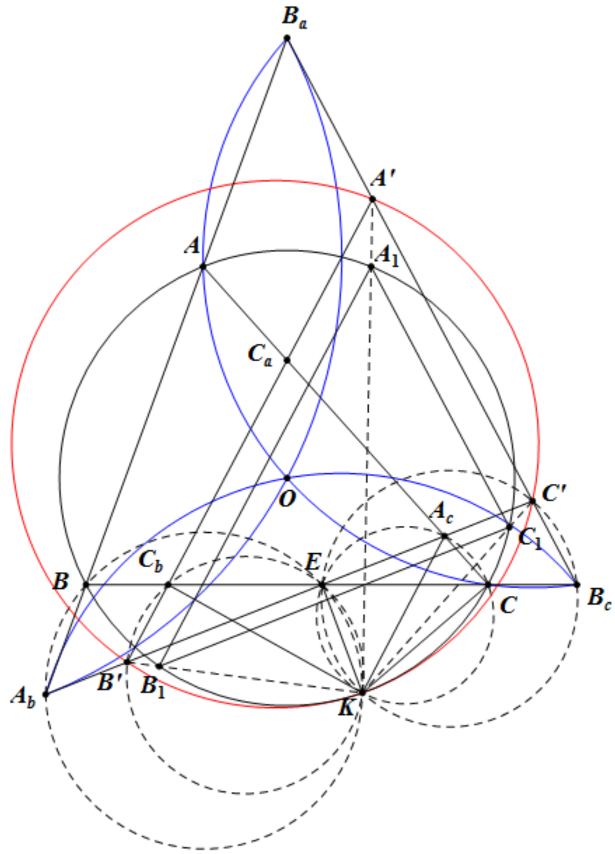
In some problems about two circles tangent to each other, the tangent point is the Anti-Steiner point or a point related to it. In many cases, defining the tangent point may help us to find the way to solve the problem and explore many other properties.

Problem 3.1 (Le Phuc Lu). *Let ABC be a triangle. Suppose that perpendicular bisector of AB cuts AC at A_1, perpendicular bisector of AC cuts AB at A_2. Similarly define B_1, B_2, C_1, C_2. Three lines A_1A_2, B_1B_2, C_1C_2 cut each other creating a triangle DEF. Prove that the circumcircle of triangle DEF is tangent to the circumcircle of triangle ABC.*

Solution. First, we have a converse Steiner's Theorem.

Lemma 3.1. *Let ABC be a triangle with orthocenter H and circumcenter O. K is a point on plane such that the reflections of K in BC, CA, AB are collinear with H. Then, K lies on the circumcenter of triangle ABC.*

Back to our main problem.



Let E be the intersection of A_bA_c and BC. Let K be the Miquel point of complete quadrilateral $AE.BA_c.A_bC$. We can easily see that O is the orthocenter of triangle AA_bA_c, hence, OH is the Steiner line of the complete quadrilateral $AE.BA_c.A_bC$. Let K_1, K_2 be the reflections of K in AB and AC, respectively, H_1 be the orthocenter of triangle EA_cC. Then, we have O, K_1, K_2, H_1 are collinear. Moreover, O and H_1 are the orthocenters of triangles C_aC_bC and EA_cC, respectively. According to the **Lemma 3.1.**, $K \in \odot(CC_aC_b)$ và $K \in \odot(CEA_c)$. Hence, K is the Miquel point of the complete quadrilateral $C_aE.C_bA_c.B'C$. Hence,

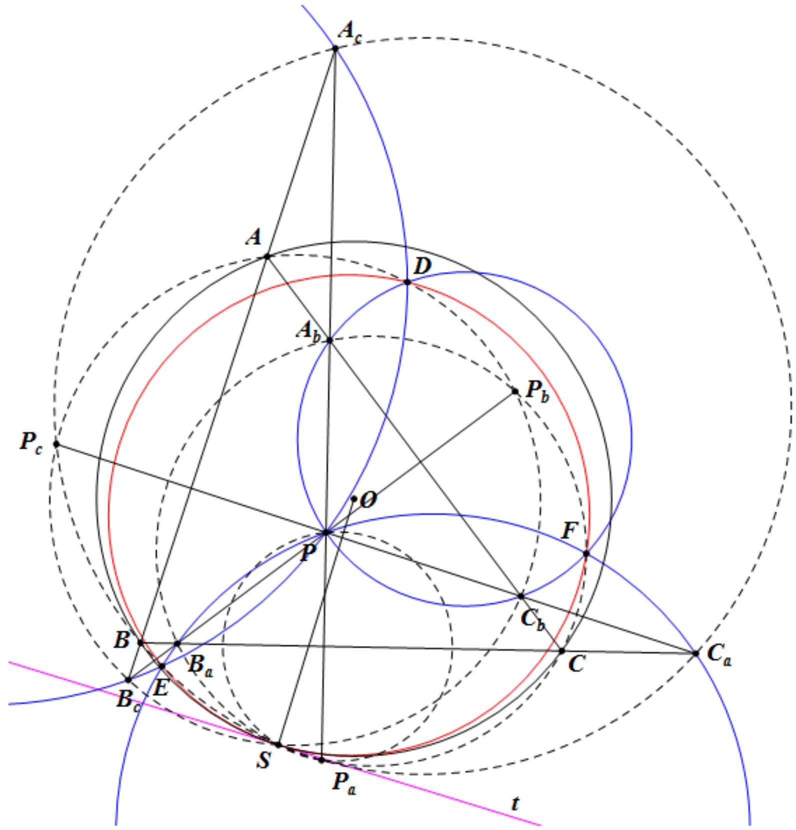
$$(KE, KB') \equiv (C_bE, C_bB') \equiv (C_bB, C_bC_a) \equiv (AB, AC)$$

Given a point A_1 on $\odot(O)$ such that $AA_1 \parallel BC$. Denote B_1, C_1 similarly. We have,

$$\begin{aligned}(KE, KB_1) &\equiv (KE, KC) + (KC, KB_1)\\ &\equiv (A_cE, A_cC) + (A_cC, AB_1)\\ &\equiv (CB, CA) + (BA, BC)\\ &\equiv (AB, AC)\end{aligned}$$

Then, we have $(KE, KB_1) \equiv (KE, KB')$. Therefore, K, B, B_1 are collinear. Similarly, K, A_1, A' are collinear, and K, C_1, C' are collinear. We can easily see that $A_1B_1 \parallel A'B', A_1C_1 \parallel A'C'$, and $B_1C_1 \parallel B'C'$. Hence, K is the center of the homothety Φ mapping $\triangle A_1B_1C_1$ to $\triangle A'B'C'$. Besides, K lie on both circles $\odot(A_1B_1C_1)$ and $\odot(A'B'C')$. Therefore, $\odot(ABC)$ is tangent to $\odot(A'B'C')$ at the Anti-Steiner point K. $\square$

Problem 3.2 (Dao Thanh Oai). *Denote triangle ABC and an arbitrary point P. The line through P and perpendicular to BC meets AB, AC at A_c, A_b, respectively. The line through P and perpendicular to CA meets BC, BA at B_a, B_c, respectively. The line through P and perpendicular to AB meets CA, CB at C_b, C_a, respectively. $\odot(PC_bA_b) \cap \odot(PA_cB_c) = \{D; P\}$, $\odot(PA_cB_c) \cap \odot(PB_aC_a) = \{E; P\}$, $\odot(PB_aC_a) \cap \odot(PA_bC_b) = \{F; P\}$. Prove that $\odot(DEF)$ and $\odot(ABC)$ are tangent to other at the Anti-Steiner point of P.*



Solution. Let S be the Anti-Steiner point of P with respect to triangle ABC. We will prove that $\odot(DEF)$ passes through point S. Indeed, let P_a, P_b, P_c be the reflections of P in BC, CA, AB, respectively. According to the **Lemma 3.1.**, since the reflections of S in AB, AC are collinear with orthocenter P of triangle AB_cC_b, $S \in \odot(AB_cC_b)$. According to the **Lemma 1.1.**, we have P_b, P_c also lie on the $\odot(AB_cC_b)$. Moreover, we have,

$$\begin{aligned}(DB_c, DC_b) &\equiv (DB_c, DP) + (DP, DC_b) \\ &\equiv (A_cB_c, A_cP) + (A_bP, A_bC_b) \\ &\equiv (AB, AC) \\ &\equiv (AB_c, AC_b)\end{aligned}$$

Hence, $D \in (AB_cC_b)$. Therefore, six points S, A, P_b, P_c, C_b, B_c are concyclic. Similarly, we have S, B, P_a, P_c, C_a, A_c are concylcic, and S, C, P_a, P_b, A_b, B_a are concyclic. Therefore,

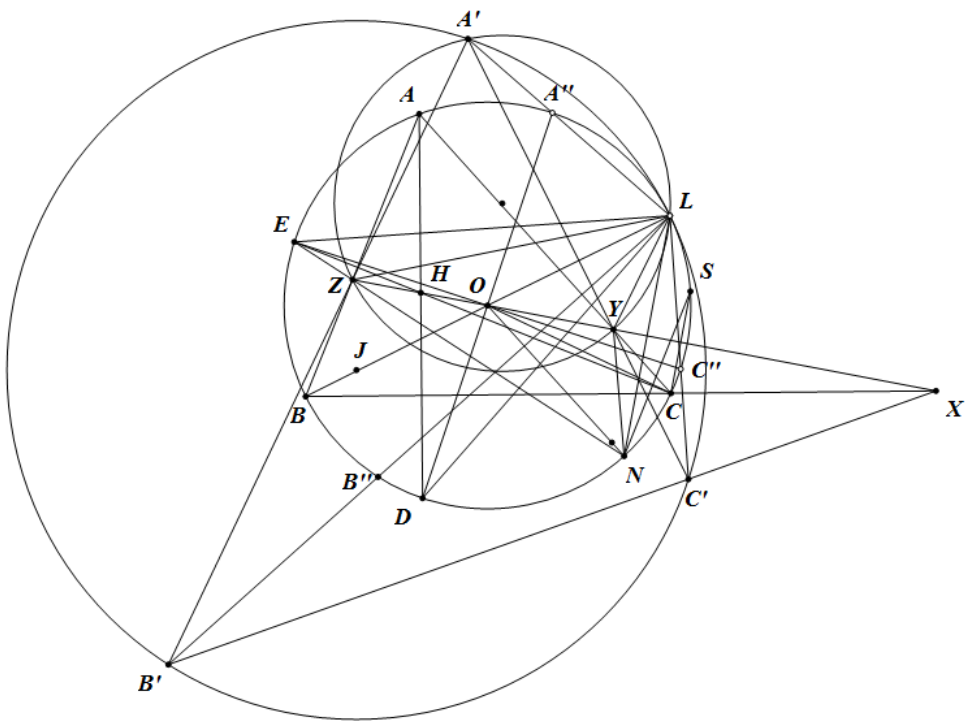
$$\begin{aligned}(DE, DF) &\equiv (DE, DP) + (DP, DF) \\ &\equiv (A_cE, A_CP) + (A_bP, A_bF) \\ &\equiv (A_cE, A_bF) \\ &\equiv (A_cE, A_cB) + (A_cB, A_bC) + (A_bC, A_bF) \\ &\equiv (SE, SB) + (AB, AC) + (SC, SF) \\ &\equiv (SE, SB) + (SB, SC) + (SC, SF) \\ &\equiv (SE, SF)\end{aligned}$$

Hence, S, D, E, F are concylic. Denote that St is the tangent at S of $\odot(DEF)$. We will show that St is tangent to $\odot(ABC)$. Indeed,

$$\begin{aligned}(St, SA) &\equiv (St, SD) + (SD, SA)\\ &\equiv (ES, DE) + (C_bD, C_bA)\\ &\equiv (ES, EA_c) + (EA_c, ED) + (C_bD, C_bA_b)\\ &\equiv (BS, BA_c) + (PA_c, PD) + (PD, PA_b)\\ &\equiv (BS, BA)\end{aligned}$$

Hence, St is tangent to $\odot(ABC)$. Therefore, $\odot(DEF)$ is tangent to $\odot(ABC)$ at the Anti-Steiner point S. □

Problem 3.3 (Nguyen Van Linh). *Let ABC is a triangle with circumcenter O and orthocenter H. OH meets BC, CA, AB at X, Y, Z respectively. The lines passing through X, Y, Z and perpendicular to OA, OB, OC respectively meet each other that create triangle $A'B'C'$. Prove that $\odot(A'B'C')$ is tangent to $\odot(O)$.*

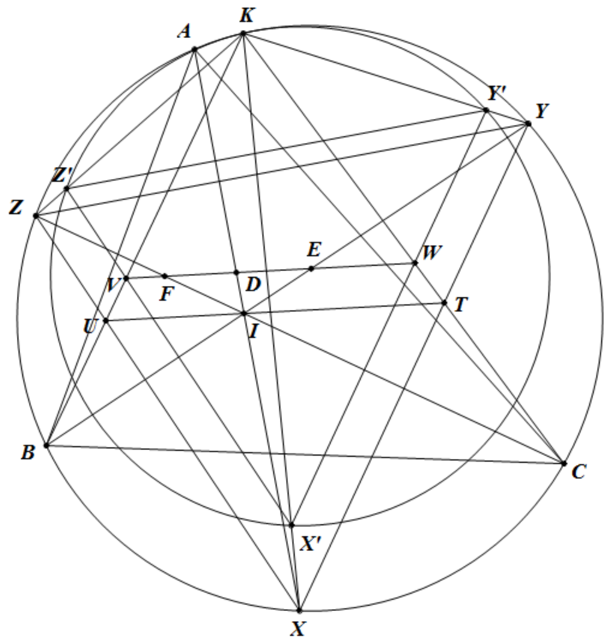


Solution. Let N be the Anti-Steiner point of the line OH with respect to triangle ABC. Denote that L is the reflection of N in OH. Hence, $L \in \odot(O)$. We have A is the excenter of triangle NYZ. Hence, $\angle YNZ = 180^o - 2\angle BAC = \angle ZA'Y$. Since, N and L are symmetric with respect to OH, $\angle YLZ = \angle YNZ$. Hence, $\angle YLZ = \angle ZA'Y$ or $L \in \odot(A'YZ)$. Lines AH, BH, CH meet $\odot(O)$ at second points D, E, F, respectively. Let $A", B", C"$ be the antipodes of D, E, F in $\odot(O)$, respectively. We can easily see that $EF \perp AO$, hence $B"C" \parallel EF \parallel B'C'$. Similarly, we can have that there is a homothety Φ mapping $\triangle A'B'C'$ to $\triangle A"B"C"$. We will show that $A'A", B'B", C'C"$ are concurrent at point L.

Indeed, we have E, Z, N are collinear, hence $\angle OEZ = \angle ONZ = \angle OLZ$. Hence, E, Z, O, L are concyclic. We can see that $A', A", L$ are collinear if and only if $\angle A'LD = 90^o$, which equivalent to $\angle A'YZ + \angle ZLD = 90^o$. Note that $YA' \perp OB$ then $\angle A'YZ + \angle ZLD = 90^o \Leftrightarrow \angle ZOB = \angle ZLD \Leftrightarrow$ $\angle AZO - \angle ABO = \angle ZLO + \angle OLD = \angle NEC" + 90^o - \angle DAL = \frac{1}{2}\angle NOC" + \angle ACB - \frac{1}{2}\angle COL$ (1). Let S be the reflection of C in OH. We have $\angle NOC = \angle LOS$, hence $\angle CHO = \angle NEC = \angle LNS$. Besides, $LN \perp OH$ and $NS \perp CH$. Hence, EC, EO are isogonal conjugate lines in $\angle NES$, or

$\angle NOC" = \angle COS$. Hence, (1) $\Leftrightarrow \angle ACB - \angle NEC = \frac{1}{2}\angle NOC" + \angle ACB - \frac{1}{2}\angle COL$. This is equivalent to $\angle COL = \angle NOC" + \angle NOC = \angle COS + \angle SOL$, which is correct. Therefore, we have $A'A", B'B", C'C"$ are concurrent at point L. Hence, L is the center of the homothety Φ. Then, $\odot(A'B'C')$ is tangent to $\odot(A"B"C")$ at point L. $\square$

Problem 3.4 (IMO Shortlist 2018 - G5). *Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.*



Solution. Denote that X, Y, Z are the midpoints of the minor arcs BC, CA, AB of $\odot(ABC)$. We can easily see that $XB = XC = XI$, $ZI = ZB = ZA$, $YI = YC = YA$. Then I is the orthocenter of triangle XYZ. Line d passing through I and parallel to DE meets XZ, XY at U, T respectively. Let K be the Anti-Steiner point of UT with respect to triangle XYZ. We can easily see that B, U, K are collinear and C, T, K are collinear. BK meets DE at V. Then,

$$(EV, EB) \equiv (IU, IB) \equiv (BI, BU) \equiv (BE, BV)$$

Then V lies on the perpendicular bisector of segment BE. Similarly, denote W be the intersection of CK and DE, then W lies on the perpendicular bisector of segment CF. Let Z', X' are the intersections of the perpendicular bisector of segment EB with KZ and KX respectively. We have,

$$\frac{\overline{KY}}{\overline{KY'}} = \frac{\overline{KT}}{\overline{KW}} = \frac{\overline{KU}}{\overline{KV}} = \frac{\overline{KZ}}{\overline{KZ'}} = \frac{\overline{KX}}{\overline{KX'}} \Rightarrow Z'Y' \parallel ZY, X'Y' \parallel XY$$

Hence $X'Y'Z'$ is the triangle Θ. Let Φ is the homothety with center K and ratio $\frac{\overline{KY'}}{\overline{KY}}$, we have

$$\Phi : X \mapsto X', Y \mapsto Y', Z \mapsto Z', \odot(XYZ) \mapsto \odot(X'Y'Z')$$

Moreover, K lies on both $\odot(XYZ)$ and $\odot(X'Y'Z')$. Therefore, $\odot(XYZ)$ is tangent to $\odot(X'Y'Z')$ at K. $\square$

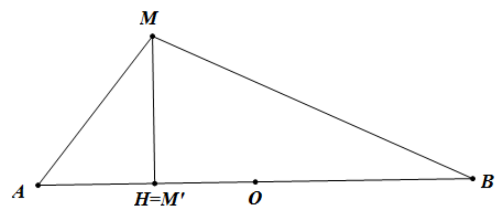
4 The orthopole of a line through the circumcenter with respect to the original triangle

First, I will introduce the definition of the orthopole.

Problem 4.1. *Given triangle ABC and line l. Let X, Y, Z be the projections of A, B, C to l, respectively. Prove that, the lines passing through X, Y, Z and perpendicular to BC, CA, AB respectively are concurrent at a point S.*

Solution. First, we have two lemmas.

Lemma 4.1. *Given the line AB and a constant k. Denote that O is the midpoint of segment AB, H is a point on segment AB such that $\overline{OH} = \dfrac{k}{2\overline{AB}}$. Then, the set of the points M such that $MA^2 - MB^2 = k$ is the line Δ passing through H and perpendicular to AB.*



Proof. We have,

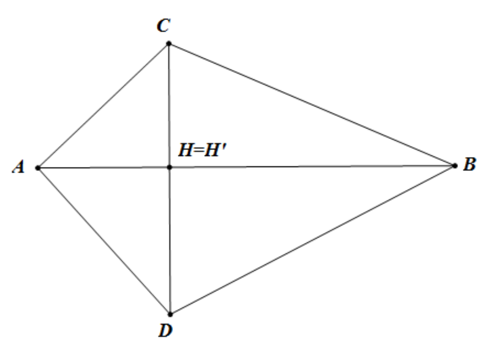
$$\begin{aligned} MA^2 - MB^2 &= (M'M^2 + M'A^2) - (M'M^2 + M'B^2) \\ &= \overline{M'A}^2 - \overline{M'B}^2 \\ &= (\overline{M'A} - \overline{M'B})(\overline{M'A} + \overline{M'B}) \\ &= \overline{BA}(\overline{M'O} + \overline{OA} + \overline{M'O} + \overline{OB}) \\ &= \overline{BA}.2\overline{M'O} \\ &= 2\overline{OM'}.\overline{AB} \end{aligned}$$

Therefore, we have

$$\begin{aligned} MA^2 - MB^2 = k &\Leftrightarrow \overline{OM'} = \frac{k}{2\overline{AB}} \\ &\Leftrightarrow \overline{OM'} = \overline{OH} \\ &\Leftrightarrow M' \equiv H \\ &\Leftrightarrow M \in \Delta \end{aligned}$$

□

Lemma 4.2 (Four-point Lemma). *Given two lines AB and CD. Then, $AB \perp CD$ if and only if $AC^2 - AD^2 = BC^2 - BD^2$.*



Proof. First, assume that $AB \perp CD$.
Let H be the intersection of AB and CD. According to the Pythagorean Theorem, we have,

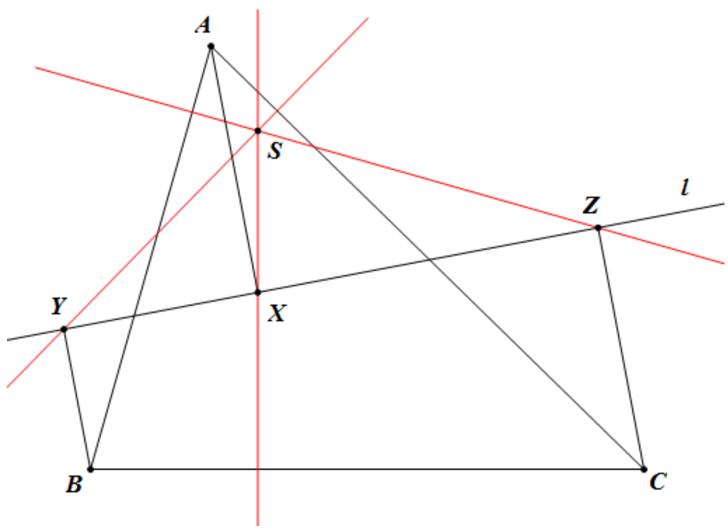
$$\begin{aligned} AC^2 - AD^2 &= (HA^2 + HC^2) - (HA^2 + HD^2) \\ &= (HB^2 + HC^2) - (HB^2 + HD^2) \\ &= BC^2 - BD^2 \end{aligned}$$

Second, assume that $AC^2 - AD^2 = BC^2 - BD^2$.
Let H, H' be the projections of A, B to CD. According to the Pythagorean Theorem, we have,

$$\begin{aligned} HC^2 - HD^2 &= (HA^2 + HC^2) - (HA^2 + HD^2) \\ &= AC^2 - AD^2 \\ &= BC^2 - BD^2 \\ &= (H'B^2 + H'C^2) - (H'B^2 + H'D^2) \\ &= H'C^2 - H'D^2 \end{aligned}$$

Then, according to the **Lemma 4.1.**, we have $AB \perp CD$. □

Back to our main problem



Let S be the intersection of the line passing through Y, Z and perpendicular to CA, AB, respectively. We will show that $SX \perp BC$. Indeed, according to the **Four-point Lemma** and the Pythagoras's theorem, we have,

$$\begin{aligned} SB^2 - SC^2 &= (SB^2 - SA^2) - (SC^2 - SA^2) \\ &= (ZB^2 - ZA^2) - (YC^2 - YA^2) \\ &= ((BY^2 + YZ^2) - (ZX^2 + XA^2)) - ((YC^2 + ZC^2) - (YX^2 + XA^2)) \\ &= (BY^2 + XY^2) - (ZC^2 + ZX^2) \\ &= (XB^2 - XC^2) \end{aligned}$$

According to the **Four-point Lemma**, we have $SX \perp BC$. □

Note. *The concurrent point S in the problem is called the orthopole of the line l with respect to the triangle ABC.*

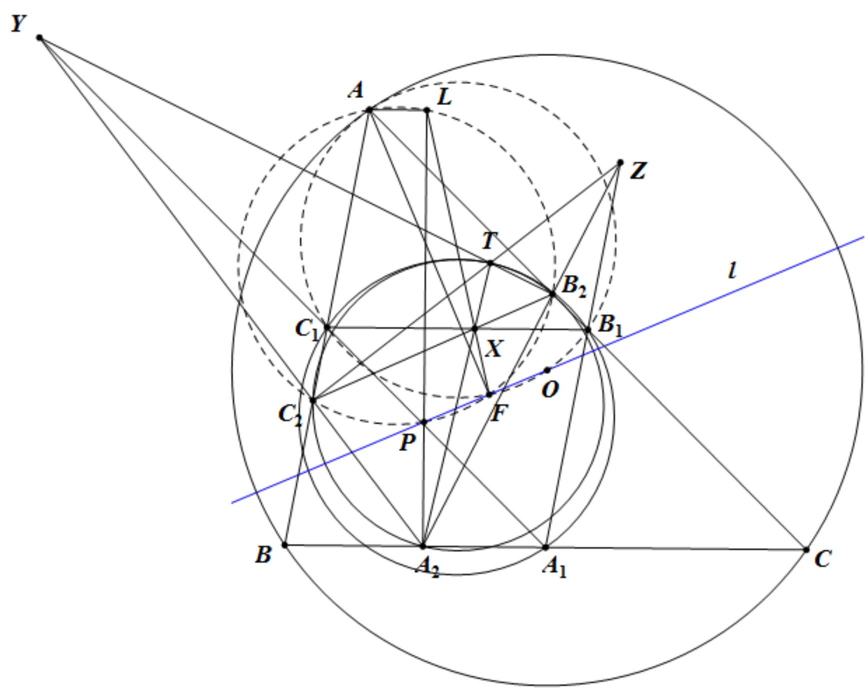
Next, I will introduce three popular theorems related to the orthopole from Fontene.

Theorem 5 (The first Fontene's Theorem). *Given triangle ABC and an arbitrary point P on the plane. Let A_1, B_1, C_1 be the midpoints of segments BC, CA, AB, respectively. Let $\triangle A_2B_2C_2$ be the Pedal triangle of P with respect to triangle ABC. Let X, Y, Z be the intersections of three pairs of lines $(B_1C_1, B_2C_2), (C_1A_1, C_2A_2), (A_1B_1, A_2B_2)$, respectively. Then, A_2X, B_2Y, C_2Z are concurrent at one common point of two circles $\odot(A_1B_1C_1)$ and $\odot(A_2B_2C_2)$.*

Theorem 6 (The second Fontene's Theorem). *Given an arbitrary point P lying on a fixed line d passing through the circumcenter O of triangle ABC. Then, the Pedal triangle of P with respect to triangle ABC always meets the Nine-point circle of triangle ABC at a fixed point.*

Theorem 7 (The third Fontene's Theorem). *Let P, Q be the two isogonal conjugate points in triangle ABC. Let O be the circumcenter of the triangle. Then, the Pedal triangle of P with respect to triangle ABC is tangent to the Nine-point circle of triangle ABC if and only if PQ passes through center O.*

We will prove those theorems in the same proof.



Proof. Let F, L be the projections of A to l and PA_2, respectively. Then, A, C_1, B_1, F, O lie on the circle with diameter AO, and A, L, C_2, B_2, P, F lie on the circle with diameter AP. Hence, F is the Miquel point of the complete quadrilateral from four lines AB, AC, C_1B_1, C_2B_2, then, X, C_1, C_2, F are concyclic. We have

$$\angle XFC_2 = \angle AC_1X = \angle ALX = \angle LXC_1$$

Hence L, X, F are collinear. Note that the circle with diameter AO is symmetric to the Nine-point circle ($\odot(A_1B_1C_1)$) of triangle ABC with respect to B_1C_1. Since, L is the reflection of A_2 in B_1C_1, the reflection T of F in B_1C_1 is the second intersection of A_2X and $\odot(A_1B_1C_1)$. Since l passes through the orthocenter O of triangle $A_1B_1C_1$, then T be the Anti-Steiner point of P with respect to triangle $A_2B_2C_2$. Similarly, we have B_2Y, C_2Z also pass through the Anti-Steiner point T. We proved the first Fontene's Theorem.

Note that the location of T on $\odot(A_1B_1C_1)$ does not depend on the location of P on line l, it just depends on the location of l passing through O. Therefore, we can prove the second Fontene's Theorem.

Now, according to a popular property of the Pedal triangle of two isogonal conjugate points, we have the center of $\odot(A_2B_2C_2)$ is the midpoint of segment PQ. Hence, if S is the second intersection of $\odot(A_1B_1C_1)$ and $\odot(A_2B_2C_2)$, S is the Anti-Steiner of Q with respect to triangle $A_1B_1C_1$. Hence, $T \equiv S$ if and only if $OP \equiv OQ$ or O, P, Q are collinear. We proved the third Fontene's Theorem. □

Comment.

- From the above proof, we notice that the orthopole of a line passing through the circumcircle of a triangle is the Anti-Steiner point of that line with respect to the median triangle (creating from three midpoints of three sides).
- Also, from the proof, if $P \equiv Q$ at the incenter of triangle ABC, the point T coincides with the Feuerbach point F_e of the triangle ABC, which is the tangent point of the incircle and the Nine-point circle of triangle ABC. Then, we have the next theorem.

Theorem 8. *The Feuerbach point is the Anti-Steiner point of the line passing the incenter and the circumcenter with respected to the tangent triangle.*

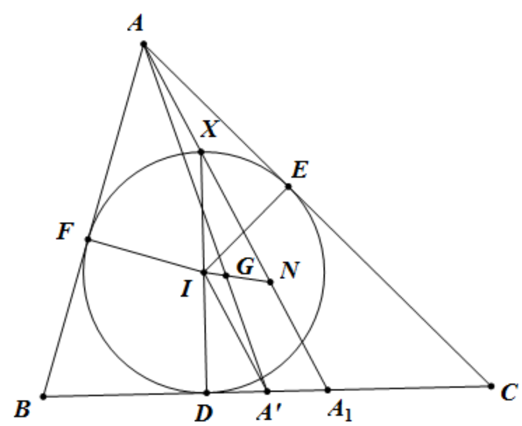
Comment. *It is a beautiful property that can be applied to solve some other problems related to the Anti-Steiner point with respect to the tangent triangle.*

Problem 4.2. *In $\triangle ABC$ let N be the Nagel point, O the circumcenter and T the Anti-Steiner point of N with respect to $\triangle ABC$. Prove that T lies on ON.*

Solution. First, we have a popular lemma

Lemma 4.3. *Let ABC be a triangle with incenter I, centroid G, and the Nagel point N. Then,*

$$\overrightarrow{GI} = -\frac{1}{2}\overrightarrow{GN}$$



Proof. Indeed, denote that D, E, F is the tangents points of the incircle with BC, CA, AB, respectively. Denote that I_a, I_b, I_c is the excenter of triangle ABC with respect to A, B, C, respectively. $\odot(I_a)$ is tangent to BC at A_1, similarly define B_1, C_1. Let X, Y, Z be the antipodes of D, E, F in the incircle, respectively. Let A', B', C' be the midpoints of segments BC, CA, AB, respectively. We have A is the center of the homothety Φ mapping $\odot(I)$ to $\odot(I_a)$. Then,

$$\Phi : \odot(I) \mapsto \odot(I_a), I \mapsto I_a, IX \mapsto I_aA_1, X \mapsto A_1$$

Then, we have A, X, A_1 are collinear. Moreover, we can easily find that $BD = CA_1 = \dfrac{BA + BC - AC}{2}$. Hence, A' is the midpoint of segment DA_1. Hence, IA_1 is the midline of triangle XDA_1. Then, $IA' \parallel AA_1$. Similarly, we have $IB' \parallel BB_1, IC' \parallel CC_1$. Let Θ be the homothety with center G and ratio $-\dfrac{1}{2}$. We have,

$$\Theta : A \mapsto A', B \mapsto B', C \mapsto C', \triangle ABC \mapsto \triangle A'B'C'$$

Since $IA' \parallel AA_1, IB' \parallel BB_1, IC' \parallel CC_1$, hence,

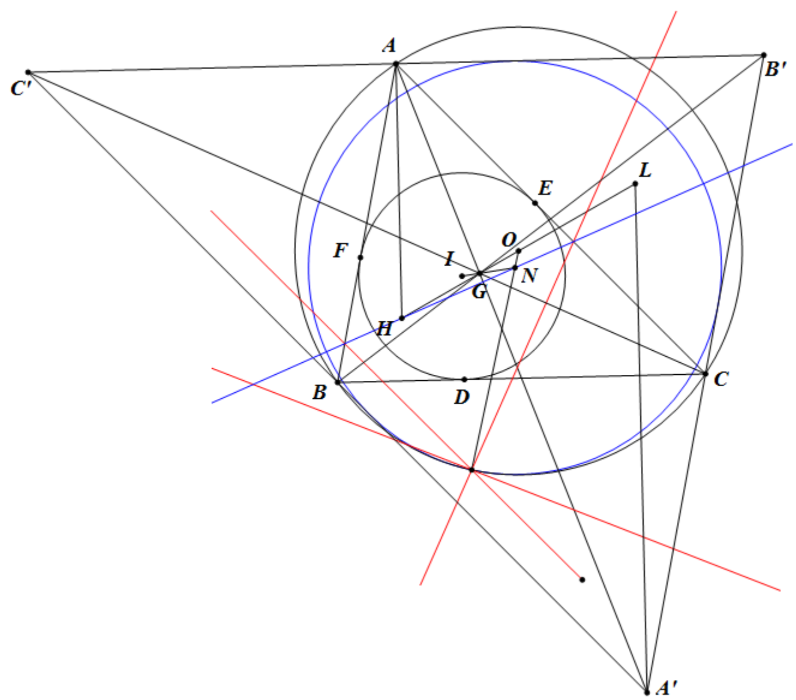
$$\Theta : AA_1 \mapsto A'I, BB_1 \mapsto B'I, CC_1 \mapsto C'I$$

Since, $A'I, B'I, C'I$ are concurrent at I and AA_1, BB_1, CC_1 are concurrent at N, we have

$$\Theta : N \mapsto I \Leftrightarrow \overrightarrow{GI} = -\frac{1}{2}\overrightarrow{GN}$$

□

Back to our main problem,



Let G be the centroid of triangle ABC. Let Φ is the homothety with center G and ratio -2, we have,

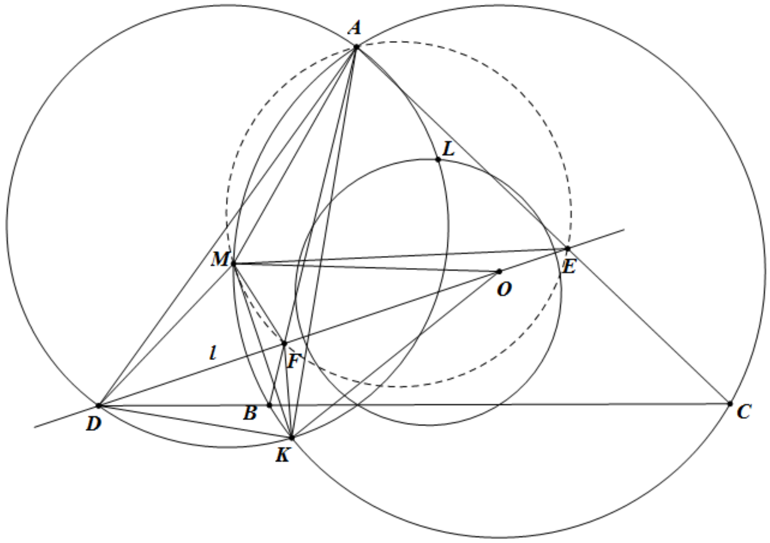
$$\Phi : A \mapsto A', B \mapsto B', C \mapsto C', I \mapsto N$$

Therefore, N is the incenter of triangle $A'B'C'$. According to the **Theorem 8.**, we have T is the Feuerbach point of triangle $A'B'C'$. Moreover, we have O is the center of the Nine-point circle of triangle $A'B'C'$, and the Nine-point circle is tangent to the incircle at the Feuerbach point T. Therefore, T lies on the line passing through two centers O and N of those circles.

□

Now, we will see some other problems related to the orthopole of a line passing through the circumcenter and the application of Fontene's Theorems.

Problem 4.3. *Let ABC be the triangle with circumcenter O. A line l passing through O meets BC, CA, AB at D, E, F respectively. Prove that three circles with diameters AD, BE, CF are concurrent at two points: one point is the orthopole of l with respect to triangle ABC, and the other is the reflection of the Miquel point of complete quadrilateral $AD.BE.CF$.*



Solution. Let M be the Miquel point of the complete quadrilateral creating from four lines AB, AC, BC, l and K be the reflection of M in l. We have,

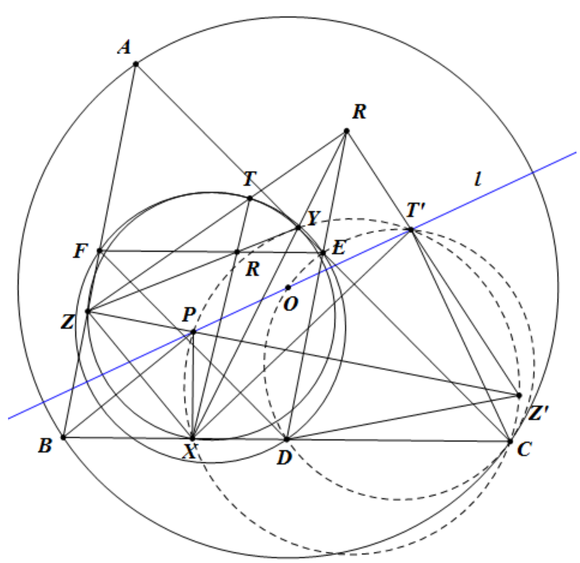
$$\angle MKA = \angle MBF = \angle MDF = \angle KDF$$

Hence, $\angle DKA = 90^o$ or K lies on the circle with diameter AD. Similarly, we have K also lies on the circles with diameters BE and CF, respectively. Let L be the orthopole of l with respect to triangle ABC. According to the Fontene's Theorem, we have L is the intersection of the Pedal circles of points D, E, F with respect to triangle ABC and the Nine-point circle of the triangle. Therefore, we can conclude that three circles with diameters AD, BE, CF are concurrent at two points: one point is the orthopole of l with respect to triangle ABC, and the other is the reflection of the Miquel point of complete quadrilateral $AD.BE.CF$

□

Problem 4.4 (Taiwan TST 2013 - Round 2). *Let $ABCD$ be a cyclic quadrilateral with circumcircle $\odot(O)$. Let l be a fixed line passing through O and P be a point varies on l. Let Ω_1, Ω_2 be the pedal circles of P with respect to $\triangle ABC, \triangle DBC$, respectively . Find the locus of $T \equiv \Omega_1 \cap \Omega_2$ (different from the projection of P on BC).*

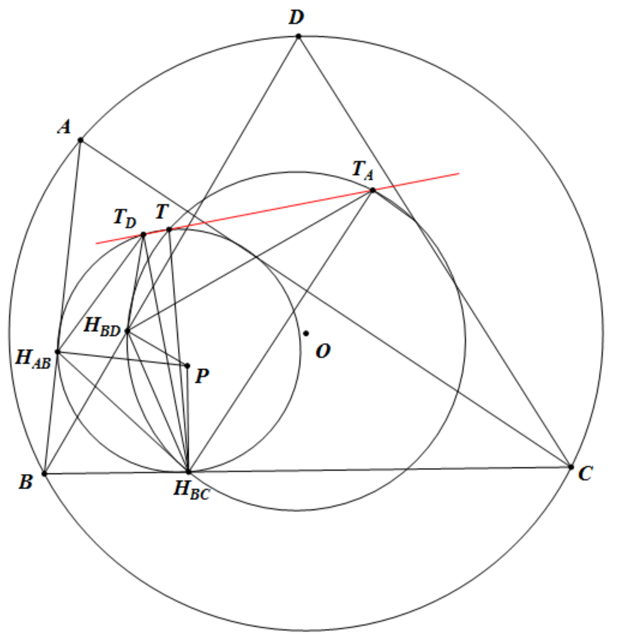
Solution. (Telv Cohl)
First, we have a lemma.



Lemma 4.4. *Given a fixed circle $\odot(O)$ and a fixed point P . Let B, C be two fixed points on $\odot(O)$ and A be a point varies on $\odot(O)$. Let T be the orthopole of OP with respect to $\triangle ABC$ and $\triangle XYZ$ be the pedal triangle of P with respect to $\triangle ABC$. Then $\angle XZT$ is fixed when A varies on $\odot(O)$*

Proof. Let D, E, F be the midpoint of BC, CA, AB, respectively. Let Z' be the reflection of Z in DE and $R \equiv DE \cap XY$. Let T' be the reflection of T in DE (i.e. the projection of C on OP) . From Fontene's Theorem we get Z, T, R are collinear. Since $C, P, X, Y, T', Z' \in \odot(CP)$ and $C, O, D, E, T' \in \odot(CO)$, so $\angle PZT = \angle Z'ZR = \angle RZ'Z = \angle T'Z'P = \angle T'CP =$ constant, hence combine $\angle XZP = \angle CBP =$ constant $\Longrightarrow \angle XZT$ is fixed when A varies on $\odot(O)$. □

Back to our main problem,



Let H_{AB} be the projection of P on AB (define H_{BC}, H_{BD}, similarly) . Let T_A, T_D be the orthopoles of ℓ with respect to $\triangle DBC, \triangle ABC$, respectively. According to Fontene's Theorem, we have $T_A \in \Omega_2, T_D \in \Omega_1$. According to the **Lemma 4.4.**, we have $\angle H_{BC}H_{AB}T_D = \angle H_{BC}H_{BD}T_A$, so $\angle T_DTH_{BC} + \angle H_{BC}TT_A = 180°$. Therefore, T_A, T, T_D are collinear. Hence the locus of T is a line passing through T_A and T_D when P varies on l . □

5 Other Anti-Steiner point related problems

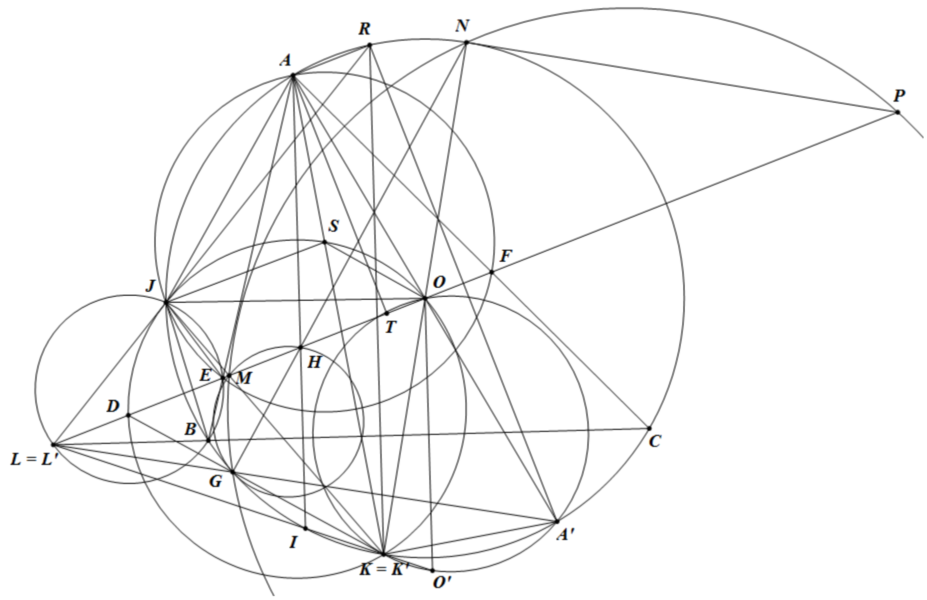
In this section, I will introduce some other beautiful problems and properties related to the Anti-Steiner point.

Problem 5.1 (DeuX Mathematics Olympiad 2020 Shortlist G5 (Level II Problem 3)). *Given a triangle ABC with circumcenter O and orthocenter H. Line OH meets AB, AC at E, F respectively. Define S as the circumcenter of AEF. The circumcircle of AEF meets the circumcircle of ABC again at J, $J \neq A$. Line OH meets circumcircle of JSO again at D, $D \neq O$ and circumcircle of JSO meets circumcircle of ABC again at K, $K \neq J$. Define M as the intersection of JK and OH and DK meets circumcircle of ABC at points K, G. Prove that circumcircle of GHM and circumcircle of ABC are tangent to each other.*

Solution. $EF \cap BC = L$. I, O' are the reflections of H and O in BC. A' is the antipode of A in circle (O). We will use six claims to solve this problem.

Claim 1. *K is the Anti-Steiner point of OH with respect to triangle ABC.*

Indeed, Let K' is the Anti-Steiner point of OH. AT is the altitude of triangle AEF. We will prove that AT, AK' are isogonal conjugate in $\angle BAC$. First, we can see that L, K, I, O' are collinear. Then, $\angle O'K'A' = \angle IAA' = \angle O'OA'$. Hence, K', O', O, A' are concyclic. Hence, $\angle AHT = \angle IO'O = \angle OA'K'$. Then, $\angle HAT = 90^o - \angle AHT = 90^o - \angle AA'K' = \angle K'AA'$. Hence, AT, AK' are isogonal conjugate in $\angle HAO$. Moreover, since AH, AO are isogonal conjugate in $\angle BAC$, AT, AK' are isogonal conjugate in $\angle BAC$. Besides, since AS, AT are isogonal conjugate in $\angle BAC$, A, S, K' are collinear. Moreover, we can easily see that $\triangle JSE \sim \triangle JOB$. Hence, $\triangle JSO \sim \triangle JEB$. Hence, $\angle JK'A = \angle JBE = \angle JOS$. Then, J, S, O, K' are concyclic. Hence, $K \equiv K'$ and K is the Anti-Steiner point of line OH.



Claim 2. *R is a point on $\odot(O)$ such that $KR \perp BC$. Then $AR \parallel OH$.*

This is a popular property of Simpson line.

Claim 3. *R, J, L are collinear.*

Indeed, we can easily see that J is the Miquel point of the complete quadrilateral $(EC.FB.AL)$. Then J, E, B, L are concyclic. $RJ \cap OH = L'$. We have, $\angle JL'E = \angle JRA = \angle JBE$. Hence, J, B, E, L' are concyclic. Then, $L' \equiv L$. Hence, R, J, L are collinear.

Claim 4. *Let N be the antipode of K. Then, $LA' \cap NH = G$*

Indeed, since $AR \parallel OH$ and $AR \perp RA'$, we have $RA' \perp OH$. Then, A' is the reflection of R in OH. Besides, since O is the midpoint of arc JK of $\odot(JOK)$, DO is the bisector of $\angle JDK$. Hence, J is the reflection of G in OH. Then, L, G, A' are collinear. Moreover, applying the Pascal's Theorem to the set of 6 points $\begin{pmatrix} G & A & K \\ I & N & A' \end{pmatrix}$, we have G, H, N are collinear.

Claim 5. *$\odot(GMN) \cap OH = \{M; P\}$. Then, PN is tangent to cirlce $\odot(O)$.*

Indeed,since $\angle ODK = \angle OJK = \angle OKJ$, $\angle OMK = \angle OKG$. Then, we have $\angle GNP = \angle GMH = \angle JMH = 180^o - \angle OMK = \angle OKG$. Hence, PN is tangent to $\odot(O)$.

Claim 6. *$\odot(MGH)$ is tangent to circle $\odot(O)$.*

Indeed, let Φ is the inversion with center H and power $\overline{HA}.\overline{HI}$. Then, we have

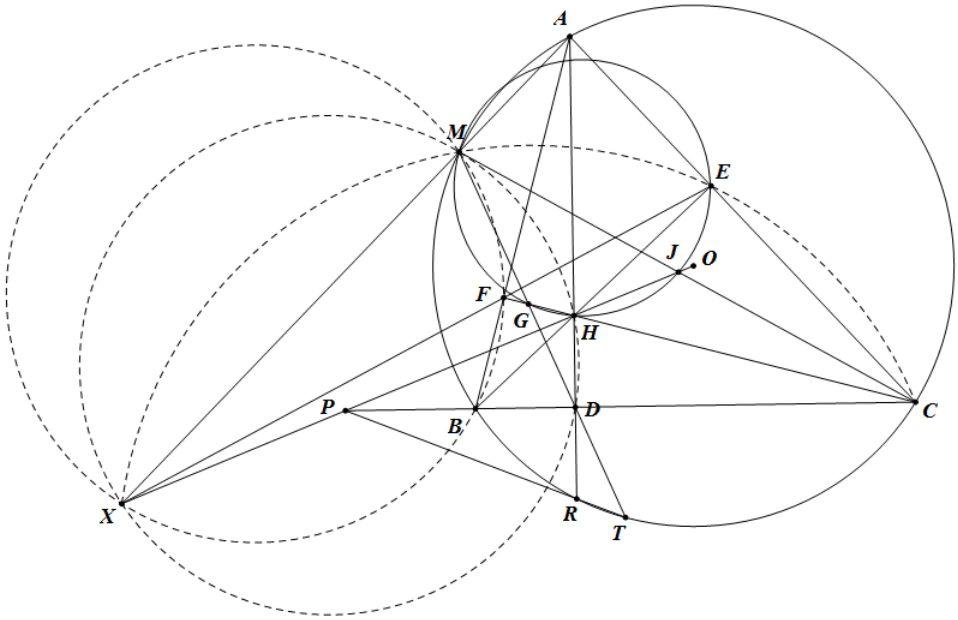
$$\Phi : H \leftrightarrow H, A \leftrightarrow I, G \leftrightarrow N, M \leftrightarrow P, \odot(O) \leftrightarrow \odot(O), \odot(MGH) \leftrightarrow PN$$

Since, PN is tangent to $\odot(O)$, $\odot(MGH)$ is tangent to $\odot(O)$. □

From **Claim 1.**, we have a corollary:

Corollary 1. *Let ABC be a triangle with orthocenter H. Let l be a line passing through H. Let K be the Anti-Steiner point of l with respect to triangle ABC. Then, line AK and the line passing through A perpendicular to l are two isogonal conjugate lines in $\angle BAC$.*

Problem 5.2. *Let Γ be the circumcircle of a triangle ABC with Euler line l. Let DEF be the orthic triangle and H the orthocenter. $X = l \cap EF, M = AX \cap \Gamma, J = l \cap MC$ and $G = CH \cap MD$. Prove that $HJMG$ are concylic.*



Solution. I will redefine the problem. Let Γ be the circumcircle of a triangle ABC with Euler line l. Let DEF be the orthic triangle and H is the orthocenter. $OH \cap BC = P$, $AH \cap \Gamma = \{A, R\}$, $PR \cap \Gamma = \{R, T\}$. We have T is the Anti-Steiner point of line l with respect to $\triangle ABC$. $TD \cap \Gamma = \{T, M\}$, $AM \cap l = X$. We will prove that X, E, F are collinear. Indeed, we have

$$(MX, MD) \equiv (MA, MT) \equiv (RA, RT) \equiv (RA, RP) \equiv (HP, HD) \equiv (HX, HD)$$

Therefore X, M, F, B are concyclic, hence,

$$(XM, XF) \equiv (BM, BF) \equiv (BM, BA)$$

Similarly, we have X, M, E, C are concyclic, hence,

$$(XM, XE) \equiv (CM, CA) \equiv (BM, BA) \equiv (XM, XF)$$

Therefore, X, E, F are collinear. Then

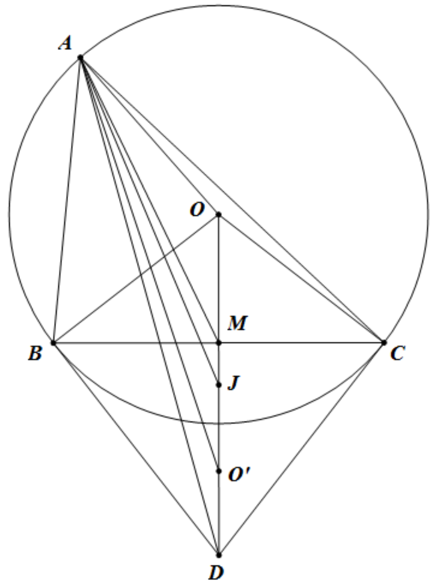
$$(HG, HJ) \equiv (HC, HP) \equiv (RP, RC) \equiv (RT, RC) \equiv (MT, MC) \equiv (MG, MJ)$$

Therefore, H, G, J, M are concyclic. □

Problem 5.3 ("Quan Hinh" Topic - April 2019 - Nguyen Duc Toan). *Triangle ABC is scalene. Let two points D and E lie on rays BC and CB respectively such that $BD = BA$ and $CE = CA$. The circumcircle of triangle ABC meets the circumcircle of triangle ADE at A and L. Let F be the midpoint of arc AB which not contain C of the circumcircle of ABC. Let K lie on CF such that $\angle FDA = \angle KDE$. $KL \cap AE = J$. Prove that J lies on the Euler line of triangle ADE.*

Solution. First, we have a lemma.

Lemma 5.1. *Let ABC be a triangle with circumcenter O. O' is the reflection of O in BC. Let J be the circumcircle of triangle BOC. Then, AJ, AO' are isogonal conjugate lines in angle $\angle BAC$.*

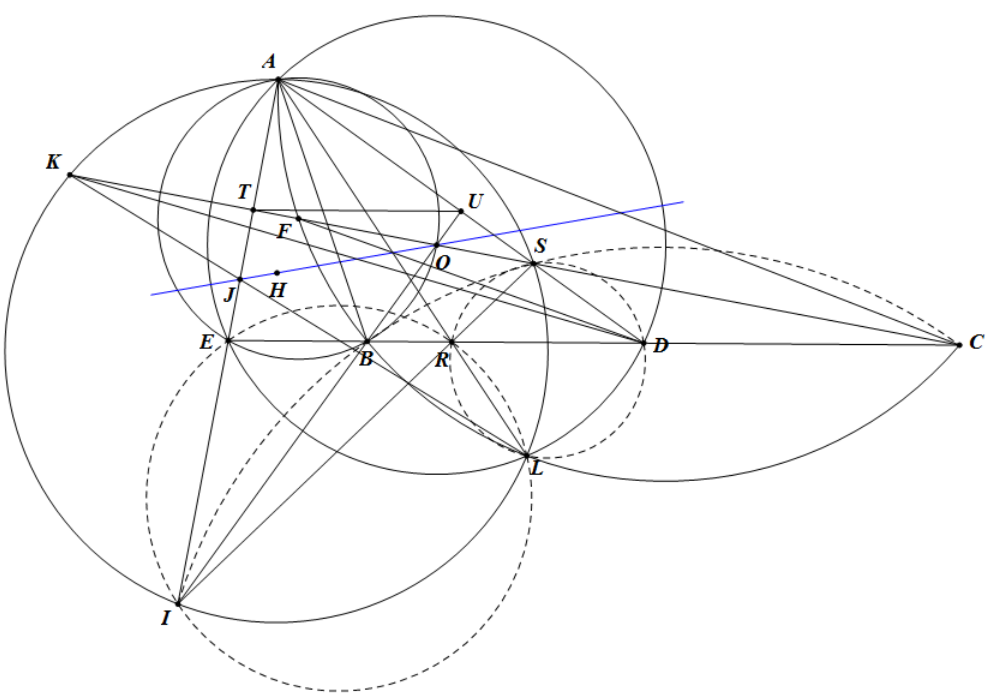


Proof. Let D be the intersection of two tangents from B and C of $\odot(O)$. Denote that M is the midpoint of segment BC. Since, AD is the symmedian, AD and AM are isogonal conjugate lines in angle $\angle BAC$. Moreover, we have

$$OM.OD = OB^2 = OA^2 = OJ.OO'$$

Hence, $\triangle OAM \sim \triangle ODA$ and $\triangle OJA \sim \triangle OAO'$. Hence, $\angle OAM = \angle ODA$ and $\angle OAJ = \angle OO'A$. Then, $\angle DAO' = \angle JAM$. Hence, AJ, AO' are two isogonal conjugate lines in angle $\angle BAC$. □

Back to our main problem,



Denote that O and H are the circumcenter and orthocenter of triangle AED, respectively. OC meets AD at S; OB meets AE at I. We can easily get that C, O, L lie on the perpendicular bisector of segment AE, and O is the incenter of triangle ABC. Then, F is the center of $\odot(AOB)$. Besides, we have

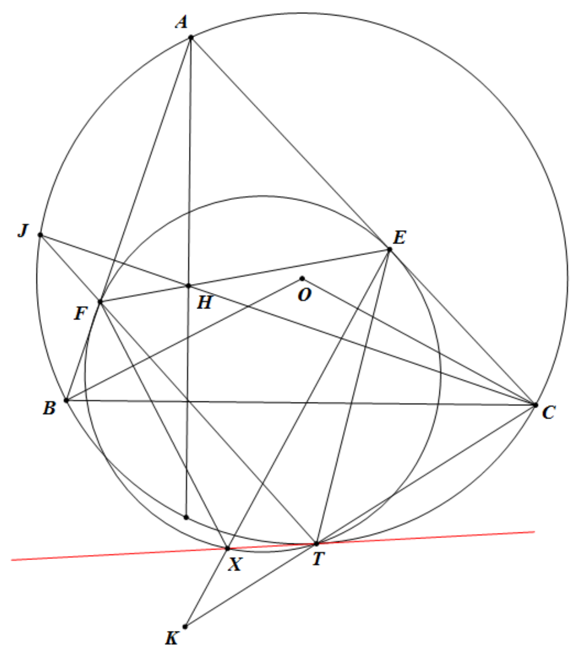
$$\angle AEB = 90^o - \frac{\angle ACB}{2} = 180^o - (90^o - \frac{\angle ACB}{2}) = \angle 180^o - \angle AOB$$

Hence, A, E, O, B are concyclic and F is the center of $\odot(AOE)$. Similarly, we have O, E, R, S, D are concyclic. According to the **Lemma 5.1.**, we have that K is the reflection of O in AE.
Now, we need to prove that L is the Anti-Steiner point of OH with respect to triangle AED.
Indeed, let L' be the Anti-Steiner point of OH with respect to triangle AED. We can easily see that O is the orthcenter of triangle AIS. Hence, L' is the Anti-Steiner point of line OH with respect to triangle AIS. Hence, $L' \in \odot(AIS)$. IS meets ED at R. Hence, L' is the Miquel point of the complete quadrilateral $AR.ES.ID$. Let T and U be the midpoints of segments AE and AD, respectively. Hence,

$$(L'I, L'A) \equiv (SI, SA) \equiv (SI, SU) \equiv (TI, TU) \equiv (EI, ER) \equiv (L'I, L'R)$$

Hence, L', R, A are collinear. We can easily prove that $OB.OI = OE^2 = OS.OC$. Hence, I, B, S, C are concyclic. Then, $RB.RC = RS.RI = RL'.RA$. Hence, A, B, C, L' are concylcic. Therefore, $L' \equiv L$ or L is the Anti-Steiner point of OH with respect to triangle ADE.
Hence, line LK is the reflection of line OH in AE. Hence, J lies on the Euler line of triangle ADE.
□

Problem 5.4 (Nguyen Van Linh). *Let ABC be a triangle with circumcenter O and orthocenter H. A line d passing through H meets BC, CA, AB at D, E, F respectively. The line through F and perpendicular to OB meets the line through E and perpendicular to OC at X. Similarly define Y, Z. Prove that X, Y, Z lie on a line tangent to $\odot(O)$.*

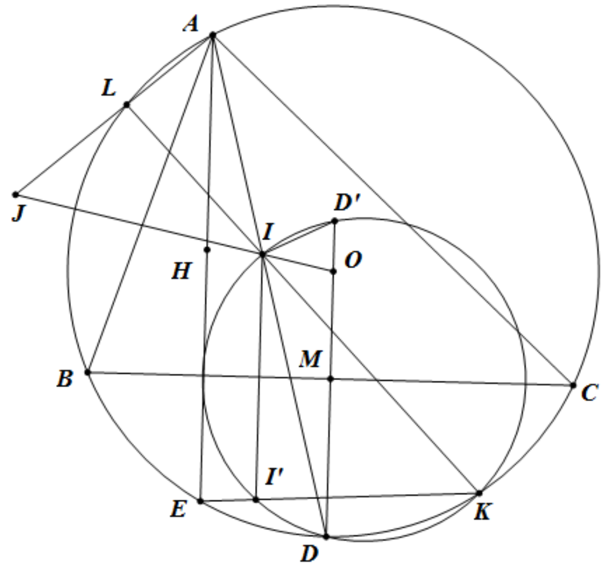


Solution. Let T be the Anti-Steiner point of line d with respect to triangle ABC. CH meets $\odot(O)$ at the second point J. Then, J, H, C are collinear. Since A is the excenter of triangle TEF, $\angle FTE = 180^o - 2\angle BAC = 180^o - \angle BOC = \angle EXF$. Hence, E, T, X, F are concyclic. Let K be the intersection of EX and TC. We have $\angle EKC = 90^o - \angle OCT = \angle TJC = \angle EHC$. Hence, E, H, K, C are concyclic. Then, $\angle TCJ = \angle XEF = \angle XTJ$. Hence, XT is tangent to $\odot(O)$. Similarly, we have X, Y, Z lie on the tangent at T of $\odot(O)$. □

Problem 5.5. *Let (I) be the incircle of triangle ABC and D, E, F be the contacts triangle. Let F_e be the Feuerbach point of triangle ABC. Let K be the orthocenter of triangle DEF. Let S be the Anti Steiner point of I with respect to triangle ABC. Prove that $KF_e \parallel IS$.*

Solution. First, we have two lemmas.

Lemma 5.2 (Ha Huy Khoi). *Let ABC be the triangle with circumcenter O, incenter I, and orthocenter H. K is the Anti-Steiner point of HI with respect to $\triangle ABC$. KI meets $\odot(O)$ at the second point L. AL meets OI at J. Then, AL is symmetric to OI with respect to the perpendicular bisector of segment AI.*



Proof. Indeed, AH meets $\odot(O)$ at the second point E. I' is the reflection of I in BC. Then, EI' passes through K. AI meets $\odot(O)$ at the second point D. Then,

$$(II', ID) \equiv (AE, AD) \equiv (KI', KD)$$

Hence, I, I', D, K are concyclic. Let D' be the reflection of D in BC, M be the midpoint of segment BC, DN be the diameter of $\odot(O)$. We have: $DI^2 = DB^2 = DM.DN = DD'.DO \Rightarrow \triangle DIO \sim \triangle DD'I$. Therefore,

$$(IA, IO) \equiv (IO, ID) \equiv (D'D, D'I) \equiv (KD, KI) \equiv (AI, AL)$$

Hence,

$$(AI, AL) \equiv (IA, IO) \equiv (IJ, IA)$$

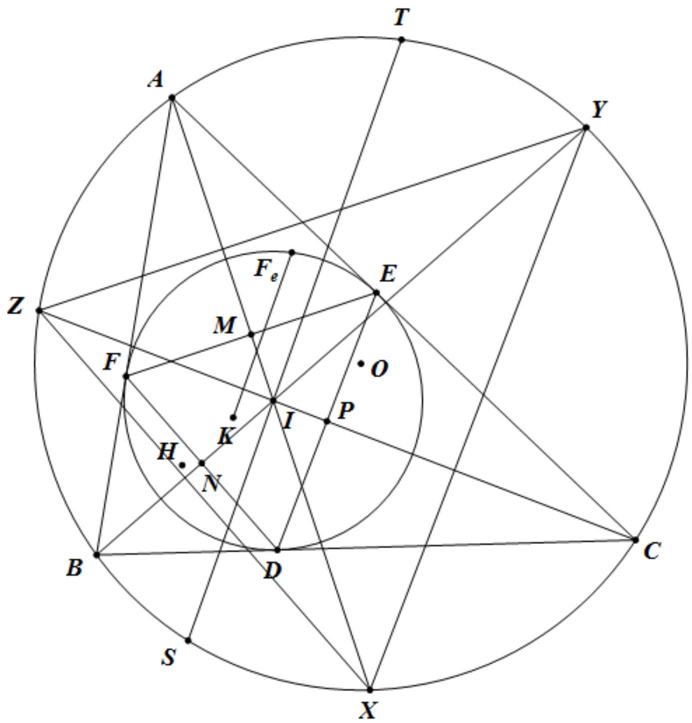
Hence, we have $JA = JI$. Hence, AL is symmetric to OI with respect to the perpendicular bisector of segment AI. □

Lemma 5.3 (Nguyen Duc Toan). *Let ABC be the triangle with circumcenter O and orthocenter H. T is the Anti-Steiner point of OH with respect to triangle ABC. Three altitudes of triangle ABC meet $\odot(O)$ at the second points D, E, F, respectively. TH meets $\odot(O)$ at the second point K. Then, K is the Anti-Steiner of H with respect to triangle DEF.*

Proof. (Ha Huy Khoi)

Let K' be the Anti-Steiner point of H with respect to triangle DEF. $K'H$ meets (O) at the second point T'. Applying **Lemma 5.2.** into the $\triangle DEF$ with its incenter H, we have DT' is symmetric to OH with respect to the perpendicular bisector of segment DH. Then, DT' is the reflection of OH in BC. Similarly, we have ET', FT' are the reflections of OH in CA, AB , respectively. Hence, T' is the Anti-Steiner point of OH with respect to $\triangle ABC$. Therefore, $T' \equiv T, K' \equiv K$. Hence, K is the Anti-Steiner point of H with respect to triangle DEF. □

Back to our main problem.



Let O, H be the circumcenter and the orthocenter of triangle ABC, X, Y, Z be the second intersections of AI, BI, CI and $\odot(O)$, respectively. Let M, N, P be the midpoint of segments EF, FD, DE respectively. Let Θ be the inversion with center I and power IE^2. We have

$$\Theta : A \leftrightarrow M, B \leftrightarrow N, C \leftrightarrow P, \odot(ABC) \leftrightarrow \odot(MNP)$$

Hence we have, I, O and the center of $\odot(MNP)$ are collinear. Hence, O lies on the Euler line of triangle DEF. Hence, O, I, K are collinear. Moreover, according to Fontene's theorem, we have F_e is the Anti-Steiner point of OK. Let T be the Anti-Steiner point of H with respect to triangle XYZ. According to **Lemma 5.3.**, we have T, I, S are collinear. Besides, we can easily get $EF \parallel YZ, FD \parallel XZ, DE \parallel XY$. Then, there is a homothety Φ that

$$\Phi : D \mapsto X, E \mapsto Y, F \mapsto Z, \triangle DEF \mapsto \triangle XYZ, \odot(DEF) \mapsto \odot(XYZ), K \mapsto I, F_e \mapsto T$$

Therefore, $KF_e \parallel IT$ or $KF_e \parallel IS$.

□

To end this article, I will introduce some practice problems related to the Anti-Steiner point.

6 Practice problems

Problem 6.1. *Let ABC be triangle with orthocenter H. D, E lies on $\odot(ABC)$ such that DE pass through H. S is an $Anti-Steiner$ point of DE. N lies on BC such that ON is perpendicular to $SE.EN$ cuts $\odot(ABC)$ at the second point F. Prove that A, O, F are conlinear.*

Problem 6.2 (Nguyen Duc Toan). *Let ABC be the triangle with circumcenter O, orthocenter H, and the center of Nine-point circle N. Let P be the Anti-Steiner point of OH with respect to the triangle. Prove that A, N, P are collinear if and only if $AP \perp OH$.*

Problem 6.3 (IMOC 2019 - G5). *Given a scalene triangle $\triangle\ ABC$ with orthocenter H and circumcenter O. The exterior angle bisector of $\angle BAC$ intersects circumcircle of $\triangle\ ABC$ at $N \neq A$. Let D be another intersection of HN and the circumcircle of $\triangle\ ABC$. The line passing through O, which is parallel to AN, intersects AB, AC at E, F, respectively. Prove that DH bisects the angle $\angle EDF$.*

Problem 6.4. *Let ABC be a triangle with orthocentre H and circumcentre O. The circle through A and B touching AC meets the circle through A and C touching AB at $X_A \neq A$. Define X_B, X_C similarly. Prove that the four circles $\odot(AX_BX_C), \odot(BX_CX_A), \odot(CX_AX_B), \odot(ABC)$ meet at the anti-Steiner point of OH in ABC.*

Problem 6.5 (Nguyen Duc Toan). *Let ABC be a triangle with the incenter I. D, E, and F are the tangent points of the incircle to BC, CA and AB respectively. ID intersects AB and AC at A_b and A_C respectively. Similarly, we define points B_a, B_c, C_a, C_b. Assume that 3 lines A_cB_c, C_aB_a, A_bC_b are pairwise intersect create triangle $A'B'C'$. Prove that the circumcircle of triangle $A'B'C'$ is tangent to the circumcircle of triangle ABC*

Problem 6.6 (Luis Gonzalez). *An arbitrary line ℓ through the circumcenter O of $\triangle ABC$ cuts AC, AB at Y, Z, respectively. The circle with diameter YZ cuts AC, AB again at M, N, respectively. Show that MN passes through the orthopole of ℓ with respect to $\triangle ABC$.*

Problem 6.7. *Let ABC be a triangle with circumcenter O and $\angle B > 90^0$. A line l passes through O cuts CA, AB at E, F such that $BE \perp CF$ at D. Let S be the orthopole of l with respect to $\triangle ABC$. Perpendicular lines from D to SB, SC cut SC, SB at M, N. Draw parallelogram $DMKN$. Prove that S is the midpoint of AK.*

Problem 6.8. *Suppose the triangle ΔABC has circumcenter O and orthocenter H. Parallel lines α, β, γ are drawn through the vertices A, B, C. Let α', β', γ' be the reflections of α, β and γ over BC, CA, AB. Then these reflections are concurrent if and only if α, β and γ are parallel to OH lines of ΔABC. In this case, their point of concurrency P is the reflection of O over the Euler Reflection Point (the Anti-Steiner Point of the Euler Line)*

Problem 6.9 (Telv Cohl). *Let I, H be the incenter, orthocenter of $\triangle ABC$, respectively . Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$ and T be the orthocenter of $\triangle DEF$. Let Fe be the Feuerbach point of $\triangle ABC$ and S be the Anti-steiner point of TFe with respect to $\triangle DEF$. Prove that $IH \perp SFe$*

Problem 6.10 (Nguyen Duc Toan). *Let ABC is a triangle with circumcircle $\odot(O)$ and Anti-median triangle $A'B'C'$ (A is the midpoint of segment $B'C'$, and similar to B, C). Let $A_1B_1C_1$ is the triangle created from three tangents of circle $\odot(O)$ at A, B, C.*

1. *Prove that $B'B_1, C'C_1, BC$ are concurrent at point X. Similarly denote Y, Z.*

2. *Prove that the orthopole of the Euler line of triangle ABC with respect to triangle XYZ lies on the circumcircle of triangle $A'B'C'$.*

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