Junior Problems

J535. Solve in positive integers the equation

$$x^3 - \frac{13}{2}xy - y^3 = 2020.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J536. Let ABC be an acute triangle. Prove that

$$\left(\frac{a+b}{\cos C}\right)^2 + \left(\frac{b+c}{\cos A}\right)^2 + \left(\frac{c+a}{\cos B}\right)^2 \geq \frac{16(a+b+c)^2}{3}$$

Proposed by Florin Rotaru, Focşani, România

J537. Solve in rational numbers the equation

$$x\lfloor x\rfloor\{x\} = 58,$$

where $\lfloor x\rfloor$ and $\{x\}$ are the greatest integer less than or equal to x and the fractional part of x, respectively.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J538. Prove that in any triangle ABC

$$\frac{(a+b)(b+c)(c+a)}{4abc} \leq 1 + \frac{R}{2r}$$

Proposed by Marius Stănean, Zalău, România

J539. Let $\alpha > 0$ be a real number. Prove that if a, b, c are real numbers in the interval $[\alpha, 30\alpha]$,

$$\frac{7}{a+6b} + \frac{7}{b+6c} + \frac{7}{c+6a} \geq \frac{6}{a+5b} + \frac{6}{b+5c} + \frac{6}{c+5a}$$

Proposed by Mihaela Berindeanu, Bucharest, România

J540. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x + \lfloor y\rfloor\right) = \lfloor f(x)\rfloor + f(y),$$

for all $x,\ y \in \mathbb{R}$.

Proposed by Besfort Shala, University of Primorska, Koper, Slovenia

Senior Problems

S535. Find all triples (p, q, r) of primes such that

$$\frac{1}{p-1} + \frac{1}{q} + \frac{1}{r+1} = \frac{1}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S536. Let $S = \{2, 4, 6, \ldots, 2020\}$ be the set of positive even integers not greater than 2020 and $T = \{3, 6, 9, \ldots, 2019\}$ be the set of positive multiples of 3 less than 2020. Evaluate

$$\sum_{A \subseteq S} \sum_{B \subseteq T} |A \cup B|.$$

Proposed by Li Zhou, Polk State College, USA

S537. Let ABC be a triangle with $\angle B = 50^\circ$. Let D be a point on the segment BC such that $\angle BAD = 30^\circ$ and $AD = BC$. Find $\angle CAD$.

Proposed by Marius Stănean, Zalău, România

S538. Let $P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$ be a polynomial with real coefficients and n an even positive integer. If $P(x)$ has n non-negative real roots, prove that

$$1 + \sqrt[n]{a_n} \leq \sqrt[n]{P(-1)} \leq 1 - \frac{a_1}{n}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S539. There is a street with n houses on the left side and n houses on the right side. Moreover, the houses which are in front of one to another are identical. We have to paint the houses with m different colors in such a way that no two neighboring houses nor two face to face houses are of the same color. In how many ways can this coloring be done?

Proposed by Mircea Becheanu, Canada

S540. Let $s(x)$ denote the sum of digits of the positive integer x. Find all positive integers n such that $s(n) = 3s(n+1)$.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Undergraduate Problems

U535. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{k}=1.$$

Evaluate

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{a_1+\cdots+a_k}{k^3}$$

Proposed by Serban Cioculescu, Găeşti, România

U536. Evaluate

$$\int x\left(\sqrt{1+x}+\sqrt{1-x}\right)\,\mathrm{d}x.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U537. Let k be a positive integer. Evaluate

$$\lim_{n\to\infty} n^{2k}\left(\frac{\arctan n^k}{n^k}-\frac{\arctan\left(n^k+1\right)}{n^k+1}\right)$$

Proposed by Dinu Ovidiu Gabriel, Bălceşti, Vâlcea, România

U538. Let $p>2$ be a prime and $r>1$ be an integer. Define the set

$$S=\{a\in\mathbb{Z}\mid a^a\equiv a\pmod{p^r},\ 2\leq a\leq p^{r-2}\}.$$

Prove that

$$|S|\leq p^{r-2}(p-1)\left(p-\varphi(p-1)\right).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U539. Evaluate

$$\int_0^1\frac{-2x^2\ln x}{\sqrt{1-x^2}\left(x^4-x^2+1\right)}dx$$

Proposed by Paolo Perfetti, Roma, Italy

U540. Let x be rational number. Prove that $\mathbb{Q}(\sqrt[3]{x})=\mathbb{Q}\left(\sqrt[3]{2}\right)$ if and only if $x=2q^3$ or $x=4q^3$, for some rational number q.

Proposed by Mircea Becheanu, Canada

Olympiad Problems

O535. Let $ABCD$ be a convex quadrilateral with an incircle. Let ω_1, ω_2, ω_3, ω_4 be the excircles of $ABCD$ tangent to segments AB, BC, CD, DA, respectively. Prove that the lengths of the internal common tangent segments to the pairs of circles (ω_1, ω_3) and (ω_2, ω_4) are equal.

Proposed by Waldemar Pompe, Warsaw, Poland

O536. Let a, b, c be the side-lengths of the sides of a triangle and let S be its area. Let R and r be the circumradius and inradius of the triangle, respectively. Prove that

$$a^2 + b^2 + c^2 \le 4S\sqrt{\frac{6r}{R}} + 3\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right].$$

Proposed by Marius Stănean, Zalău, România

O537. Let $i_1 < \cdots < i_l$ and $j_1 \le \cdots \le j_m$ be nonnegative integers such that

$$2^{i_1} + \cdots + 2^{i_l} = 2^{j_1} + \cdots + 2^{j_m}.$$

Prove that $l \le m$.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O538. Find the greatest constant λ such that the inequality

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \lambda \ge \frac{2}{3}(1+\lambda)\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)$$

holds for all positive real numbers a, b, c.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O539. Let $A_1A_2B_1B_2C_1C_2$ be a convex hexagon in which all angles are obtuse. Let $A_1A_2 \cap B_1B_2 = C$, $B_1B_2 \cap C_1C_2 = A$, and $C_1C_2 \cap A_1A_2 = B$. Let O be the circumcenter of ABC. Suppose that $\angle B_2OC_1 = \angle BAC$, $\angle C_2OA_1 = \angle CBA$, and $\angle A_2OB_1 = \angle ACB$. Prove that

$$A_1A_2 + B_1B_2 + C_1C_2 \le A_2B_1 + B_2C_1 + C_2A_1.$$

Proposed by Dominik Burek, Krakow, Poland

O540. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a}{\sqrt[3]{4(b^6+c^6)} + 7bc} + \frac{b}{\sqrt[3]{4(c^6+a^6)} + 7ca} + \frac{c}{\sqrt[3]{4(a^6+b^6)} + 7ab} + \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} \ge \frac{7}{12}.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam