A Tetrahedron Whose Faces Have Equal Area Is "Equifacial"

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1. Introduction. A tetrahedron is called *equifacial* if all of its faces are congruent triangles, which happens if and only if every edge is congruent to the opposite one. The word "isosceles was in use instead of "equifacial" (see, for example, [1]), but nowadays this is not the case anymore–equifacial is, indeed, a more appropriate name. Specifically, the tetrahedron $ABCD$ is equifacial when $[AB] \equiv [CD]$, $[AC] \equiv [BD]$, and $[AD] \equiv [BC]$, in which case $\triangle ABC \equiv \triangle DCB \equiv \triangle CDA \equiv \triangle BAD$. (We denote by $[XY]$ the line segment joining the points X and Y, and by XY the length of this segment. XY also denotes the straight line determined by points X and Y–we believe there will be no confusion by using the same notation for length and line. The sign "$\equiv$" designates the congruence of geometrical figures.) As one would expect, such a geometric configuration has a lot of nice properties. For instance, to name only a few of them, note that: the faces of an isosceles tetrahedron have equal perimeters and equal areas (of course!), the trihedral angles of such a tetrahedron are congruent and the sum of the measures of the plane angles of each of these trihedra is $180°$, the inscribed and circumscribed spheres of an equifacial tetrahedron are concentric, the bimedians are mutually perpendicular and any of them is perpendicular to the edges the midpoints of which it joins, and so on. It is worth noting that these properties actually *characterize* the equifacial tetrahedron, that is, their converses also hold. As a (very simple) example we invite the reader to show that, if the faces of a tetrahedron have equal perimeters, then the tetrahedron is equifacial. In other words, the only possibility for the faces of a tetrahedron to have equal perimeters is that they be congruent triangles. Indeed, it is not hard at all to justify this claim (not so evident, however, as its direct correspondent formulated above, stating that the faces of an equifacial tetrahedron have equal perimeters). If we denote by a, b, c, x, y, z the lengths of the edges $[BC]$, $[AC]$, $[AB]$, $[AD]$, $[BD]$, and $[CD]$ respectively, we are given that

$$a+b+c=a+y+z=b+x+z=c+x+y,$$

and we think that the reader will find an easy exercise to infer from these equations that $a=x$, $b=y$ and $c=z$ follow. Moreover, the careful reader will note that, for instance, $a=x$ and $b=y$ follow only by knowing that $a+b+c=c+x+y$ and $a+y+z=b+x+z$. Thus we can prove that, if in the tetrahedron $ABCD$ we have $P_{ABC}=P_{ABD}$ and $P_{ACD}=P_{BCD}$ (where P_{XYZ} denotes the perimeter of triangle XYZ), then we must have $[AC] \equiv [BD]$ and $[AD] \equiv [BC]$.

We will see a similar phenomenon when dealing with the converse of the property concerning the equality of areas of faces. Although it is obvious (and trivial) that the areas of the faces of an equifacial tetrahedron are equal, it is not clear at all why the converse of this statement would be true (it is more likely that one finds surprising that this converse holds). It is the fact that if the areas of the faces of a tetrahedron are all equal, then each edge of the tetrahedron must be congruent to its opposite edge (and, consequently, the faces are congruent triangles) that is the matter of this note. More specifically, we intend to give a less known (and, basically, algebraic) proof for the following

Proposition 1. If in a tetrahedron $ABCD$ we have $A_{ABC}=A_{ABD}=A_{ACD}=A_{BCD}$ (where A_{XYZ} stands for the area of the triangle XYZ), then we have $[AB] \equiv [CD]$, $[AC] \equiv [BD]$, and $[AD] \equiv [BC]$ (and, thus, the tetrahedron is equifacial).

Various proofs of this folkloric (and, maybe, not enough known) result can be found in the indicated references. The equivalence of some of the above mentioned properties of an equifacial tetrahedron is treated in [7]. As we just said, we further provide an algebraic proof of Proposition 1 (with an inevitable geometric flavor). We do not know any reference in the literature for this proof–however we believe that somebody else might have thought of it until now. Before we proceed with our demonstration, let us remind one of Euler's theorems of elementary geometry, which generalizes Apollonius's theorem.

2. Euler's quadrilateral theorem and some of its consequences. In this section we prove the following

Proposition 2. (Euler's quadrilateral theorem) Given a quadrilateral $ABCD$, and the midpoints M and N of its diagonals $[AC]$ and $[BD]$ respectively, the following relation holds:

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2.$$

Actually, the same is true for *any* points A, B, C, D in space, whenever M and N are the midpoints of $[AC]$ and $[BD]$.

Proof. There are many proofs, all based on the same ideas; we choose the vector way, since it indubitably shows that the result is true for any four points in the space (not necessarily for the vertices of a convex quadrilateral). We first note that

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{BN},$$

and, also,

$$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CD} + \overrightarrow{DN}.$$

By adding together these equalities we get

$$2\overrightarrow{MN} = \overrightarrow{AB} + \overrightarrow{CD},$$

as

$$\overrightarrow{MA} + \overrightarrow{MC} = \overrightarrow{BN} + \overrightarrow{DN} = \overrightarrow{0}.$$

By squaring we get

$$4MN^2 = AB^2 + CD^2 + 2\overrightarrow{AB} \cdot \overrightarrow{CD}$$

(where "$\cdot$" denotes the scalar product). However

$$2\overrightarrow{AB} \cdot \overrightarrow{CD} = 2\overrightarrow{AB} \cdot \left(\overrightarrow{AD} - \overrightarrow{AC}\right) = 2\overrightarrow{AB} \cdot \overrightarrow{AD} - 2\overrightarrow{AB} \cdot \overrightarrow{AC} =$$

$$= \left(AB^2 + AD^2 - BD^2\right) - \left(AB^2 + AC^2 - BC^2\right),$$

and the desired equality follows. Note that

$$2\overrightarrow{AB} \cdot \overrightarrow{AC} = AB^2 + AC^2 - BC^2$$

is just another way to put the cosine law (and it easily follows by squaring $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$; here and above the squaring is, of course, in the sense of the scalar product).

From Euler's relation for quadrilaterals one immediately infers Apollonius's theorem, according to which the sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of its diagonals (as the midpoints of the diagonals of a parallelogram coincide):

Corollary 1. If $ABCD$ is a parallelogram, then

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$$

holds.

The reader will probably remark (or has already noticed) that this is equivalent to the fact that for the length of the median $[AM]$ of a triangle ABC one has

$$4AM^2 = 2(AB^2 + AC^2) - BC^2.$$

Moreover, a converse of the first corollary is also valid, namely we have

Corollary 2. (a) If A, B, C, D are points in space for which

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$$

then A, B, C, D are the vertices of a parallelogram, with diagonals $[AC]$ and $[BD]$. Of course, it is possible for the parallelogram to be degenerate, that is, all the points A, B, C, D might be collinear, such that the midpoints of $[AC]$ and $[BD]$ coincide.

(b) In particular, if we denote $a = BC$, $b = AC$, $c = AB$, $x = AD$, $y = BD$, $z = CD$, and we have $x = a$, $y = b$, and $z^2 = 2(a^2 + b^2) - c^2$, then A, B, C, D are the vertices of a parallelogram (with diagonals $[AB]$ and $[CD]$).

We skip the proofs of the corollaries, as we are convinced that the reader has already understood why do they hold. Nevertheles, since the parallelogram came into our attention, let us notice that the areas of triangles ABC, ABD, ACD, and BCD are all equal whenever $ABCD$ is a parallelogram. However, if the parallelogram $ABCD$ is not a rectangle, then only the equalities $AB = CD$ and $AD = BC$ hold ($AC = BD$ fails). This is one reason for which the property from Proposition 1 seems a little counterintuitive: it remains not true when the tetrahedron degenerates into a (plane) quadrilateral.

We have now all that we need to prove Proposition 1, if we still remember Heron's formula for the area of a triangle in terms of the lengths of its sides–which we do immediately below.

3. Proof of Proposition 1. We keep the above notations $BC = a$, $AC = b$, $AB = c$, $AD = x$, $BD = y$, and $CD = z$. By Heron's formula we have

$$A_{ABC} = \frac{1}{4}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}$$

and, of course, similar formulas for the areas of the other faces of the tetrahedron hold. Thus, by the hypothesis, we know that

$$2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 = 2c^2x^2 + 2c^2y^2 + 2x^2y^2 - x^4 - y^4 - c^4 =$$

$$= 2b^2x^2 + 2b^2z^2 + 2x^2z^2 - b^4 - x^4 - z^4 = 2a^2y^2 + 2a^2z^2 + 2y^2z^2 - a^4 - y^4 - z^4,$$

and we want to prove that $x = a$, $y = b$, and $z = c$ follow from these equalities. Note that we consider the tetrahedron not to be degenerate (not to be a plane quadrilateral). However, it will be clear from the proof that the only alternative to the conclusion $x = a$, $y = b$, and $z = c$ (if A, B, C, D are allowed to be coplanar) is that the four points be (in some order) the vertices of a parallelogram.

If we denote $u = x^2 - a^2$, $v = y^2 - b^2$, and $w = z^2 - c^2$, we have

$$4(b^2 + c^2 - a^2)u - 2(u - v)(u - w) =$$

$$= (2c^2x^2 + 2c^2y^2 + 2x^2y^2 - x^4 - y^4 - c^4) + (2b^2x^2 + 2b^2z^2 + 2x^2z^2 - b^4 - x^4 - z^4) -$$
$$-(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4) - (2a^2y^2 + 2a^2z^2 + 2y^2z^2 - a^4 - y^4 - z^4) = 0.$$

Consequently

$$2(b^2 + c^2 - a^2)u = (u - v)(u - w),$$

and in the same way we get

$$2(a^2 + c^2 - b^2)v = (v - u)(v - w),$$

and

$$2(a^2 + b^2 - c^2)w = (w - u)(w - v).$$

Suppose first that we have, for example, $b^2 + c^2 - a^2 = 0$. The first of the above three equalities tells us that either $u = v$, or $u = w$. Assuming $u = v$, we obtain from the second equality that either $v = 0$ or $a^2 + c^2 - b^2 = 0$. The relation $a^2 + c^2 - b^2 = 0$ leads, together with

$b^2+c^2-a^2=0$ to $c=0$, so this cannot happen. If $v=0$, then $u=v=0$, implying $x=a$ and $y=b$. Further, the third equality becomes

$$2(a^2+b^2-c^2)w=w^2$$

and it gives either $w=0$, or $w=2(a^2+b^2-c^2)$. For $w=0$ we have $z=c$ and we are done. If $w=2(a^2+b^2-c^2)$, that is $z^2=2(a^2+b^2)-c^2$, then part (b) of Corollary 2 applies (do not forget that we already know $x=a$ and $y=b$), therefore $ACBD$ is a parallelogram–which is not allowed by hypothesis. If, on the other hand, we had $u=w$ (instead of $u=v$), then we would get in the same manner that either $x=a$, $z=c$, then $y=b$ (and the proof ends), or $x=a$, $z=c$, and $y^2=2(a^2+c^2)-b^2$ (which, by the second part of Corollary 2, means that $ABCD$ is a parallelogram).

Any of the cases $a^2+c^2-b^2=0$ or $a^2+b^2-c^2=0$ can be treated similarly, thus it only remains to see what happens when $b^2+c^2-a^2\neq 0$, $a^2+c^2-b^2\neq 0$, and $a^2+b^2-c^2\neq 0$. In this situation we can put the three equalities from above in the forms

$$-2u=\frac{(u-v)(w-u)}{b^2+c^2-a^2},$$

$$-2v=\frac{(u-v)(v-w)}{a^2+c^2-b^2},$$

and

$$-2w=\frac{(v-w)(w-u)}{a^2+b^2-c^2}.$$

If we multiply them with $v-w$, $w-u$, and $u-v$ respectively, then add up the resulted equations, we arrive at

$$(u-v)(v-w)(w-u)\left(\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2}\right)=0.$$

Because

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2}=$$
$$=\frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2)}$$

is not equal to 0 (as the numerator is 16 times the square of the area of the triangle ABC), one of the following must hold: $u=v$, or $v=w$, or $w=u$.

If, say, $u=v$, then from the first two equations we get $u=v=0$, and from the third either $w=0$ or $w=2(a^2+b^2-c^2)$ follows, therefore we have $x=a$, $y=b$ and $z=c$ (or $x=a$, $y=b$ and $z^2=2(a^2+b^2)-c^2$, which, as we have already seen, leads to $ACBD$ being a parallelogram). Any of the other two analogous cases ($v=w$, or $w=u$) similarly yields the desired conclusion, and the announced algebraic proof (with an inevitable geometric fragrance) of Proposition 1 is now complete.

4. Final remarks. 1) Note that an equifacial tetrahedron has all its faces acute-angled triangles. We invite the reader to prove this fact, or to read about it in Ross Honsberger's book [6]. Thus none of the cases $b^2+c^2-a^2=0$, $a^2+c^2-b^2=0$, $a^2+b^2-c^2=0$ can actually happen. If one could find a simple way to show that they are not possible only by knowing that the tetrahedron's faces have equal areas, then the proof would be considerably shortened.

2) As said in the beginning, the following slightly more general statement holds:

Proposition 3. If, in tetrahedron $ABCD$, we have $A_{ABC}=A_{ABD}$ and $A_{ACD}=A_{BCD}$, then $[AC]\equiv[BD]$ and $[AD]\equiv[BC]$ hold.

We invite the interested reader to think about an algebraic proof for Proposition 3, or to read one in our book [1]. Note that the proof we presented there also uses Euler's quadrilateral theorem, besides the inevitable algebraic manipulations starting from the given equalities of the areas expressed by Heron's formula.

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