## Junior problems

J535. Solve in positive integers the equation

$$x^3 - \frac{13}{2}xy - y^3 = 2020.$$

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

*Solution by Polyahedra, Polk State College, USA*

Clearly, $(x, y) = (13, 2)$ is a solution. We show that it is the only one. Taking the equation modulo 13 we get $x^3 - y^3 \equiv 5$. Since $\{0, \pm 1, \pm 5\}$ is the complete set of cubic residues modulo 13, we must have either $(x, y) = (13m, 13n - r)$ or $(13m + r, 13n)$, with $m \ge n \ge 1$ and $r \in \{7, 8, 11\}$. If $x = 13m + r$ and $y = 13n$, then the left side of the equation equals

$$(x-y)^3 + \left(3(x-y) - \frac{13}{2}\right)xy \ge 7^3 + \left(21 - \frac{13}{2}\right)(20)(13) > 2020.$$

Consider then $x = 13m$ and $y = 13n - r$. If $n \ge 2$ then we similarly have

$$7^3 + \left(21 - \frac{13}{2}\right)(26)(15) > 2020.$$

Therefore, $n = 1$. If $m \ge 2$, then $(x - y)^3 \ge 20^3 > 2020$. Hence $m = 1$, and it is easy to check that only $r = 11$ works.

*Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Taes Padhihary, Disha Delphi Public School, India.*

J536. Let $ABC$ be an acute triangle. Prove that

$$\left(\frac{a+b}{\cos C}\right)^2+\left(\frac{b+c}{\cos A}\right)^2+\left(\frac{c+a}{\cos B}\right)^2\geq\frac{16\,(a+b+c)^2}{3}.$$

*Proposed by Florin Rotaru, Focşani, România*

*Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam*
By the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left(\frac{b+c}{\cos A}\right)^2+\left(\frac{c+a}{\cos B}\right)^2+\left(\frac{a+b}{\cos C}\right)^2 &\geq\frac{1}{3}\left(\frac{b+c}{\cos A}+\frac{c+a}{\cos B}+\frac{a+b}{\cos C}\right)^2\\ &=\frac{1}{3}\left(\frac{(b+c)^2}{(b+c)\cos A}+\frac{(c+a)^2}{(c+a)\cos B}+\frac{(a+b)^2}{(a+b)\cos C}\right)^2\\ &\geq\frac{1}{3}\left(\frac{(b+c+c+a+a+b)^2}{(b+c)\cos A+(c+a)\cos B+(a+b)\cos C}\right)^2\\ &=\frac{1}{3}\left(\frac{4(a+b+c)^2}{(b\cos C+c\cos B)+(c\cos A+a\cos C)+(a\cos B+b\cos A)}\right)^2\\ &=\frac{1}{3}\left(\frac{4(a+b+c)^2}{a+b+c}\right)^2\\ &=\frac{16(a+b+c)^2}{3}\end{aligned}$$

as desired.

*Also solved by Daniel Văcaru, Piteşti, Romania; Taes Padhihary, Disha Delphi Public School, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.*

J537. Solve in rational numbers the equation

$$x\lfloor x\rfloor\{x\} = 58,$$

where $\lfloor x\rfloor$ and $\{x\}$ are the greatest integer less than or equal to $x$ and the fractional part of $x$, respectively.

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

*Solution by HyunBin Yoo, South Korea*

Let $x = \alpha + \beta$ where $\alpha$ is an integer and $\beta$ is a rational number which satisfies $0 < \beta < 1$. Let $\beta = \frac{n}{m}$ where $\gcd(n, m) = 1$. Then $x\lfloor x\rfloor\{x\} = (\alpha + \frac{n}{m}) \cdot \alpha \cdot \frac{n}{m} = \frac{\alpha n(\alpha m+n)}{m^2} = 58$.

Since $\gcd(m, n) = 1$, $\gcd(m^2, n) = 1$ and $\gcd(m^2, \alpha m + n) = 1$. Therefore, $\alpha$ must be a multiple of $m^2$ in order for the fraction to be an integer. Let $\alpha = m^2 k$, where $k$ is a nonzero integer.

Then the above equation becomes $(m^3 k + n) \cdot k \cdot n = 58$. If $k > 0$, $(m^3 k + n)$ is the biggest out of the three terms so the possible cases are $58 \cdot 1 \cdot 1$, $29 \cdot 2 \cdot 1$ and $29 \cdot 1 \cdot 2$. Out of these three, only $29 \cdot 1 \cdot 2$ results in an integral $m = 3$.

On the other hand, when $k < 0$, none of the possible combinations results in $m$ being an integer. So, the only solution is when $m = 3$, $k = 1$ and $n = 2$. In conclusion, the only rational solution $x = m^2 k + \frac{n}{m} = 9 + \frac{2}{3} = \frac{29}{3}$.

*Also solved by Polyahedra, Polk State College, USA; Alina Craciun, Miron Costin Theoretical High School, Pașcani, România.*

J538. Prove that in any triangle $ABC$

$$\frac{(a+b)(b+c)(c+a)}{4abc} \le 1 + \frac{R}{2r}.$$

*Proposed by Marius Stănean, Zalău, Romania*

*Solution by Arkady Alt, San Jose, CA, USA*

Let $s$ be semiperimeter of a triangle. Since $ab+bc+ca = s^2+4Rr+r^2$, $abc = 4Rrs$ then $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc = 2s\left(s^2+4Rr+r^2\right) - 4Rrs = 2s\left(s^2+2Rr+r^2\right)$ and

$$\begin{aligned} \frac{(a+b)(b+c)(c+a)}{4abc} \le 1 + \frac{R}{2r} &\iff \\ (a+b)(b+c)(c+a) \le 4abc + \frac{2abcR}{r} &\iff \\ 2s\left(s^2+2Rr+r^2\right) \le 16Rrs + 8R^2 s &\iff \\ s^2+2Rr+r^2 \le 4R^2+8Rr &\iff \\ s^2 \le 4R^2+6Rr-r^2&. \end{aligned}$$

Since $s^2 \le 4R^2+4Rr+3r^2$ (Gerretsen's Inequality) and $2r \le R$ (Euler's Inequality) we get $4R^2+6Rr-r^2-s^2 = 4R^2+6Rr-r^2-\left(4R^2+4Rr+3r^2\right)+\left(4R^2+4Rr+3r^2-s^2\right) = 2r(R-2r)+\left(4R^2+4Rr+3r^2-s^2\right) \ge 0.$

*Also solved by Polyahedra, Polk State College, USA; Taes Padhihary, Disha Delphi Public School, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Mihaly Bencze, Brasov, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam.*

J539. Let $\alpha > 0$ be a real number. Prove that if $a, b, c$ are real numbers in the interval $[\alpha, 30\alpha]$, then

$$\frac{7}{a+6b} + \frac{7}{b+6c} + \frac{7}{c+6a} \geq \frac{6}{a+5b} + \frac{6}{b+5c} + \frac{6}{c+5a}.$$

*Proposed by Mihaela Berindeanu, Bucharest, România*

*Solution by Polyahedra, Polk State College, USA*

Since $30b \geq a$,

$$\frac{7}{a+6b} - \frac{6}{a+5b} + \frac{1}{42a} - \frac{1}{42b} = \frac{(a-b)^2(30b-a)}{42ab(a+6b)(a+5b)} \geq 0.$$

Summing with the other two analogous inequalities completes the proof.

*Also solved by Taes Padhihary, Disha Delphi Public School, India.*

J540. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+\lfloor y\rfloor)=\lfloor f(x)\rfloor+f(y),$$

for all $x,y\in\mathbb{R}$.

*Proposed by Besfort Shala, University of Primorska, Koper, Slovenia*

*First solution by Polyahedra, Polk State College, USA*

Setting $x=y=0$ yields $0=\lfloor f(0)\rfloor$, so $f(0)=c$ for some $c\in[0,1)$. Setting $x=0$ gives $f(\lfloor y\rfloor)=f(y)$ for all real $y$. Let $k=\lfloor f(1)\rfloor$. As an induction hypothesis, suppose that $m$ is a nonnegative integer and $f(m)=km+c$. Setting $x=1$ and $y=m$ we get $f(1+m)=k+km+c=k(1+m)+c$.

Therefore, for all nonnegative integer $m$, $f(m)=km+c$, and then letting $x=m$ and $y=-m$, we also have $c=f(m-m)=km+f(-m)$, so $f(-m)=k(-m)+c$ as well.

Hence, for all real $y$, $f(y)=f(\lfloor y\rfloor)=k\lfloor y\rfloor+c$. Finally, if $f(x)=k\lfloor x\rfloor+c$ for any integer $k$ and $c\in[0,1)$, then both sides of the functional equation equal $k\lfloor x\rfloor+k\lfloor y\rfloor+c$ for all real $x,y$.

*Second solution by Joel Schlosberg, Bayside, NY, USA*

For a function $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+\lfloor y\rfloor)=\lfloor f(x)\rfloor+f(y)$ for all $x,y\in\mathbb{R}$, let $c=f(1)-f(0)$ and $d=f(0)$. Then

$$\lfloor d\rfloor=\lfloor f(0)\rfloor=f(0+\lfloor 0\rfloor)-f(0)=0$$

so $d\in[0,1)$, and

$$f(1)=\lfloor f(0)\rfloor+f(1)=f(0+\lfloor 1\rfloor)=f(1+\lfloor 0\rfloor)=\lfloor f(1)\rfloor+f(0)$$

so $c=\lfloor f(1)\rfloor$ is an integer.

Since $f(1)=c\lfloor 1\rfloor+d$ and if $f(n)=c\lfloor n\rfloor+d$ for some $n\in\mathbb{N}$,

$$f(n+1)=f(1+\lfloor n\rfloor)=\lfloor f(1)\rfloor+f(n)=c+c\lfloor n\rfloor+d=c\lfloor n+1\rfloor+d$$

so by induction, $f(n)=c\lfloor n\rfloor+d$ for all $n\in\mathbb{N}$. Also, $f(0)=c\lfloor 0\rfloor+d$ and for $n\in\mathbb{N}$

$$f(-n)=f(n+\lfloor -n\rfloor)-\lfloor f(n)\rfloor=d-\lfloor c\lfloor n\rfloor+d\rfloor=c\lfloor -n\rfloor+d$$

so $f(z)=c\lfloor z\rfloor+d$ for all $z\in\mathbb{Z}$.

Then for $y\in\mathbb{R}$,

$$f(y)=f(0+\lfloor y\rfloor)-\lfloor f(0)\rfloor=f(\lfloor y\rfloor)=c\lfloor y\rfloor+d.$$

Conversely, if $f(x)=c\lfloor x\rfloor+d$ for constants $c\in\mathbb{Z}$ and $d\in[0,1)$,

$$\begin{aligned}f(x+\lfloor y\rfloor)&=c\lfloor x+\lfloor y\rfloor\rfloor+d=c\lfloor x\rfloor+c\lfloor y\rfloor+d\\&=\lfloor c\lfloor x\rfloor+d\rfloor+c\lfloor y\rfloor+d=\lfloor f(x)\rfloor+f(y).\end{aligned}$$

*Also solved by Arkady Alt, San Jose, CA, USA; Alina Craciun, Miron Costin Theoretical High School, Pașcani, România; Taes Padhihary, Disha Delphi Public School, India.*

## Senior problems

S535. Find all triples $(p, q, r)$ of primes such that

$$\frac{1}{p-1} + \frac{1}{q} + \frac{1}{r+1} = \frac{1}{2}$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Li Zhou, Polk State College, USA*

Clearly, $(p, q, r) = (5, 5, 19)$, $(13, 3, 11)$, and $(43, 7, 2)$ are solutions. We show that they are the only ones. First, we must have $p \geq 5$ and $q \geq 3$.

If $p = 5$, then the equation becomes $(q-4)(r-3) = 16$, yielding only $(q, r) = (5, 19)$. If $p = 7$, then the equation becomes $(q-3)(r-2) = 9$, yielding no solution. Consider $p \geq 11$.

If $q = 3$, then the equation becomes $(p-7)(r-5) = 36$, yielding only $(p, r) = (13, 11)$. If $q = 5$, then the equation becomes $(3p-13)(3r-7) = 100$, yielding no solution. Consider $q \geq 7$.

If $r \geq 3$, then

$$\frac{1}{p-1} + \frac{1}{q} + \frac{1}{r+1} \leq \frac{1}{10} + \frac{1}{7} + \frac{1}{4} < \frac{1}{2}.$$

Thus $r = 2$, and the equation becomes $(p-7)(q-6) = 36$, yielding only $(p, q) = (43, 7)$.

*Also solved by Joel Schlosberg, Bayside, NY, USA; Taes Padhihary, Disha Delphi Public School, India.*

S536. Let $S = \{2, 4, 6, \dots, 2020\}$ be the set of positive even integers not greater than 2020 and $T = \{3, 6, 9, \dots, 2019\}$ be the set of positive multiples of 3 less than 2020. Evaluate

$$\sum_{A\subseteq S}\sum_{B\subseteq T}|A\cup B|.$$

*Proposed by Li Zhou, Polk State College, USA*

*Solution by the author*

We first notice that $|S| = 1010$, $|T| = 673$, and $|S\cap T| = 336$. For each $i\in S$, half of all $2^{|S|}$ subsets of $S$ contain $i$, so

$$\sum_{A\subseteq S}\sum_{B\subseteq T}|A| = |S|\cdot 2^{|S|-1}\cdot 2^{|T|} = 1010\cdot 2^{1682}.$$

Likewise,

$$\sum_{A\subseteq S}\sum_{B\subseteq T}|B| = 2^{|S|}\cdot |T|\cdot 2^{|T|-1} = 673\cdot 2^{1682}.$$

Similarly, for each $i\in S\cap T$, half of all $2^{|S|}$ subsets of $S$ contain $i$ and half of all $2^{|T|}$ subsets of $T$ contain $i$, thus

$$\sum_{A\subseteq S}\sum_{B\subseteq T}|A\cap B| = |S\cap T|\cdot 2^{|S|-1}\cdot 2^{|T|-1} = 336\cdot 2^{1681}.$$

Finally, from the beloved PIE we get

$$\sum_{A\subseteq S}\sum_{B\subseteq T}|A\cup B| = 2^{|S|+|T|-1}\left(|S|+|T|\right) - 2^{|S|+|T|-2}|S\cap T| = 1515\cdot 2^{1682}.$$

S537. Let $ABC$ be a triangle with $\angle B = 50^\circ$. Let $D$ be a point on the segment $BC$ such that $\angle BAD = 30^\circ$ and $AD = BC$. Find $\angle CAD$.

*Proposed by Marius Stănean, Zalău, Romania*

*First solution by Li Zhou, Polk State College, USA*

Let $E$ be the reflection of $D$ across $AB$, then $\angle AEB = \angle ADB = 100^\circ = \angle CBE$ and $AE = AD = CB$, so $AEBC$ is an isosceles trapezoid. Therefore, $\angle CAB = \angle EBA = 50^\circ$, thus $\angle CAD = 20^\circ$.

*Second solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA*

Applying the Law of Sines to $\Delta BAD$ yields

$$2BD = \frac{BD}{\sin 30^\circ} = \frac{AD}{\sin 50^\circ} = \frac{BC}{\sin 50^\circ};$$

thus,

$$BC = 2\sin 50^\circ BD \quad \text{and} \quad DC = (2\sin 50^\circ - 1)BD.$$

Now, let $x$ denote the measure of $\angle CAD$. Applying the Law of Sines to $\Delta CAD$ yields

$$\frac{2\sin 50^\circ BD}{\sin(100^\circ - x)} = \frac{(2\sin 50^\circ - 1)BD}{\sin x}$$

or

$$\frac{2\sin 50^\circ}{2\sin 50^\circ - 1} = \frac{\sin(100^\circ - x)}{\sin x} = \sin 100^\circ \cot x - \cos 100^\circ = \cos 10^\circ \cot x + \sin 10^\circ.$$

Next, multiply the numerator and denominator of the fraction on the left side by $\cos 50^\circ$ to obtain

$$\begin{aligned}
\frac{2\sin 50^\circ}{2\sin 50^\circ - 1} &= \frac{\sin 100^\circ}{\sin 100^\circ - \cos 50^\circ} = \frac{\cos 10^\circ}{\cos 10^\circ - \cos 50^\circ} \\
&= \frac{\cos 10^\circ}{\cos(30^\circ - 20^\circ) - \cos(30^\circ + 20^\circ)} \\
&= \frac{\cos 10^\circ}{\sin 20^\circ};
\end{aligned}$$

thus,

$$\cot x = \frac{\cos 10^\circ - \sin 10^\circ \sin 20^\circ}{\cos 10^\circ \sin 20^\circ}.$$

But,

$$\cos 10^\circ = \cos(20^\circ - 10^\circ) = \cos 20^\circ \cos 10^\circ + \sin 20^\circ \sin 10^\circ,$$

so

$$\cos 10^\circ - \sin 10^\circ \sin 20^\circ = \cos 20^\circ \cos 10^\circ$$

and

$$\cot x = \cot 20^\circ.$$

Therefore, $\angle CAD = 20^\circ$.

*Also solved by Joel Schlosberg, Bayside, NY, USA; Ivko Dimitric, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Alina Craciun, Miron Costin Theoretical High School, Pașcani, România; Taes Padhihary, Disha Delphi Public School, India; Arighna Pan, Nabadwip Vidyasagar College, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

S538. Let $P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ be a polynomial with real coefficients and $n$ an even positive integer. If $P(x)$ has $n$ non-negative real roots, prove that

$$1 + \sqrt[n]{a_n} \le \sqrt[n]{P(-1)} \le 1 - \frac{a_1}{n}.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution by Joel Schlosberg, Bayside, NY, USA*

By comparing coefficients of $(x - r_1)(x - r_2) \cdot \ldots \cdot (x - r_n) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ (or via Viete's formulas),

$$1 + \sqrt[n]{a_n} = 1 + \prod_{i=1}^{n} r_i^{1/n}$$

and

$$1 - \frac{a_1}{n} = 1 + \frac{1}{n}\sum_{i=1}^{n} r_i$$

while $\sqrt[n]{P(-1)} = \prod_{i=1}^{n}(1 + r_i)^{1/n}$.

According to Mahler's inequality, for any $q_1, \ldots, q_n, r_1, \ldots, r_n \ge 0$, $\prod_{i=1}^{n} q_i^{1/n} + \prod_{i=1}^{n} r_i^{1/n} \le \prod_{i=1}^{n}(q_i + r_i)^{1/n}$ so for $q_1 = \cdots = q_n = 1$,

$$1 + \sqrt[n]{a_n} = 1 + \prod_{i=1}^{n} r_i^{1/n} \le \prod_{i=1}^{n}(1 + r_i)^{1/n} = \sqrt[n]{P(-1)}.$$

By the arithmetic mean-geometric mean inequality,

$$\sqrt[n]{P(-1)} = \prod_{i=1}^{n}(1 + r_i)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n}(1 + r_i) = 1 + \frac{1}{n}\sum_{i=1}^{n} r_i = 1 - \frac{a_1}{n}.$$

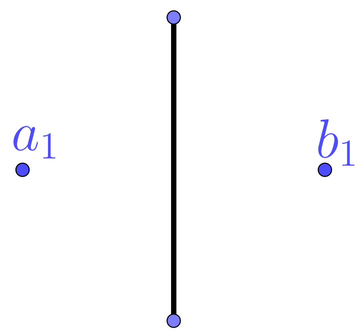
*Also solved by Li Zhou, Polk State College, USA; Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Taes Padhihary, Disha Delphi Public School, India; Arighna Pan, Nabadwip Vidyasagar College, India.*

S539. There is a street with $n$ houses on the left side and $n$ houses on the right side. Moreover, the houses which are in front of one to another are identical. We have to paint the houses with $m$ different colors in such a way that no two neighboring houses nor two face to face houses are of the same color. In how many ways can this coloring be done?

*Proposed by Mircea Becheanu, Canada*

*Solution by Nicole Lipschitz and Arturo Acunã, San José, Costa Rica*

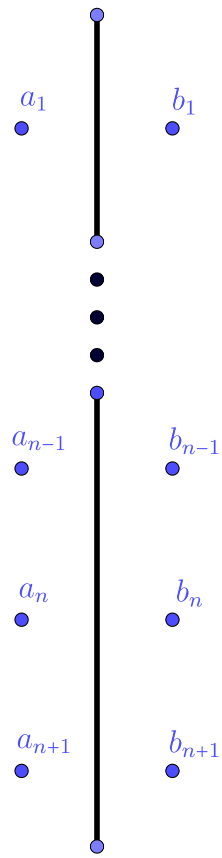
Let $n = 1$. Then, the number of ways to color the houses would be $m(m-1)$, as we will have m options for the first house to be colored, lets call it $a_1$, and $(m-1)$ options for the one infront, $b_1$, since it must be a different color.




Now, we will prove that, for any 2n houses (n on each side of the street) and m colors, the number of ways to color them (lets call it $k(n)$) is:

$$m(m-1)((m-1)+(m-2)^2)^{n-1}.$$

Assume this is true for all positive integers smaller or equal to n and lets name the houses on the right hand side $a_1, a_2, \cdots, a_{n+1}$, and the ones on the left hand side $b_1, b_2, \cdots, b_{n+1}$. Therefore, we know that there are $k(n)$ ways to color the first n houses on both sides of the street, so we only have the houses $a_{n+1}$ and $b_{n+1}$ left to color.




Lets color first $a_{n+1}$. Since the adjecent house, $a_n$, was already colored, there are $(m-1)$ posible ways to color $a_{n+1}$, yet we will separate it in two cases. On one hand, if the color of $a_{n+1}$ is the same as $b_n$, then there are $(m-1)$ color options for $b_{n+1}$ (since it must be different from the house infront and adjecent to itself, which are both colored in the same way). On the other hand, if the color of $a_{n+1}$ is any of the other $(m-2)$ colors, there are $(m-2)$ color options for $b_{n+1}$. So, altogether, there are $(m-1)+(m-2)^2$ ways to color $a_{n+1}$ and $b_{n+1}$.

Therefore, the number of ways to color the 2(n+1) houses would be equal to:

$$\begin{aligned} k(n)\cdot((m-1)+(m-2)^2) &= m(m-1)((m-1)+(m-2)^2)^{n-1}\cdot((m-1)+(m-2)^2) \\ &= m(m-1)((m-1)+(m-2)^2)^n \end{aligned}$$

Hence, we have concluded our proof for this problem.

*Also solved by Li Zhou, Polk State College, USA.*

S540. Let $s(x)$ denote the sum of digits of the positive integer $x$. Find all positive integers $n$ such that $s(n) = 3s(n+1)$.

*Proposed by Titu Andreescu, USA and Marian Tetiva, România*

*Solution by Joel Schlosberg, Bayside, NY, USA*

For a positive integer $n$, let $z$ be the number of rightmost 9s in the decimal representation of $n$, and let $m$ be the integer formed when the $z$ 9s are removed (if $z = 0$, $m = n$; if all of the digits of $n$ are 9, $m = 0$). The decimal representation of $n+1$ replaces the $z$ rightmost 9s by zeroes, adds 1 to the rightmost digit of $m$ or changes $m$ from zero to one, and leaves any other digits of $m$ unchanged. Therefore

$$s(n) = s(m) + 9z$$

and

$$s(n+1) = s(m) + 1.$$

If $s(n) = 3s(n+1)$, $s(m) = \frac{9z-3}{2}$. Since $s(m)$ is an integer, $z$ is odd (and thus nonzero). Conversely, if $z$ is an odd positive integer and $s(m) = \frac{9z-3}{2}$, $s(n) = 3s(n+1)$.

*Also solved by Li Zhou, Polk State College, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.*

## Undergraduate problems

U535. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{k}=1.$$

Evaluate

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{a_1+\ldots+a_k}{k^3}.$$

*Proposed by Florin Stanescu, România*

*Solution by Joel Schlosberg, Bayside, NY, USA*

By the Stolz-Cesaro theorem,

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}a_k/k}{\sum_{k=1}^{n}1}=1\implies\lim_{n\to\infty}\frac{a_n}{n}=1\implies\lim_{n\to\infty}\frac{\sum_{k=1}^{n}a_k}{\sum_{k=1}^{n}k}=1.$$

Let

$$m_n=\min_{k\in\{n+1,\ldots,2n\}}\frac{a_1+\cdots+a_k}{k^2}$$

and

$$M_n=\max_{k\in\{n+1,\ldots,2n\}}\frac{a_1+\cdots+a_k}{k^2}.$$

Since $a_1+\cdots+a_k$ is asymptotic to $1+\cdots+k=\frac{1}{2}k(k+1)$, both $m_n$ and $M_n$ tend to $1/2$ as $n\to\infty$. Since

$$m_n\sum_{k=n+1}^{2n}\frac{1}{k}\leq\sum_{k=n+1}^{2n}\frac{a_1+\cdots+a_k}{k^3}\leq M_n\sum_{k=n+1}^{2n}\frac{1}{k},$$

and the well-known approximation $\sum_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+O(1/n)$ with $\gamma$ the Euler-Mascheroni constant implies that $\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k}=\ln 2$, by the squeeze theorem

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{a_1+\cdots+a_k}{k^3}=\frac{\ln 2}{2}.$$

*Also solved by Arkady Alt, San Jose, CA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France.*

U536. Evaluate

$$\int x(\sqrt{1+x}+\sqrt{1-x})\,dx.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*First solution by Henry Ricardo, Westchester Area Math Circle*
We use integration by parts:

$$\begin{aligned}\int \overbrace{x}^{u}\overbrace{(\sqrt{1+x}+\sqrt{1-x})\,dx}^{dv} &= \frac{2x}{3}\left((1+x)^{3/2}-(1-x)^{3/2}\right)-\frac{2}{3}\int (1+x)^{3/2}-(1-x)^{3/2}\,dx\\ &= \frac{2x}{3}(1+x)\sqrt{1+x}-\frac{2x}{3}(1-x)\sqrt{1-x}\\ &\quad+\frac{4}{15}\left((1+x)^{5/2}+(1-x)^{5/2}\right)+C\\ &= \frac{2(1-x)^{5/2}}{5}-\frac{2(1-x)^{3/2}}{3}+\frac{2(1+x)^{5/2}}{5}-\frac{2(1+x)^{3/2}}{3}+C\\ &= \frac{(6x^2+2x-4)\sqrt{1+x}}{15}+\frac{(6x^2-2x-4)\sqrt{1-x}}{15}+C.\end{aligned}$$

*Second solution by Henry Ricardo, Westchester Area Math Circle*
We write

$$I = \int x(\sqrt{1+x}+\sqrt{1-x})\,dx = \int x\sqrt{1+x}\,dx + \int x\sqrt{1-x}\,dx = I_1 + I_2.$$

In $I_1$, make the substitution $t^2=1+x$, and in $I_2$ let $u^2=1-x$. This gives us

$$\begin{aligned} I = I_1 + I_2 &= \int (2t^4-2t^2)\,dt + \int (2u^4-2u^2)\,du\\ &= \frac{2}{5}t^5-\frac{2}{3}t^3+\frac{2}{5}u^5-\frac{2}{3}u^3+C\\ &= \frac{2(1+x)^2\sqrt{1+x}}{5}-\frac{2(1+x)\sqrt{1+x}}{3}+\frac{2(1-x)^2\sqrt{1-x}}{5}-\frac{2(1-x)\sqrt{1-x}}{3}+C\\ &= \frac{(6x^2+2x-4)\sqrt{1+x}}{15}+\frac{(6x^2-2x-4)\sqrt{1-x}}{15}+C.\end{aligned}$$

*Also solved by Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, India; Alina Craciun, Miron Costin Theoretical High School, Pașcani, România; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; G. C. Greubel, Newport News, VA, USA; Mihaly Bencze, Brasov, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.*

U537. Let $k$ be a positive integer. Evaluate

$$\lim_{n\to\infty} n^{2k}\left(\frac{\arctan n^k}{n^k}-\frac{\arctan(n^k+1)}{n^k+1}\right).$$

*Proposed by Dinu Ovidiu Gabriel, Bălceşti, Vâlcea, România*

*Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA*

With

$$\arctan n^k = \frac{\pi}{2} - \arctan\frac{1}{n^k} = \frac{\pi}{2} - \frac{1}{n^k} + O\left(\frac{1}{n^{3k}}\right)$$

and

$$\arctan(n^k+1) = \frac{\pi}{2} - \arctan\frac{1}{n^k+1} = \frac{\pi}{2} - \frac{1}{n^k+1} + O\left(\frac{1}{(n^k+1)^3}\right),$$

it follows that

$$\frac{\arctan n^k}{n^k}-\frac{\arctan(n^k+1)}{n^k+1} = \frac{\pi}{2}\cdot\frac{1}{n^k(n^k+1)} - \frac{2n^k+1}{n^{2k}(n^k+1)^2} + O\left(\frac{1}{n^{4k}}\right),$$

and

$$n^{2k}\left(\frac{\arctan n^k}{n^k}-\frac{\arctan(n^k+1)}{n^k+1}\right) = \frac{\pi}{2}\cdot\frac{n^{2k}}{n^k(n^k+1)} - \frac{2n^k+1}{(n^k+1)^2} + O\left(\frac{1}{n^{2k}}\right).$$

Thus,

$$\lim_{n\to\infty} n^{2k}\left(\frac{\arctan n^k}{n^k}-\frac{\arctan(n^k+1)}{n^k+1}\right) = \frac{\pi}{2}.$$

*Also solved by Alina Craciun, Miron Costin Theoretical High School, Pașcani, România; Moubinool Omarjee, Lycée Henri IV, Paris, France; Olimjon Jalilov, National University of Uzbekistan, Tashkent, Uzbekistan; Arkady Alt, San Jose, CA, USA.*

U538. Let $p > 2$ be a prime and $r > 1$ be an integer. Define the set

$$S = \{a \in \mathbb{Z} | a^a \equiv a \pmod{p^r}, 2 \leq a \leq p^r - 2\}.$$

Prove that

$$|S| \leq p^{r-2}(p-1)(p - \varphi(p-1)).$$

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

*Solution by the author*

A the outset, we prove that if $x^x \equiv x \pmod{p^r}$ then $\gcd(x,p) = 1$, otherwise, since $x(x^{x-1} - 1) \equiv 0 \pmod{p^r}$. Since $x \not\equiv 0 \pmod{p^r}$, then $x^x \equiv 1 \pmod{p^r}$. Thus, $\gcd(x,p) = 1$. Now, let us prove the following lemma.

*Lemma:* Let $x \in S$, then $x$ is not a primitive root mod $p^r$

*Proof:* Assume as a contraction that $p^{r-1}(p-1)$ divides $x-1$. Write $x = k(p^{r-1}(p-1))$, for some positive integer $k$, then $k(p^{r-1}(p-1)) \leq p^r - 3$, thus $k = 1$. Therefore, $x = p^r - p^{r-1} + 1$. Since $v_p((p^r - p^{r-1} + 1)^{2p} - 1) = r$, we find that $x^{2p} \equiv 1 \pmod{()p^r}$. Hence, $2p$ is divisible by $p^{r-1}(p-1)$ and $p^{r-1}(p-1) \leq 2p$. It would be absurd unless $p = 3, r = 2$. In that case $x = 7$ is the only solution and 7 is not a primitive root mod 9. That is, since $\text{ord}_9^7 = 3$ and we are done.

Back to our problem, since the total number of primitive roots modulo $p^r$ is $\varphi(p^{r-1}(p-1)) = p^{r-2}(p-1)\varphi(p-1)$, we find that

$$|S| \leq p^{r-1}(p-1) - p^{r-2}(p-1)\varphi(p-1) = p^{r-2}(p-1)(p - \varphi(p-1)).$$

U539. Calculate

$$\int_0^1 \frac{-2x^2\ln x}{\sqrt{1-x^2}(x^4-x^2+1)}dx$$

*Proposed by Paolo Perfetti, Roma, Italy*

*Solution by the author*

$$\int_0^1 \frac{-2x^2\ln x}{\sqrt{1-x^2}(x^4-x^2+1)}dx = \int_0^1 \frac{-2x^2\ln x}{\sqrt{1-x^2}}\frac{1}{(1-x\sqrt{1-x^2})(1+x\sqrt{1-x^2})}dx =$$
$$= \int_0^1 \frac{-x^2\ln x}{\sqrt{1-x^2}}\left(\frac{1}{1-x\sqrt{1-x^2}}+\frac{1}{1+x\sqrt{1-x^2}}\right)dx$$

$x=\cos t$ with $-\pi/2\le t\le 0$ yields

$$\int_0^1 \frac{-x^2\ln x}{\sqrt{1-x^2}}\frac{dx}{1-x\sqrt{1-x^2}} = \int_{-\pi/2}^0 \frac{-(\cos t)^2\ln(\cos t)}{-\sin t(1+\sin t\cos t)}(-\sin t)dt$$

$x=\cos t$ with $0\le t\le \pi/2$ yields

$$\int_0^1 \frac{-x^2\ln x}{\sqrt{1-x^2}}\frac{dx}{1+x\sqrt{1-x^2}} = -\int_0^{\pi/2} \frac{-(\cos t)^2\ln(\cos t)}{\sin t(1+\sin t\cos t)}(-\sin t)dt$$

By summing we get

$$\int_{-\pi/2}^{\pi/2} \frac{-(\cos t)^2\ln(\cos t)}{1+\sin t\cos t}dt$$

By $x=\arctan t$ we get

$$\int_{-\infty}^{+\infty} \frac{\ln(1+t^2)}{2(1+t^2)(1+t+t^2)}dt$$

By introducing the complex functions $F_k(z)=\dfrac{\mathrm{Ln}(z-t_k)}{(1+z^2)^2}$, $k=0,1,\ t_0=i,\ t_1=-i\ z\in\mathbf{C}$, $\mathrm{Ln}(z)=\mathrm{Ln}(\rho e^{it})\doteq \ln\rho+it$, we can write

$$\int_{-\infty}^{+\infty} \frac{\mathrm{Ln}(t-t_0)}{2(1+t^2)(1+t+t^2)}dt + \int_{-\infty}^{+\infty} \frac{\mathrm{Ln}(t-t_1)}{2(1+t^2)(1+t+t^2)}dt$$

To evaluate the first integral, we cut the complex plane along the set $\mathrm{Im}z=[i,+\infty)$ and we take $-3\pi/2\le \mathrm{Arg}z\le\pi/2$. We perform the *clockwise* integral over the two curves

$$\gamma_1(t)=\{z\in\mathbf{C}: z=t,\ -r\le t\le r\},$$
$$\gamma_2(t)=\{z\in\mathbf{C}: z=re^{-it},\ -\pi\le t\le 0\},$$

$$\lim_{r\to\infty}\int_{\gamma_1\cup\gamma_2} F_0(z)dz = -2\pi i Res(F_0(-i))$$

The integral over $\gamma_2$ tends to zero by observing that $|z-t_0|\le|z|+1$

$$|F_0(z)| = \left|\frac{\mathrm{Ln}(z-t_0)}{(1+z^2)(1+z+z^2)}\right| \le \frac{\ln(|z|+1)+\pi}{||z|^2-1|\,(|z|^2-|z|-1)} \le C\frac{\ln(|z|+1)+\pi}{|z|^4}$$

if $|z|\ge r_C>0$ and $r_C$ is large enough. Thus we have

$$\left|\int_{\gamma_2} F_0(z)dz\right| = \left|\int_{-\pi}^0 F_0(re^{-it})(-i)re^{-it}dt\right| = r\pi C\frac{\ln(r+1)+\pi}{r^4}\to 0$$

Clearly

$$\lim_{r\to\infty}\int_{\gamma_1} F_0(z)dz = \int_{-\infty}^{+\infty}\frac{\ln(t-i)}{(1+t^2)(1+t+t^2)}dt$$

and analogously occurs for $F_1(z)$.

$$\mathrm{Res}(F_0(-i)) = \frac{\mathrm{Ln}(-2i)}{-2i\cdot -i} = \frac{\mathrm{Ln}(-2i)}{-2} = \frac{\ln 2}{-2} + \frac{i\pi}{4}$$

$$\mathrm{Res}(F_0(e^{\frac{-2\pi i}{3}})) = \frac{\mathrm{Ln}(e^{\frac{-2i\pi}{3}} - i)}{(1+e^{\frac{-4i\pi}{3}})(2e^{\frac{-2i\pi}{3}}+1)} = \frac{\mathrm{Ln}(\frac{-1}{2} - i\frac{\sqrt{3}}{2} - i)}{(1+e^{\frac{-4i\pi}{3}})(2e^{\frac{-2i\pi}{3}}+1)} =$$
$$\frac{\frac{1}{2}\ln(2+\sqrt{3}) - \frac{7i\pi}{12}}{(\frac{1}{2}+i\frac{\sqrt{3}}{2})(-i\sqrt{3})} = \frac{\ln(2+\sqrt{3}) - \frac{7i\pi}{6}}{-i\sqrt{3}+3} = \frac{1}{12}(i\sqrt{3}+3)(\ln(2+\sqrt{3}) - \frac{7i\pi}{6})$$

Doing analogous calculation for the integral with $k=1$ we get (in this case we cut in the complex plane in $\mathrm{Im}z = [-i,+\infty)$, $-\pi/2 \le \mathrm{Arg}z \le 3\pi/2$ and the path

$$\gamma_3(t) = \{z\in\mathbf{C} : z = t,\ -r\le t\le r\},$$
$$\gamma_4(t) = \{z\in\mathbf{C} : z = re^{it},\ 0\le t\le \pi\},$$

is run *counterclockwise*

$$\mathrm{Res}F_1(i) = \frac{\mathrm{Ln}(2i)}{2i\cdot i} = \frac{\ln 2 + \frac{i\pi}{2}}{-2}$$

$$\mathrm{Res}F_1(e^{\frac{2i\pi}{3}}) = \frac{\mathrm{Ln}(e^{\frac{2i\pi}{3}}+i)}{(1+e^{\frac{4i\pi}{3}})(1+2e^{\frac{2i\pi}{3}})} = \frac{\frac{1}{2}\ln(2+\sqrt{3}) + i\frac{7\pi}{12}}{(\frac{1}{2}-i\frac{\sqrt{3}}{2})i\sqrt{3}}$$
$$= \frac{1}{12}(3-i\sqrt{3})(\ln(2+\sqrt{3}) + \frac{7i\pi}{6})$$

Finally the integral is

$$-\frac{1}{2}\cdot 2\pi i\left(\frac{-\ln 2}{2} + \frac{i\pi}{4} + \frac{1}{12}(i\sqrt{3}+3)(\ln(2+\sqrt{3}) - \frac{7i\pi}{6})\right) +$$
$$\frac{1}{2}\cdot 2\pi i\left(\frac{-\ln 2}{2} - \frac{i\pi}{4} + \frac{1}{12}(3-i\sqrt{3})(\ln(2+\sqrt{3}) + \frac{7i\pi}{6})\right) = \frac{-\pi^2}{12} + \frac{\pi}{2\sqrt{3}}\ln(2+\sqrt{3})$$

U540. Let $x$ be a rational number. Prove that $\mathbb{Q}(\sqrt[3]{x}) = \mathbb{Q}(\sqrt[3]{2})$ if and only is $x = 2q^3$ or $x = 4q^3$, for some rational number $q$.

*Proposed by Mircea Becheanu, Canada*

*First solution by Gabriel Dospinescu, France*

It is clear that the condition $x = 2q^3$ or $x = 4q^3$, with $q \in \mathbb{Q}$ is sufficient. We will prove that the condition is necessary as well. Let $a, b, c$ be rational numbers such that $\sqrt[3]{x} = a + b\sqrt[3]{2} + c\sqrt[3]{4}$. We introduce the rational polynomial $P(X) = a + bX + cX^2$. Then $P(X)^3 - x = F(X)$ is also a rational polynomial and $F(\sqrt[3]{2}) = 0$. Since $X^3 - 2$ is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$, it follows that $X^3 - 2 | F(X)$. Therefore, we also have $F(\omega\sqrt[3]{2}) = 0$, where $\omega^2 + \omega + 1 = 0$. From $P(\omega\sqrt[3]{2})^3 = x$ we obtain $P(\omega\sqrt[3]{2}) \in \{\sqrt[3]{x}, \omega\sqrt[3]{x}, \omega^2\sqrt[3]{x}\}$. We distinguish three cases:

*Case 1:* $P(\omega\sqrt[3]{2}) = \sqrt[3]{x}$. It means that

$$a + b\sqrt[3]{2}\omega + c\sqrt[3]{4}\omega^2 = a + b\sqrt[3]{2} + c\sqrt[3]{4}.$$

Then we have

$$b + c\sqrt[3]{2} = b\omega + c\sqrt[3]{2}\omega^2.$$

taking the complex conjugation one obtains also

$$b + c\sqrt[3]{2} = b\omega^2 + c\sqrt[3]{2}\omega.$$

By adding these equalities and using the realtion $\omega + \omega^2 = -1$ we obtain:

$$2b + 2c\sqrt[3]{2} = -b - c\sqrt[3]{2}.$$

This gives $b + c\sqrt[3]{2} = 0$ which means that $b = c = 0$ or $\sqrt[3]{2}$ is rational. This case is impossible.

*Case 2:* $P(\omega\sqrt[3]{2}) = \omega\sqrt[3]{x}$. It means that

$$a + b\sqrt[3]{2}\omega + c\sqrt[3]{4}\omega^2 = a\omega + b\sqrt[3]{2}\omega + c\sqrt[3]{4}\omega \Leftrightarrow a + c\sqrt[3]{4}\omega^2 = a\omega + c\sqrt[3]{4}\omega \Leftrightarrow$$

$$\Leftrightarrow a(1-\omega) = c\sqrt[3]{4}\omega(1-\omega) \Leftrightarrow a = c\sqrt[3]{4}\omega.$$

From this we have $a = c = 0$ which gives $\sqrt[3]{x} = b\sqrt[3]{2}$ and $x = 2b^3$.

*Case 3:* $P(\omega\sqrt[3]{2}) = \omega^2\sqrt[3]{x}$. It means that

$$a + b\sqrt[3]{2}\omega + c\sqrt[3]{4}\omega^2 = a\omega^2 + b\sqrt[3]{2}\omega^2 + c\sqrt[3]{4}\omega^2 \Leftrightarrow a + b\sqrt[3]{2}\omega = a\omega^2 + b\sqrt[3]{2}\omega^2.$$

WE multiply the last equality by $\omega$ to obtain

$$a + b\sqrt[3]{2} = a\omega + b\sqrt[3]{2}\omega^2 \Leftrightarrow a(1-\omega) = b\sqrt[3]{2}(\omega^2 - 1) \Leftrightarrow a = -b\sqrt[3]{2}(\omega + 1) \Leftrightarrow$$

$$a = b\sqrt[3]{2}\omega^2.$$

This shows that $a = b = 0$ and then $x = 4c^3$.

*Second solution by Corneliu Mănescu-Avram, Ploieşti, Romania*
Note that $x \neq 0$, otherwise $\mathbb{Q}(\sqrt[3]{x}) = \mathbb{Q}(0) = \mathbb{Q}$.
If $x = 2q^3$, with $q \in \mathbb{Q}^*$, then $\mathbb{Q}(\sqrt[3]{x}) = \mathbb{Q}(q\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2})$.
If $x = 4q^3$, with $q \in \mathbb{Q}^*$, then $\mathbb{Q}(\sqrt[3]{x}) = \mathbb{Q}\left(\frac{2q}{\sqrt[3]{2}}\right) = \mathbb{Q}(\sqrt[3]{2})$.

Conversely, suppose that $\mathbb{Q}(\sqrt[x]{3}) = \mathbb{Q}(\sqrt[3]{2})$ for some $q \in \mathbb{Q}^*$. The set $\{1, \mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{4})\}$ is the base of the vector space $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$. This means that $\mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} | a, b, c \in \mathbb{Q}\}$ and that the equality $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0, a, b, c \in \mathbb{Q}$ implies that $a = b = c = 0$. From $\sqrt[3]{x} \in \mathbb{Q}(\sqrt[3]{2})$ we deduce that there are $a, b, c \in \mathbb{Q}$ not all zero, such that $\sqrt[3]{x} = a + b\sqrt[3]{2} + c\sqrt[3]{4}$. Then we have to prove that $a = b = 0$ or $a = c = 0$. Calculating third powers, we have

$$x = a^3 + 2b^3 + 4c^3 + 12abc + 3(a^2b + 2b^2c + 2c^2a)\sqrt[3]{2} + 3(ab^2 + 2bc^2 + ca^2)\sqrt[3]{4},$$

whence, we deduce that

$$a^2b + 2b^2c + 2c^2a = 0 \text{ and } ab^2 + 2bc^2 + ca^2 = 0 \tag{1}$$

If $b = c = 0$, then $\sqrt[3]{x} = a \in \mathbb{Q}$, contradiction.
If $c = 0$, then the equalities (1) become $a^2b = ab^2 = 0$, therefore $a = 0, b \neq 0$.
If $b = 0$, then the equalities (1) become $c^2a = ca^2 = 0$, therefore $a = 0, c \neq 0$.

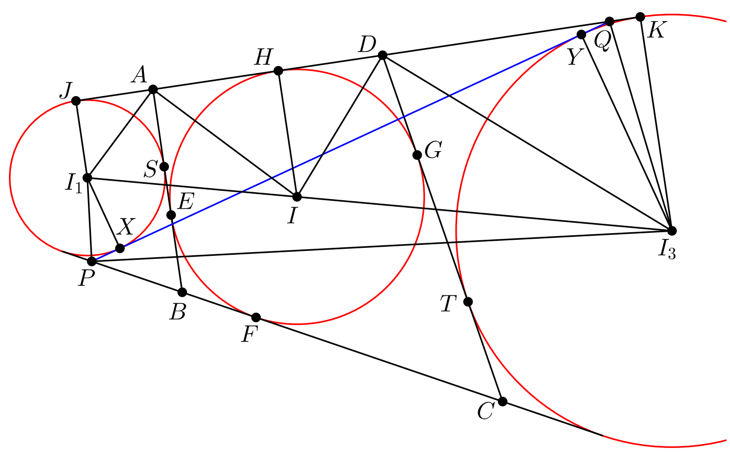
Indeed, if we put $b/a = u, c/b = v$ and $c/a = uv$ and dividing equalities (1) by $a^3$ one obtains $u + 2u^3v + 2u^2v^2 = 0$ and $uv + u^2 + 2u^3v^2 = 0$. Now, we multiply the first equality by $v$ and comparing the two equalities one have $u^2 = 2u^2v^3$. This is a contradiction, as $u$ and $v$ are rational numbers.

**Olympiad problems**

O535. Let $ABCD$ be a convex quadrilateral with an incircle. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the excircles of $ABCD$ tangent to segments $AB, BC, CD, DA$, respectively. Prove that the lengths of the internal common tangent segments to the circles $(\omega_1, \omega_3)$ and $(\omega_2, \omega_4)$ are equal.

*Proposed by Waldemar Pompe, Warsaw, Poland*

*Solution by Li Zhou, Polk State College, USA*




Let $I$, $r$, and $s$ be the incenter, inradius, and semi-perimeter of $ABCD$, and $I_i$ and $r_i$ be the center and radius of $\omega_i$. Label other points as in the figure. First, it is easy to see that $AJ = AS = BE = BF$, $AH = AE = BS$, $DH = DG = CT$, and $DK = DT = CG = CF$. Next, since $\triangle JI_1A \sim \triangle HAI$, $JI_1/AJ = HA/IH$, that is, $r_1 = AH \cdot BE/r$. Likewise, $r_3 = CF \cdot DG/r$. Therefore,

$$r_1 r_3 = \frac{AH \cdot BE \cdot CF \cdot DG}{r^2} = \frac{AE \cdot BF \cdot CG \cdot DH}{r^2} = r_2 r_4.$$

Now, $\triangle PXI_1 \sim \triangle I_3YP$, so $PX/XI_1 = I_3Y/YP$, that is, $PX \cdot PY = r_1 r_3$. Similarly, $QX \cdot QY = r_1 r_3$. Therefore, $PX(PQ - QY) = (PQ - PX)QY$, thus $PX = QY$. Hence, $PX + PY = QX + QY = QJ + QK = JK = s$, that is, $PX$ and $QY$ are the smaller zero of the quadratic polynomial $x^2 - sx + r_1 r_3 = x^2 - sx + r_2 r_4$. Finally, $XY = XQ - QY = JK - 2QK = s - 2QY$, completing the proof.

O536. Let $a, b, c$ be the side-lengths of a triangle and let $S$ be its area. Let $R$ and $r$ be the circumradius and inradius of the triangle, respectively. Prove that

$$a^2 + b^2 + c^2 \le 4S\sqrt{\frac{6r}{R}} + 3\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right].$$

*Proposed by Marius Stănean, Zalău, Romania*

*First solution by the author*

The inequality can be rewritten as

$$16(ab + bc + ca) - 5(a + b + c)^2 \le 4S\sqrt{\frac{6r}{R}}.$$

But $ab + bc + ca = s^2 + r^2 + 4Rr$, so this becomes

$$s^2 + sr\sqrt{\frac{6r}{R}} \ge 4r^2 + 16Rr,$$

or

$$\frac{s^2}{R^2} + \left(\frac{s}{R}\right)\left(\frac{r}{R}\right)\sqrt{\frac{6r}{R}} \ge \frac{4r^2}{R^2} + \frac{16r}{R}.$$

If we denote $x^2 = 1 - \frac{2r}{R} \in [0, 1)$, then by Blundon's Inequality

$$\frac{s^2}{R^2} \ge 2 + 5(1-x^2) - \frac{(1-x^2)^2}{4} - 2x^3 = \frac{(1-x)(x+3)^3}{4}.$$

Hence, it suffices to prove that

$$\frac{(1-x)(x+3)^3}{4} + \frac{1-x^2}{2}\sqrt{\frac{3(1-x^2)(1-x)(x+3)^3}{4}} \ge (1-x^2)^2 + 8(1-x^2),$$

or

$$\frac{1-x^2}{2}\sqrt{\frac{3(1-x^2)(1-x)(x+3)^3}{4}} \ge \frac{(x-1)^2(x+3)(5x+3)}{4},$$

or

$$(x-1)^2(x+3)^2\left[3(x+1)^3(x+3) - (5x+3)^2\right] \ge 0,$$

that is

$$x^2(x-1)^2(x+3)^2(3x^2 + 18x + 11) \ge 0,$$

clearly true. The equality holds when $x = 0$ so when the triangle is equilateral.

*Note:* The inequality is a slightly stronger version of Hadwiger-Finsler Reverse Inequality.

*Second solution by Li Zhou, Polk State College, USA*

Let $s = (a+b+c)/2$. From Gerretsen's inequality $s^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$ we have $s^2 \geq 16Rr - 5Rr/2 = 27Rr/2$. Using the well-known relations $S = rs$ and $ab + bc + ca = 4Rr + r^2 + s^2$ we get

$$\begin{aligned}
&4S\sqrt{\frac{6r}{R}} + 3\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] - a^2 - b^2 - c^2 \\
&= 4rs\sqrt{\frac{6r}{R}} + 20s^2 - 16\left(4Rr + r^2 + s^2\right) \\
&\geq 4rs\sqrt{\frac{6r}{R}} - 36r^2 = 4r\sqrt{\frac{6r}{R}}\left(s - \sqrt{\frac{27Rr}{2}}\right) \geq 0.
\end{aligned}$$

*Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.*

O537. Let $i_1 < \cdots < i_l$ and $j_1 \le \cdots \le j_m$ be nonnegative integers such that

$$2^{i_1} + \cdots + 2^{i_l} = 2^{j_1} + \cdots + 2^{j_m}.$$

Prove that $l \le m$.

*Proposed by Titu Andreescu, USA and Marian Tetiva, România*

*First solution by Li Zhou, Polk State College, USA*

If $l > m$, then there is a minimal $k \in \{1, \ldots, m\}$ such that $j_k > i_k$, thus

$$2^{j_1} + \cdots + 2^{j_{k-1}} \le 2^{i_1} + \cdots + 2^{i_{k-1}} < 2^{i_{k-1}+1} \le 2^{i_k} < 2^{j_k}.$$

Therefore, in modulo $2^{j_k}$, $2^{j_1} + \cdots + 2^{j_m} \equiv 2^{j_1} + \cdots + 2^{j_{k-1}}$, while $2^{i_1} + \cdots + 2^{i_l}$ has a residue greater than or equal to $2^{i_1} + \cdots + 2^{i_k}$, which is a desired contradiction.

*Second solution by Joel Schlosberg, Bayside, NY, USA*

Given any finite sum $2^{j_1} + \cdots + 2^{j_m}$ where $j_1, \ldots, j_m$ are nonnegative integers, replace any identical summands $2^j, 2^j$ with $2^{j+1}$. If the replacement process is iterated, eventually no duplicates will remain, since the number of terms in the sum is a positive integer which decreases with each substitution. Rearrange the terms of the resulting sum in increasing order as $2^{i_1} + \cdots + 2^{i_l}$, so that $i_1 < \ldots < i_l$. Each substitution leaves the total sum unchanged, so $2^{i_1} + \cdots + 2^{i_l} = 2^{j_1} + \cdots + 2^{j_m}$.

The digits in the binary representation of $2^{i_1} + \cdots + 2^{i_l}$ uniquely determine $i_1, \ldots, i_l$, so the replacement process applied to any sum $2^{j_1} + \cdots + 2^{j_m}$ equaling $2^{i_1} + \cdots + 2^{i_l}$ will yield the same sequence of increasing values $i_1, \ldots, i_l$. Since the number of terms in the sum is nonincreasing during the process, $l \le m$.

O538. Find the greatest constant $\lambda$ such that the inequality

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\lambda \geq \frac{2}{3}(1+\lambda)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

holds for all positive real numbers $a, b, c$.

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution by the author*

Multiplying the both sides by $ab+bc+ca$, the inequality becomes

$$a^2+b^2+c^2+\lambda(ab+bc+ca) \geq \frac{2}{3}(1+\lambda)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)(ab+bc+ca),$$

or

$$a^2+b^2+c^2+\lambda(ab+bc+ca) \geq \frac{2}{3}(1+\lambda)\left(a^2+b^2+c^2+\frac{abc}{b+c}+\frac{abc}{c+a}+\frac{abc}{a+b}\right),$$

or

$$(1-2\lambda)(a^2+b^2+c^2)+3\lambda(ab+bc+ca) \geq 2(1+\lambda)abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right),$$

or

$$(1-2\lambda)(a^2+b^2+c^2-ab-bc-ca) \geq (1+\lambda)\sum_{\text{cyc}} a\left(\frac{2bc}{b+c}-\frac{b+c}{2}\right),$$

or

$$\sum_{\text{cyc}}\left(1-2\lambda+\frac{(1+\lambda)a}{b+c}\right)(b-c)^2 \geq 0,$$

or

$$S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0$$

where

$$S_a = 1-2\lambda+\frac{(1+\lambda)a}{b+c},$$

$$S_b = 1-2\lambda+\frac{(1+\lambda)b}{c+a},$$

$$S_c = 1-2\lambda+\frac{(1+\lambda)c}{a+b}.$$

Now we take $c=b$ then obtain

$$(S_b+S_c)(a-b)^2 \geq 0.$$

This is equivalent to

$$S_b + S_c \geq 0,$$

or

$$2 - 4\lambda + \frac{2(1+\lambda)b}{a+b} \geq 0,$$

or

$$\frac{a+2b}{2a+b} \geq \lambda,$$

or

$$\frac{t+2}{2t+1} \geq \lambda \quad (t = \frac{a}{b}).$$

This is true for all $t > 0$ if and only if

$$\lambda \leq \inf_{t>0} \frac{t+2}{2t+1} = \frac{1}{2}.$$

Next, we will show that the inequality holds for $\lambda = \frac{1}{2}$. Indeed, for $\lambda = \frac{1}{2}$, the inequality becomes

$$\sum_{\text{cyc}} \frac{a(b-c)^2}{b+c} \geq 0$$

which is clearly true and we are done.

*Also solved by Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.*

O539. Let $A_1A_2B_1B_2C_1C_2$ be a convex hexagon, in which all angles are obtuse. Let $A_1A_2 \cap B_1B_2 = C$, $B_1B_2 \cap C_1C_2 = A$ and $C_1C_2 \cap A_1A_2 = B$. Let $O$ be the circumcenter of $ABC$. Suppose that $\angle B_2OC_1 = \angle BAC$, $\angle C_2OA_1 = \angle CBA$ and $\angle A_2OB_1 = \angle ACB$. Prove that

$$A_1A_2 + B_1B_2 + C_1C_2 \leq A_2B_1 + B_2C_1 + C_2A_1.$$

*Proposed by Dominik Burek, Krakow, Poland*

*Solution by the author*

We will use the following fact: *let $ABC$ be an acute triangle with circumcenter $O$. Let $X$ and $Y$ be points on sides $AB$ and $AC$, respectively, with $\angle YOX = \angle BAC$. Prove that the perimeter of triangle $AXY$ is greater than or equal to $BC$.*

Indeed, let $X'$ be a point such that $\triangle OAX \equiv \triangle OBX'$. Let $Y'$ be a point such that $\triangle OAY \equiv \triangle OCY'$. Then $OX' = OX$, $OY = OY'$, and

$$\begin{aligned} \angle X'OY' &= \angle BOC - \angle BOX' - \angle Y'OC = 2\angle BAC - \angle AOX - \angle YOA \\ &= 2\angle BAC - \angle YOX = \angle BAC. \end{aligned}$$

It follows that $\triangle OXY \equiv \triangle OX'Y'$. In particular, $XY = X'Y'$. Hence

$$\text{Perimeter}(AXY) = AX + XY + YA = BX' + X'Y' + Y'C \geq BC.$$

O540. Let $a, b, c$ be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq\frac{7}{12}.$$

*Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam*

*Solution by the author*

We have $b^6+c^6=(b^2+c^2)(b^4-b^2c^2+c^4)=(b^2+c^2)\left[(b^2+c^2)^2-(bc\sqrt{3})^2\right]=(b^2+c^2)(b^2-bc\sqrt{3}+c^2)(b^2+bc\sqrt{3}+c^2)$.

By AM-GM for three positive real numbers we have:

$$\sqrt[3]{4(b^6+c^6)}=\sqrt[3]{(b^2+c^2)\cdot 2(2+\sqrt{3})(b^2-bc\sqrt{3}+c^2)\cdot 2(2-\sqrt{3})(b^2+bc\sqrt{3}+c^2)}\leq\frac{9b^2-12bc+9c^2}{3}\Leftrightarrow$$

$$\sqrt[3]{4(b^6+c^6)}\leq 3b^2-4bc+3c^2\Leftrightarrow$$

$$\sqrt[3]{4(b^6+c^6)}+7bc\leq 3b^2+3bc+3c^2\Leftrightarrow$$

$$\frac{1}{\sqrt[3]{4(b^6+c^6)}+7bc}\geq\frac{1}{3(c^2+bc+c^2)}\Leftrightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}\geq\frac{a}{3(b^2+bc+c^2)}$$

Similarly for $b$ and $c$. Now,

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\geq\frac{a}{3(b^2+bc+c^2)}+\frac{b}{3(c^2+ca+a^2)}+\frac{c}{3(a^2+ab+b^2)}\tag{1}$$

Next, by Cauchy-Schwarz inequality, we have:

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}=\frac{a^2}{ab^2+abc+ac^2}+\frac{b^2}{bc^2+bca+ba^2}+\frac{c^2}{ca^2+cab+cb^2}\geq$$

$$\frac{(a+b+c)^2}{(ab^2+abc+ac^2)+(bc^2+bca+ba^2)+(ca^2+cab+cb^2)}$$

which equals to

$$\frac{a+b+c}{ab+bc+ca}$$

It follows that

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\geq\frac{a+b+c}{ab+bc+ca}=\frac{3}{ab+bc+ca}\tag{2}$$

From (1) and (2) we conclude that

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\geq\frac{1}{ab+bc+ca}\tag{3}$$

By AM-GM inequality for five positive real numbers: $\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\geq 5\sqrt[5]{\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot a^2\cdot a^2}=5\sqrt[5]{a^5}=5a\Leftrightarrow 3\sqrt[3]{a}+2a^2\geq 5a\Leftrightarrow 3\sqrt[3]{a}\geq 5a-2a^2$, similarly for $b$ and $c$.

Hence,

$$3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}) \geq 5(a+b+c)-2(a^2+b^2+c^2) \Leftrightarrow$$
$$3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}) \geq 15-2(a^2+b^2+c^2)) \Leftrightarrow$$
$$3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1) \geq 18-2(a^2+b^2+c^2) = 2(a+b+c)^2-2(a^2+b^2+c^2) \Leftrightarrow$$
$$3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1) \geq 2(a^2+b^2+c^2+2ab+2bc+2ca)-2(a^2+b^2+c^2) \Leftrightarrow$$
$$3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1) \geq 4(ab+bc+ca) \Leftrightarrow$$

$$3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}) \geq 4(ab+bc+ca)-3 \tag{4}$$

From (3) and (4) we conclude that

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12} \geq$$
$$\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12} \tag{5}$$

By AM-GM we get

$$\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9} \geq 2\sqrt{\frac{1}{ab+bc+ca}\cdot\frac{ab+bc+ca}{9}} = 2\sqrt{\frac{1}{9}} = \frac{2}{3}$$

(5) $\Leftrightarrow$

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab} \geq \frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12} \geq \frac{2}{3}-\frac{1}{12} = \frac{7}{12}$$

and we are done. Equality occurs when $a=b=c=1$.