Junior problems

J541. Solve in positive real numbers the system of equations

$$\begin{cases} (x-\sqrt{xy})(x+3y) & = & 8(9+8\sqrt{3}) \\ (y-\sqrt{xy})(3x+y) & = & 8(9-8\sqrt{3}) \end{cases}$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Adding the two equations we get $(\sqrt{x}-\sqrt{y})^4 = 144$. Subtracting the two equations we get $(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})^3 = 128\sqrt{3}$. Therefore, $(\sqrt{x}-\sqrt{y}, \sqrt{x}+\sqrt{y}) = (2\sqrt{3}, 4)$ or $(-2\sqrt{3}, -4)$, from which we have

$$x = (2+\sqrt{3})^2 = 7+4\sqrt{3} \text{ and } y = (2-\sqrt{3})^2 = 7-4\sqrt{3}.$$

It is easy to check that they indeed satisfy the system.

Also solved by Chistopher Lee, Singapore American School, Singapore; Ace Kim, Northern Valley Regional High School at Old Tappan, NJ, USA; Dumitru Barac, Sibiu, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Taes Padhihary, Disha Delphi Public School, India; HyunBin Yoo, South Korea; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Evripides P. Nastou, 6th High School, Nea Smyrni, Greece; Arkady Alt, San Jose, CA, USA.

J542. Let $ABCD$ be a unit square. Points M and N lie on sides BC and CD, respectively, such that $\angle MAN = 45^\circ$. Prove that

$$1 \le MC + NC \le 4 - 2\sqrt{2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Polyahedra, Polk State College, USA

Let $x = \angle BAM$ and $y = \angle DAN$. We have $MC + NC = 2 - BM - DN = 2 - \tan x - \tan y$. Since $\tan x + \tan y = (1 - \tan x \tan y)\tan 45^\circ \le 1$, $MC + NC \ge 1$.

By Jensen's inequality, $\tan x + \tan y \ge 2\tan(45^\circ/2) = 2\sqrt{2} - 2$, thus $MC + NC \le 4 - 2\sqrt{2}$.

Also solved by Chistopher Lee, Singapore American School, Singapore; Vicente Vicario García, Sevilla, Spain; Taes Padhihary, Disha Delphi Public School, India; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Joel Schlosberg, Bayside, NY, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; HyunBin Yoo, South Korea; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Arkady Alt, San Jose, CA, USA.

J543. Let a and b be positive real numbers. Prove that

$$\left|a^5 - b^5\right| = ab \max\left(a^3, b^3\right)$$

if and only if

$$\left|a^3 - b^3\right| = ab \min(a, b).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Joel Schlosberg, Bayside, NY, USA

If $a = b$, both quantities are zero.

If $a \neq b$,

$$|a^5 - b^5| - ab\max(a^3, b^3) = \begin{cases} (a^2 - ab + b^2)(a^3 - b^3 - ab^2) \text{ if } a > b \\ (a^2 - ab + b^2)(b^3 - a^3 - a^2 b) \text{ if } a < b. \end{cases}$$

Since $a^2 - ab + b^2 = \frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2 > 0$,

$$|a^5 - b^5| - ab\max(a^3, b^3) = 0 \iff \begin{cases} |a^3 - b^3| - ab \cdot b = 0 \text{ if } a > b \\ |a^3 - b^3| - ab \cdot a = 0 \text{ if } a < b. \end{cases}$$

Also solved by Chistopher Lee, Singapore American School, Singapore; Polyahedra, Polk State College, USA; Arkady Alt, San Jose, CA, USA; Taes Padhihary, Disha Delphi Public School, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; HyunBin Yoo, South Korea; Titu Zvonaru, Comănești, Romania; Ace Kim, Northern Valley Regional High School at Old Tappan, NJ, USA.

J544. Let a,b,c,x,y,z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{a}{a+2bx}+\frac{b}{b+2cy}+\frac{c}{c+2az}\geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Arkady Alt, San Jose, CA, USA

By replacing for convenience (a,b,c) with $\left(a^2,b^2,c^2\right)$, where $a,b,c>0$ we obtain for the proof equivalent inequality

$$\frac{a^2}{a^2+2b^2x}+\frac{b^2}{b^2+2c^2y}+\frac{c^2}{c^2+2a^2z}\geq 1,\text{where } x,y,z>0 \text{ and } x+y+z=3.$$

Consecutively applying Cauchy Inequality and AM-GM Inequality we obtain

$$\sum_{cyc}\frac{a^2}{a^2+2b^2x}=\sum_{cyc}\frac{\frac{a^2}{b^2}}{\frac{a^2}{b^2}+2x}\geq\frac{\left(\sum_{cyc}\frac{a}{b}\right)^2}{\sum_{cyc}\left(\frac{a^2}{b^2}+2x\right)}=\frac{\left(\sum_{cyc}\frac{a}{b}\right)^2}{\sum_{cyc}\frac{a^2}{b^2}+6}=\frac{\sum_{cyc}\frac{a^2}{b^2}+2\sum_{cyc}\frac{a}{c}}{\sum_{cyc}\frac{a^2}{b^2}+6}\geq$$

$$\frac{\sum_{cyc}\frac{a^2}{b^2}+2\cdot 3\sqrt[3]{\prod_{cyc}\frac{a}{c}}}{\sum_{cyc}\frac{a^2}{b^2}+6}=\frac{\sum_{cyc}\frac{a^2}{b^2}+6}{\sum_{cyc}\frac{a^2}{b^2}+6}=1.$$

Also solved by Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Taes Padhihary, Disha Delphi Public School, India; Polyahedra, Polk State College, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania.

J545. Let a, b, c be distinct positive real numbers such that

$$\left(a+\frac{b^2}{a-b}\right)\left(a+\frac{c^2}{a-c}\right)=4a^2.$$

Prove that $a^2 > bc$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Polyahedra, Polk State College, USA
Let ϕ be the golden ratio $(1+\sqrt{5})/2$. We prove the stronger claim that $a^2 > \phi^2 bc$. Suppose that $a^2 \le \phi^2 bc$. Write the equation as $(a^3+b^3)(a^3+c^3) = 4a^2(a^2-b^2)(a^2-c^2)$, or equivalently,

$$0 = 3a^6 - 4a^4(b^2+c^2) - a^3(b^3+c^3) + 4a^2b^2c^2 - b^3c^3.$$

By the AM-GM inequality, $b^2+c^2 > 2bc$ and $b^3+c^3 > 2bc\sqrt{bc} \ge 2\phi^{-1}abc$, thus

$$0 < 3a^6 - (8+2\phi^{-1})a^4bc + 4a^2b^2c^2 - b^3c^3 = (a^2-\phi^2bc)\left(3a^4-(3-\phi)a^2bc+\phi^{-2}b^2c^2\right) \le 0,$$

a contradiction.

Second solution by HyunBin Yoo, South Korea
The original equation equals $\left(\dfrac{a^2-ab+b^2}{a-b}\right)\left(\dfrac{a^2-ab+c^2}{a-c}\right) = 4a^2 \cdots (1)$.
Since $4a^2 > 0$, the two terms on the left side must have the same sign.
$a^2-ab+b^2 = \left(a-\dfrac{b}{2}\right)^2 + \dfrac{3}{4}b^2 > 0$ and $a^2-ac+c^2 = \left(a-\dfrac{c}{2}\right)^2 + \dfrac{3}{4}c^2 > 0$ means that both $a-b$ and $a-c$ must either be positive or negative.
In other words, ($a<b$ and $a<c$) or ($a>b$ and $a>c$).

$$(1) \Leftrightarrow \left(\frac{(a-b)^2+ab}{a-b}\right)\left(\frac{(a-c)^2+ac}{a-c}\right) = 4a^2$$

$$\Leftrightarrow \left(a-b+\frac{ab}{a-b}\right)\left(a-c+\frac{ac}{a-c}\right) = 4a^2$$

Due to the AM-GM inequality, $a-b+\dfrac{ab}{a-b} \ge 2\sqrt{(a-b)\cdot\dfrac{ab}{a-b}} = 2\sqrt{ab}$. Equality occurs when $b = \dfrac{(3\pm\sqrt{5})a}{2}$.
Applying this to the other term gives $a-c+\dfrac{ac}{a-c} \ge 2\sqrt{ac}$ and equality when $c = \dfrac{(3\pm\sqrt{5})a}{2}$.

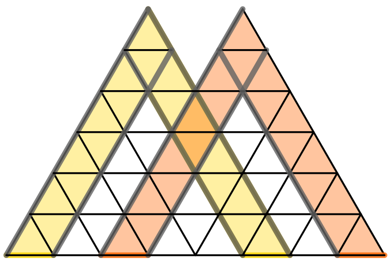
Note that since b and c are distinct and both of them are either bigger or smaller than a, at least one of the two equalities cannot be met for any (b, c).
So $4a^2 = \left(a-b+\dfrac{ab}{a-b}\right)\left(a-c+\dfrac{ac}{a-c}\right) > 2\sqrt{ab}\cdot 2\sqrt{ac}$.

$$\therefore 4a^2 > 2\sqrt{ab}\cdot 2\sqrt{ac} = 4a\sqrt{bc}$$

Dividing both sides by $4a$ then squaring results in $a^2 > bc$.

Also solved by Chistopher Lee, Singapore American School, Singapore; Ace Kim, Northern Valley Regional High School at Old Tappan, NJ, USA; Arkady Alt, San Jose, CA, USA; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Taes Padhihary, Disha Delphi Public School, India; Polyahedra, Polk State College, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J546. For $m \geq n \geq 0$, let AM_n^m be the *AwesomeMath figure of degree* (m, n), formed by two equilateral triangles of side m, overlapping in an equilateral triangle of side n. Assume that the triangles are subdivided into equilateral triangles of side 1. For example, the figure depicts AM_4^6. Count the number of parallelograms in AM_n^m.



Proposed by Li Zhou, Polk State College, USA

Solution by Polyahedra, Polk State College, USA

We count them by three types: A *Lefty* (L) is bounded by two horizontal lines and two lines of slope $-\sqrt{3}$; A *Righty* (R) is bounded by two horizontal lines and two lines of slope $\sqrt{3}$; A *Tiptoey* (T) is bounded by two lines of slope $\sqrt{3}$ and two lines of slope $-\sqrt{3}$.

In a single equilateral triangle of side m, which is the same as AM_m^m, there are $\binom{m+2}{4}$ T's: Just extend each of the four bounding sides of a T 1 unit below the base of AM_m^m to yield 4 endpoints out of $m+2$ possible points. By symmetry, the numbers of L's and R's in an AM_m^m are the same as well. By PIE, the number of parallelograms that are contained entirely within one or entirely within the other equilateral triangle of side m is $6\binom{m+2}{4} - 3\binom{n+2}{4}$.

It remains to count the parallelograms not entirely contained within one or the other triangles of side m. There are no such T. By symmetry, it suffices to count the number of such L's. There are $\binom{n+1}{2}$ choices for the two horizontal sides of such an L. Then its upper-left corner has $m - n$ choices on its top side and its lower-right corner also has $m - n$ choices on its bottom side. Therefore, there are $\binom{n+1}{2}(m-n)^2$ such L's, and the same number of such R's. Summing everything up we arrive at the answer

$$6\binom{m+2}{4} - 3\binom{n+2}{4} + 2\binom{n+1}{2}(m-n)^2.$$

Also solved by Chistopher Lee, Singapore American School, Singapore; Taes Padhihary, Disha Delphi Public School, India.

Senior problems

S541. Prove that for each positive integer n the number

$$3^{3^{n+1}+3}+3^{3^n+2}+1$$

is composite.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by David E. Manes, Oneonta, NY, USA

Let $N_n = 3^{3^{n+1}+3}+3^{3^n+2}+1$. If $n=1$, then

$$N_1 = 3^{9+3}+3^{3+2}+1 = 531\,685 = 7(5\cdot 11\cdot 1381)\equiv 0 \pmod 7.$$

Assume inductively if k is a positive integer such that $k\geq 1$, then $N_k\equiv 0 \pmod 7$. Then

$$N_{k+1} = 3^{3^{k+2}+3}+3^{3^{k+1}+2}+1 = 3^{3^{k+2}}\cdot 27+3^{3^{k+1}}\cdot 9+1.$$

Observe that

$$3^{3^{k+2}}\cdot 27 = \left(3^3\right)^{3^{k+1}}\cdot 27 \equiv (-1)^{3^{k+1}}(-1)\equiv 1 \pmod 7$$
$$3^{3^{k+1}}\cdot 9 = \left(3^3\right)^{3^{k}}\cdot 9 \equiv (-1)^{3^{k}}\cdot 2\equiv -2 \pmod 7.$$

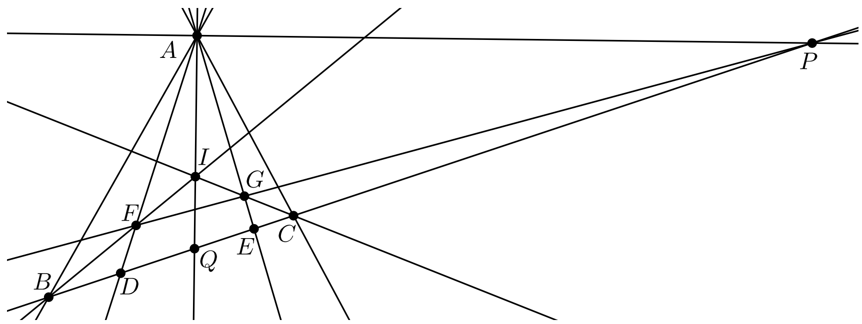
Therefore, $N_{k+1}\equiv 1-2+1\equiv 0 \pmod 7$ so that 7 is a divisor of N_{k+1}. Hence, N_{k+1} is composite and, by induction, N_n is composite for each positive integer n.

Also solved by Vicente Vicario García, Sevilla, Spain; Marie-Nicole Gras, Le Bourg d'Oisans, France; Li Zhou, Polk State College, USA; HyunBin Yoo, South Korea; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

S542. Let ABC be a triangle with $AB \neq AC$ and let I be its incenter. Points D and E are taken on side BC such that $\angle DAB = \angle EAC$. Lines AD and BI intersect in F, lines AE and CI intersect in G, and lines BC and FG intersect in P. Prove that $AP \perp AI$.

Proposed by Mihai Miculita, Oradea, Romania

Solution by Li Zhou, Polk State College, USA



Suppose that AI intersects BC at Q. Applying Menelaus' theorem to $\triangle ADE$ with transversal FG, we get

$$\frac{DP}{EP} = \frac{AG}{GE} \cdot \frac{FD}{AF} = \frac{AC}{EC} \cdot \frac{BD}{AB} = \frac{AQ \sin \angle QAE}{EQ \sin \angle EAC} \cdot \frac{DQ \sin \angle BAD}{AQ \sin \angle DAQ} = \frac{DQ}{EQ},$$

that is, $(D, E; P, Q)$ is a harmonic bundle. Therefore, $AP \perp AQ$ by the right-angle-and-bisector lemma (see Evan Chen, *Euclidean Geometry in Math. Olympiads*, MAA, 2016, Lemma 9.18, p. 177).

Also solved by Taes Padhihary, Disha Delphi Public School, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Titu Zvonaru, Comănești, Romania.

S543. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2abc}{ab+bc+ca} \geq \frac{11}{3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA

Letting $X=(ab+bc+ca)/abc$, we see that the left-hand side of the proposed inequality can be expressed as $X+2/X$, $X>0$. Then

$$X+\frac{2}{X} \geq \frac{11}{3} \iff (X-\frac{2}{3})(X-3) \geq 0 \iff 0<X \leq \frac{2}{3} \text{ or } X \geq 3.$$

But the AGM inequality gves us $\sqrt[3]{abc} \leq (a+b+c)/3=1$, so that $abc \leq (abc)^{2/3}$; and $\sqrt{(ab+bc+ca)/3} \geq \sqrt[3]{abc}$ yields $ab+bc+ca \geq 3(abc)^{2/3}$. Thus $X \geq 3(abc)^{2/3}/(abc)^{2/3}=3$, and we are done. Equality holds if and only if $a=b=c=1$.

Also solved by Marie-Nicole Gras, Le Bourg d'Oisans, France; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Titu Zvonaru, Comănești, Romania; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; HyunBin Yoo, South Korea; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Yuchen Fan; Arkady Alt, San Jose, CA, USA.

S544. Let ABC be a triangle. Prove that

$$\frac{\cos A}{\sin^2 A} + \frac{\cos B}{\sin^2 B} + \frac{\cos C}{\sin^2 C} \geq \frac{R}{r}.$$

Proposed by An Zhenping, Xianyang Normal University, China

First solution by Arkady Alt, San Jose, CA, USA

$$\sum_{cyc} \frac{\cos A}{\sin^2 A} \geq \frac{R}{r} \iff \sum_{cyc} \frac{\cos A}{4R^2 \sin^2 A} \geq \frac{1}{4Rr} \iff \sum_{cyc} \frac{\cos A}{a^2} \geq \frac{s}{4Rrs} \iff$$

$$\sum_{cyc} \frac{\cos A}{a^2} \geq \frac{s}{abc} \iff \sum_{cyc} \frac{2bc\cos A}{a} \geq 2s \iff \sum_{cyc} \frac{b^2+c^2-a^2}{a} \geq a+b+c \iff$$

$$\sum_{cyc} \frac{b^2+c^2}{a} \geq 2\left(a+b+c\right),$$

where the latter inequality holds since

$$\sum_{cyc} \frac{b^2}{a} \geq \sum_{cyc} \left(2b-a\right) = a+b+c$$

and

$$\sum_{cyc} \frac{c^2}{a} \geq \sum_{cyc} \left(2c-a\right) = a+b+c.$$

Second solution by Taes Padhihary, Disha Delphi Public School, India

Using Trigonometric Manipulations, we obtain

$$\frac{\cos A}{\sin^2 A} = \frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{4\sin^2 \frac{A}{2} \cos^2 \frac{A}{2}} = \frac{1}{4}\left(\csc^2 \frac{A}{2} - \sec^2 \frac{A}{2}\right).$$

Summing them all up and using identities, we obtain that

$$LHS \geq \frac{\csc \frac{A}{2} \csc \frac{B}{2} \csc \frac{C}{2}}{4} = \frac{R}{r},$$

as claimed.

Also solved by Marie-Nicole Gras, Le Bourg d'Oisans, France; Telemachus Baltsavias, Kerameies Junior High School, Kefallonia, Greece; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Scott H. Brown., Auburn University Montgomery, Montgomery, AL, USA; Titu Zvonaru, Comănești, Romania.

S545. Let x, y, z be nonnegative real numbers such that $x^2+y^2+z^2+xyz=4$. Prove that

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\geq\frac{1}{4}+\frac{4}{(x+y)(y+z)(z+x)}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Arkady Alt, San Jose, CA, USA

First, note that only one number from x, y, z can be equal zero, because otherwise we get division by zero. So, further $x, y, z \geq 0$ and $xy+yz+zx>0$. Since $x, y, z \geq 0$ and $x^2+y^2+z^2+xyz=4$ implies $x, y, z \in [0, 2]$ then denoting$\alpha := \arccos\frac{x}{2}, \beta := \arccos\frac{y}{2}, \gamma := \arccos\frac{z}{2}$ we obtain

$(x, y, z) = (2\cos\alpha, 2\cos\beta, 2\cos\gamma)$,where $\alpha, \beta, \gamma \in [0, \pi/2]$ and $x^2+y^2+z^2+xyz=4 \iff$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1, \tag{1}$$

$$\sum_{cic}\frac{1}{(x+y)^2}\geq\frac{1}{4}+\frac{4}{\prod_{cyc}(x+y)} \iff \sum_{cic}\frac{1}{(\cos\alpha+\cos\beta)^2}\geq 1+\frac{2}{\prod_{cyc}(\cos\alpha+\cos\beta)}.$$

Since $\sum_{cic}\frac{1}{(\cos\alpha+\cos\beta)^2}\geq\sum_{cic}\frac{1}{(\cos\alpha+\cos\beta)(\cos\beta+\cos\gamma)}$ then remains to prove inequality $\sum_{cic}\frac{1}{(\cos\alpha+\cos\beta)(\cos\beta+\cos\gamma)}\geq 1+\frac{2}{\prod_{cyc}(\cos\alpha+\cos\beta)} \iff$

$\sum_{cic}(\cos\gamma+\cos\alpha)\geq\prod_{cyc}(\cos\alpha+\cos\beta)+2 \iff$

$$2\sum_{cic}\cos\alpha\geq 2+\sum_{cic}\cos\alpha\cdot\sum_{cic}\cos\alpha\cos\beta-\cos\alpha\cos\beta\cos\gamma. \tag{2}$$

where α, β, γ can be considered as angles of some non obtuse triangle because (1) and $\alpha, \beta, \gamma \in [0, \pi/2]$ implies $\alpha+\beta+\gamma=\pi$ and also only one angle from α, β, γ can be equal $\pi/2$.

Let R, r and s be circumradius, inradius and semiperimeter of this triangle.

Since $\cos\alpha+\cos\beta+\cos\gamma=1+\frac{r}{R}, \cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha=\frac{s^2+r^2-4R^2}{4R^2}$, $\cos\alpha\cos\beta\cos\gamma=\frac{s^2-(2R+r)^2}{4R^2}$ then (2) becomes

$$2\left(1+\frac{r}{R}\right)\geq 2+\left(1+\frac{r}{R}\right)\frac{s^2+r^2-4R^2}{4R^2}-\frac{s^2-(2R+r)^2}{4R^2} \iff$$

$$\frac{2r}{R}\geq\left(1+\frac{r}{R}\right)\frac{s^2+r^2-4R^2}{4R^2}-\frac{s^2-(2R+r)^2}{4R^2}=\frac{r\left(2Rr+r^2+s^2\right)}{4R^3} \iff$$

$$s^2\leq 8R^2-2Rr-r^2.$$

Since $s^2\leq 4R^2+4Rr+3r^2$ (Gerretsen's Inequality) and $R\geq 2r$ (Euler's Inequality) then $8R^2-2Rr-r^2-s^2=2(R-2r)(2R+r)+\left(4R^2+4Rr+3r^2-s^2\right)\geq 0.$

Also solved by Marie-Nicole Gras, Le Bourg d'Oisans, France; Taes Padhihary, Disha Delphi Public School, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S546. Solve in real numbers the system of equations

$$x^3 - 2xyz + y^3 = \frac{1}{2}$$
$$y^3 - 2xyz + z^3 = 1$$
$$z^3 - 2xyz + x^3 = -\frac{3}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by David E. Manes, Oneonta, NY, USA

The solution (x,y,z) for the system of equations is $(x,y,z) = \left(\frac{-2}{\sqrt[3]{14}}, \frac{3}{\sqrt[3]{14}}, \frac{-1}{\sqrt[3]{14}}\right)$. One checks that these values do satisfy the three equations.

Solving for $2xyz$ in each of the three equations yields

$$2xyz = x^3 + y^3 - \frac{1}{2} = y^3 + z^3 - 1 = z^3 + x^3 + \frac{3}{2}.$$

Then $x^3 + y^3 - 1/2 = y^3 + z^3 - 1$ implies $x^3 = z^3 - 1/2$. From $x^3 + y^3 - 1/2 = z^3 + x^3 + 3/2$, we get $y^3 = z^3 + 2$. Adding the three given equations, one obtains

$$x^3 + y^3 + z^3 = 3xyz.$$

Therefore, $z^3 - (1/2) + z^3 + 2 + z^3 = 3\sqrt[3]{z^3 - (1/2)}\sqrt[3]{z^3 + 2} \cdot z$. Simplifying, we get

$$z^3 + 1/2 = z\sqrt[3]{z^3 + (3/2)z^3 - 1}.$$

Cubing both sides of this equation, one obtains

$$z^9 + (3/2)z^6 + (3/4)z^3 + 1/8 = z^9 + (3/2)z^6 - z^3.$$

Therefore, $(7/4)z^3 = -1/8$ or $z^3 = -1/14$ so that $z = -1/\sqrt[3]{14}$. Then $x^3 = z^3 - 1/2 = -8/14$ implies $x = -2/\sqrt[3]{14}$ and $y^3 = z^3 + 2 = (-1/14) + 28/14 = 27/14$ implies $y = 3/\sqrt[3]{14}$. This completes the solution.

Also solved by Vicente Vicario García, Sevilla, Spain; Marie-Nicole Gras, Le Bourg d'Oisans, France; Arkady Alt, San Jose, CA, USA; Taes Padhihary, Disha Delphi Public School, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; HyunBin Yoo, South Korea; Yuchen Fan.

U541. Let R be a (non necessary commutative) ring which contains $\mathbb{Q}$ as a subring and in which every non invertible element is a divisor of zero. Assume that x, y are elements of R such that $xy = yx$ and $x^m = y^n = 1$, where m, n are pairwise prime positive integers. Prove that $1 + x + y$ is invertible in R.

Proposed by Mircea Becheanu, Canada

Solution by the author

The ring $\mathbb{Q}$ is contained in the center of R. Then the ring $\mathbb{Q}[x, y]$ is commutative. Assume by contradiction that $1 + x + y$ is not invertible. If $1 + x + y \neq 0$ it is a zero divisor, then there exists $a \in R$, $a \neq 0$, such that $(1 + x + y)a = 0$. The same is true if $1 + x + y = 0$. Hence we have $(1 + x)a = -ya$. We multiply this equality by $1 + x$ to obtain:

$$(1 + x)^2 a = (1 + x)(1 + x)a = (1 + x)(-ya) = -y(1 + x)a = y^2 a.$$

Again multiply by $1 + x$ and obtain:

$$(1 + x)^3 a = (1 + x)y^2 a = y^2(1 + x)a = -y^3 a.$$

By induction, for every n, we have:

$$(1 + x)^n a = (-1)^n y^n a.$$

For n given in the problem we obtain $[(1 + x)^n - (-1)^n]a = 0$. Because $x^m = 1$ we also have $(x^m - 1)a = 0$. Using the hypothesis $\gcd(m, n) = 1$, it is not difficult to see that the polynomials $(1+X)^n-(-1)^n$ and X^m-1 are relatively rime in the ring $\mathbb{Q}[X]$. Then, there exists a linear combination of rational polynomials such that

$$[(1 + X)^n - (-1)^n]f(X) + (X^m - 1)g(X) = 1.$$

Therefore, we have in R the identity $[(1 + x)^n - (-1)^n]f(x) + (x^m - 1)g(x) = 1$. Fom here, we obtain

$$a = [f(x)(1 + x)^n - (-1)^n]a + g(x)(x^m - 1)a = 0.$$

This is a contradiction.

U542. Evaluate

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2}}+\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}+\sqrt{2}}\right).$$

Proposed by Toyesh Prakash Sharma, Agra, I ndia

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA

We use the Squeeze Theorem to show that the limit equals 2.

The fact that $f(x)=1/(\sqrt{x}+\sqrt{2})$ is convex and decreasing gives us the inequalities

$$\int_0^{n+1}\frac{dx}{\sqrt{x}+\sqrt{2}} < \sum_{k=0}^{n}\frac{1}{\sqrt{k}+\sqrt{2}} < \frac{1}{\sqrt{2}}+\int_0^{n}\frac{dx}{\sqrt{x}+\sqrt{2}}. \tag{1}$$

Some elementary substitutions give us

$$\int\frac{dx}{\sqrt{x}+\sqrt{2}} = -2\sqrt{2}\ln(\sqrt{x}+\sqrt{2}) + 2\sqrt{x} + C,$$

so that we have, after evaluating the integrals and dividing (1) through by $\sqrt{n}$,

$$\begin{aligned}2\left\{\frac{\sqrt{2}\ln 2}{\sqrt{n}}-\frac{\sqrt{2}\ln(\sqrt{n+1}+\sqrt{2})}{\sqrt{n}}+\sqrt{\frac{n+1}{n}}\right\} &< \frac{1}{\sqrt{n}}\sum_{k=0}^{n}\frac{1}{\sqrt{k}+\sqrt{2}}\\ &< \frac{1}{\sqrt{2n}}+2\left\{\frac{\sqrt{2}\ln 2}{\sqrt{n}}-\frac{\sqrt{2}\ln(\sqrt{n}+\sqrt{2})}{\sqrt{n}} + 1\right\}.\end{aligned}$$

Since the lower and upper bounds of the given sequence tend to 2 as $n\to\infty$, we have our claimed result.

Also solved by Vicente Vicario García, Sevilla, Spain; Arkady Alt, San Jose, CA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, USA; Olimjon Jalilov, Tashkent, Uzbekistan; Yuchen Fan.

U543. Let n be a positive integer. Evaluate

$$\lim_{x\to 0}\frac{1}{x^{n+1}}\left(\int_0^x e^{t^n}\,dt - x\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
By L'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x^{n+1}}\left(\int_0^x e^{t^n}\,dt - x\right) &= \lim_{x\to 0}\frac{e^{x^n}-1}{(n+1)x^n}\\ &= \lim_{x\to 0}\frac{nx^{n-1}e^{x^n}}{(n+1)nx^{n-1}}\\ &= \lim_{x\to 0}\frac{e^{x^n}}{n+1} = \frac{1}{n+1}.\end{aligned}$$

Second solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
As $x \to 0$.

$$\int_0^x e^{t^n}\,dt = \int_0^x \left(1 + t^n + O\left(t^{2n}\right)\right)\,dt = x + \frac{x^{n+1}}{n+1} + O\left(x^{2n+1}\right).$$

Thus,

$$\frac{1}{x^{n+1}}\left(\int_0^x e^{t^n}\,dt - x\right) = \frac{1}{n+1} + O\left(x^n\right),$$

and

$$\lim_{x\to 0}\frac{1}{x^{n+1}}\left(\int_0^x e^{t^n}\,dt - x\right) = \frac{1}{n+1}.$$

Also solved by Vicente Vicario García, Sevilla, Spain; Taes Padhihary, Disha Delphi Public School, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Franklin Maxfield, Utah Valley University, USA; Olimjon Jalilov, Tashkent, Uzbekistan; Westchester Area Math Circle, Purchase, NY, USA; Yuchen Fan.

U544. Find all real numbers x such that the sequence $(\cos 2^n x)_{n\geq 1}$ converges.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Li Zhou, Polk State College, USA

If it converges to the limit L, then by the double-angle formula $L = 2L^2 - 1$, so $L = 1$ or $-1/2$.

For $L = 1$, we must have $x = k\pi/2^m$, where k is an integer and m is a nonnegative integer. For $L = -1/2$, we must have $x = k\pi/(3\cdot 2^m)$, where k is an integer not divisible by 3 and m is a nonnegative integer.

Also solved by Taes Padhihary, Disha Delphi Public School, India.

U545. Prove that

$$\int_{e}^{4e} \frac{dx}{\ln x - \ln 2} \geq \frac{90e}{34\ln 2 + 15}.$$

Proposed by Olimjon Jalilov, Tashkent, Uzbekistan

Solution by Li Zhou, Polk State College, USA

Let $t = \ln x - \ln 2 - 1$, then $x = 2e^{1+t}$, $dx = 2e^{1+t}dt$, and the integral becomes

$$2e\int_{-\ln 2}^{\ln 2} \frac{e^t}{1+t}dt > 2e\int_{-\ln 2}^{\ln 2} 1dt = 4e\ln 2 > \frac{90e}{34\ln 2 + 15}.$$

Also solved by Yuchen Fan.

U546. Let p be an odd prime and let $n > 2$ be an integer. For any permutation f of the set $\{1, 2, \ldots, n\}$, let $I(f)$ denote the number of inversions of f. Let A_j denote the number of permutations f such that $I(f) \equiv j \pmod p$, for all $0 \le j \le p-1$. Prove that $A_1 = A_2 = \cdots = A_{p-1}$ if and only if $p \le n$.

Proposed by Shubhrajit Bhattacharya, Chennai Mathematical Institute, India

Solution by Li Zhou, Polk State College, USA

Let a_k be the number of $f \in S_n$ with $I(f) = k$, then $A_j = \sum_{k \equiv j \pmod p} a_k$ and a_k has the well-known generating function

$$G(x) = \sum_{f \in S_n} x^{I(f)} = \sum_{k=0}^{\binom{n}{2}} a_k x^k = (1+x)\left(1+x+x^2\right)\cdots\left(1+x+\cdots+x^{n-1}\right).$$

Let $\xi = e^{2\pi i/p}$. If $p \le n$, then $G(\xi)$ has $1 + \xi + \cdots + \xi^{p-1} = 0$ as a factor, thus

$$0 = G(\xi) = \sum_{k=0}^{\binom{n}{2}} a_k \xi^k = A_0 + A_1\xi + \cdots + A_{p-1}\xi^{p-1}.$$

Since $1 + x + \cdots + x^{p-1}$ is the minimal polynomial for ξ, $A_0 = A_1 = \cdots = A_{p-1}$.

Conversely, suppose that $A_1 = A_2 = \cdots = A_{p-1} = m$. Since $a_k = a_{\binom{n}{2}-k}$ for all k and $\binom{n}{2} \equiv j \pmod p$ for some $j \in \{1, 2, \ldots, p-1\}$, we have $A_0 = A_j$. Therefore $G(\xi) = m\left(1 + \xi + \cdots + \xi^{p-1}\right) = 0$, so $p \le n$.

Also solved by Taes Padhihary, Disha Delphi Public School, India.

Olympiad problems

O541. Let a, b, c be the lengths of the sides of a triangle, and let S be its area. Let R and r be the circumradius and inradius of the triangle, respectively. Prove that

$$\cot^2 A + \cot^2 B + \cot^2 C \geq \frac{1}{5}\left(31 - \frac{52r}{R}\right).$$

Proposed by Titu Andreescu, USA and Marius Stănean, Romania

Solution by the authors

The inequality can be rewritten as

$$\frac{(ab+bc+ca)^2 - 2abc(a+b+c)}{16s^2r^2} \geq \frac{23}{10} - \frac{13r}{5R}.$$

But $ab+bc+ca = s^2 + r^2 + 4Rr$, so this becomes

$$\frac{s^4 + 2(r^2+4Rr)s^2 + (r^2+4Rr)^2 - 16Rrs^2}{16s^2r^2} \geq \frac{23}{10} - \frac{13r}{5R},$$

$$\frac{s^2}{16R^2} + \frac{R^2}{16s^2}\left(\frac{r^2}{R^2} + \frac{4r}{R}\right)^2 + \frac{r^2}{8R^2} - \frac{r}{2R} \geq \frac{23r^2}{10R^2} - \frac{13r^3}{5R^3}.$$

Hence, we need to prove that $f\left(\frac{s^2}{R^2}\right) \geq 0$, where

$$f\left(\frac{s^2}{R^2}\right) = \frac{s^2}{16R^2} + \frac{R^2}{16s^2}\left(\frac{r^2}{R^2} + \frac{4r}{R}\right)^2 + \frac{r^2}{8R^2} - \frac{r}{2R} - \frac{23r^2}{10R^2} + \frac{13r^3}{5R^3}.$$

Because

$$\frac{s^2}{R^2} \geq \frac{r^2}{R^2} + \frac{4r}{R},$$

we deduce that f is a increasing function.

If we denote $x^2 = 1 - \frac{2r}{R} \in [0,1)$, then by Blundon's Inequality

$$\frac{s^2}{R^2} \geq 2 + 5(1-x^2) - \frac{(1-x^2)^2}{4} - 2x^3 = \frac{(1-x)(x+3)^3}{4}.$$

Hence, it suffices to prove that

$$f\left(\frac{(1-x)(x+3)^3}{4}\right) \geq 0,$$

that is

$$\frac{(1-x)(x+3)^3}{64} + \frac{(1-x^2)^2(9-x^2)^2}{64(1-x)(x+3)^3} + \frac{(1-x^2)^2}{32} - \frac{1-x^2}{4} - \frac{23(1-x^2)^2}{40} + \frac{13(1-x^2)^3}{40} \geq 0.$$

After some computations, we can rewrite these last inequalities as

$$\frac{x^2(1-x)\left[13x^4 + 52x^3 + (6x-1)^2\right]}{40(x+3)} \geq 0,$$

clearly true. The equality holds when $x = 0$, so when the triangle is equilateral.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Marie-Nicole Gras, Le Bourg d'Oisans, France.

O542. Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{1}{x^3+y^3+z^3}+\frac{24}{xy+yz+zx} \geq 81.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Let $t := xy+yz+zx$. Since $x^3+y^3+z^3 = 3xyz - 3(x+y+z)(xy+yz+zx) + (x+y+z)^3 =$
$3xyz - 3t + 1 \leq t^2 - 3t + 1$ (because $3xyz = 3xyz(x+y+z) \leq (xy+yz+zx)^2 = t^2$)
then

$$\frac{1}{x^3+y^3+z^3}+\frac{24}{xy+yz+zx} - 81 = \frac{1}{t^2-3t+1}+\frac{24}{t} - 81.$$

Since $3t = 3(xy+yz+zx) \leq (x+y+z)^2 = 1$ then

$$\frac{1}{t^2-3t+1}+\frac{24}{t} - 81 = \frac{(1-3t)\left(27t^2+24-80t\right)}{t\left(t^2+(1-3t)\right)} \geq 0$$

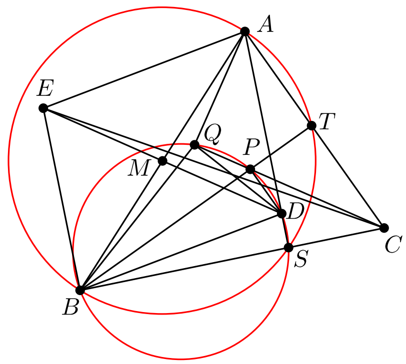
because $27t^2 - 80t + 24 > 27t^2 - 81t + 24 = 3(1-3t)(8-3t) \geq 0$.

Also solved by Marie-Nicole Gras, Le Bourg d'Oisans, France; Taes Padhihary, Disha Delphi Public School, India; Dao Quang Anh, Archimedes Dong Anh School, Dong Anh, Ha Noi, Vietnam; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O543. Let ABC be a triangle. Point M is the midpoint of side AB and D lies inside the triangle. Let E be the reflection of D with respect to M. Inside triangle ABC a point P is chosen such that DP and AC are parallel and $\angle CBP = \angle DAC$. Prove that $\angle ACP = \angle BCE$.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Li Zhou, Polk State College, USA



Let $S = AD \cap BC$ and $T = BP \cap AC$. Since $\angle ADP = \angle DAC = \angle CBP$, the points B, S, D, P lie on a circle ω and B, S, T, A lie on another circle Ω. Suppose that ω intersects CP at a second point Q. By the power of point, $CP \cdot CQ = CS \cdot CB = CT \cdot CA$, thus

$$\angle CAQ = \angle CPT = \angle BPQ = \angle BDQ.$$

Also, $\angle DBQ = \angle DPC = \angle ACQ$, so $\triangle DBQ \sim \triangle ACQ$. Since $BDAE$ is a parallelogram,

$$\frac{BQ}{CQ} = \frac{BD}{CA} = \frac{EA}{CA}.$$

Next,

$$\angle BQC = \angle BDS + \angle ADP = \angle EAD + \angle DAC = \angle EAC.$$

Hence, $\triangle BQC \sim \triangle EAC$, from which the claim follows.

Also solved by Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Taes Padhihary, Disha Delphi Public School, India; Yuchen Fan.

O544. Find all triples of positive integers (a,b,p), with p prime, such that

$$\frac{2^a+2^b}{a+b}=a^p+b^p.$$

Proposed by Karthik Vedula, James S. Rickards High School, Tallahassee, USA

Solution by the author

$(a,b,p)=(1,1,p)\ \forall$ primes p

WLOG, let $a\geq b$. Note that $(a+b)(a^p+b^p)$ is even, but $a+b\equiv a^p+b^p \pmod 2$, so both $a+b$ and a^p+b^p are even. Now, suppose that $a\neq b$. Clearly, we have $v_2(2^a+2^b)=b$, and we have

$$v_2((a+b)(a^p+b^p))=v_2(a+b)+v_2(a^p+b^p)$$

However, for odd primes p, we have

$$b=2v_2(a+b)+v_2(a^{p-1}-a^{p-2}b+\cdots+b^{p-1})$$

1. If a,b are both odd, the term $a^{p-1}-a^{p-2}b+\cdots+b^{p-1}$ is odd, and we have $b=2v_2(a+b)$. However, this is a contradiction to the original assumption that b is odd.

2. If a,b are both even, then $2^a+2^b\equiv 2 \pmod 3$. However, we have that $a^p+b^p\equiv a+b \pmod 3$, so the equation reduces to $(a+b)^2\equiv 2 \pmod 3$, a contradiction.

This means that our original assumptions of $a\neq b$ and $p\neq 2$ were wrong, so we must have at least one of $a=b$ and/or $p=2$.

$a=b$, then the equation turns into $\frac{2^a}{a}=2a^p \implies 2^{a-1}=a^{p+1}$. This means that a is a power of 2, so let $a=2^c$. We now have

$$2^{a-1}=2^{c(p+1)} \implies a-1=c(p+1)=2^c-1$$

Clearly $c=0$ works (for all p) and $c=1$ fails. Now, suppose that $c\geq 2$. We have $c|2^c-1$. Let p denote the smallest prime factor of p. We have

$$p|2^c-1 \implies \text{ord}_p 2|c,p-1 \implies \text{ord}_p 2|\gcd(p-1,c)$$

However, since p is the smallest prime factor of c, the only factor of c less than p is 1. This means $\gcd(p-1,c)=1$, so $\text{ord}_p 2=1 \implies p|2^1-1=1$, contradiction. This means that p cannot exist, and $c=0$ is the only possibility in this case. This implies $a=b=1$ and p has no restrictions, which clearly works. $p=2$, then the equation turns into

$$2^a+2^b=(a+b)(a^2+b^2) \implies 2^a+2^b|a^4-b^4$$

We have already resolved the case where $a = b$, so we can WLOG $a > b$. However, this means that $2^a + 2^b \leq a^4 - b^4 \implies 2^a < a^4 \implies a < 16$. Since $a^2 + b^2 \equiv a + b \pmod{2}$, and their product is even, a and b have the same parity:

a, b are both odd: Since $a^2 + b^2 \equiv 2 \pmod{4}$, we have $v_2(2^a + 2^b) = v_2(a + b) + 1 = b$. This means that $2^{b-1} | a + b$. Note that the maximum value of $a + b$ is $15 + 14 = 29$, so $2^{b-1} < 32 \implies b < 6$. If $b = 5$, then $16 | a + b \implies a = 11$, which does not work. If $b = 3$, then $2^a + 8 = (a + 3)(a^2 + 9)$ and $4 | a + 3$. The only values that satisfy the second condition are $a = 5, 9, 13$. The only one of these values which satisfy $a + 3 | 2^a + 8$ is $a = 5$, but it does not satisfy the original equation.

This means that we must have $b = 1$ and $2^a + 2 = (a + 1)(a^2 + 1) \implies 2^a + 1 = a^3 + a^2 + a \implies 2^a > a^3 \implies a \geq 10$. This means that $a = 11, 13, 15$, but it is easy to see that none of these values satisfy $a | 2^a + 1$, so there is no solution in any odd case.

This means that a, b are both even, so $a, b \in \{2, 4, 6, 8, 10, 12, 14\}$. Note that $a + b$ divides $2^a + 2^b = 2^b(2^{a-b} + 1)$. Since $a - b$ is positive and even, then the LHS is not a multiple of 3, so $a + b$ and $a^2 + b^2$ are not multiples of 3:

Suppose that neither a nor b are multiples of 3. This means that $a \equiv b \equiv 1, 2 \pmod{3}$ and $a^2 + b^2 \equiv 2 \pmod{3}$. Since $2^a + 2^b \equiv 2 \pmod{3}$, we have $a + b \equiv 1 \pmod{3}$. Therefore, $a \equiv b \equiv 2 \pmod{3}$. This means that $a, b \in \{2, 8, 14\}$. Note that if $b = 2$, then $v_2(2^a + 2^b) = 2$, but $v_2(a^2 + b^2) \geq 2$ and $v_2(a + b) \geq 1$, contradiction. This means that $(a, b) = (14, 8)$. However, $v_2(2^a + 2^b) = 8$ and $v_2((a + b)(a^2 + b^2)) = 3$, contradiction.

This means that at least one of a, b is a multiple of 3, but clearly not both.

One of them is 6: WLOG $b = 6$. We have $2^a + 64 = (a + 6)(a^2 + 36) \implies 2^a > a^3 + 100 \implies a \geq 11 \implies a = 12, 14$. But, 3 cannot divide a, b, so the only possible value of a is 14, but this fails as the LHS is not a multiple of 5, while the RHS is. One of them is 12: WLOG $b = 12$. We have $2^a + 2^{12} = (a + 12)(a^2 + 144)$. Note that taking the equation modulo 3 gives $2 \equiv a^3 \pmod{3} \implies a \equiv 2 \pmod{3} \implies a \in \{2, 8, 14\}$. With these options, it is clear that $2^a + 2^{12}$ is never a multiple of 13, but when $a = 8$ and $a = 14$, it has a factor of $8^2 + 144 = 208 = 13 \cdot 16$ and $14 + 12 = 26 = 13 \cdot 2$, contradiction. This means $a = 2$ is the only possible solutions, but this fails.

This means that the case $p = 2$ and $a \neq b$ has no solutions.

Therefore, our only solutions are $(a, b, p) = (1, 1, p)$ for any prime p, which clearly work as both the LHS and RHS are 2.

O545. Let a and b be integers with $a > 2$ and $\gcd(a,b) = 1$. Prove that for any positive integer n there are infinitely many positive integers k such that $(ak+b)^n$ divides $\binom{2k}{k.}$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Take primes p, q such $p \equiv -1 \pmod{a}, q \equiv -b \equiv c \pmod{a}$ where $0 < c < a$ and morevoer $p > \max\{b, n\}$ and $\frac{ap+2c}{2} < q < ap + c$. It follows that

$$k = \frac{q-c}{a}p + \frac{pc-b}{a}, 2k = p^2 + \left(\frac{2q-2c}{a} - p\right)p + \frac{2pc-2b}{a}.$$

Hence, $2S_p(k) - S_p(2k) = p - 1 \geq n$. Also

$$k = \frac{p+b-a}{a}q + \frac{(a-1)q-b}{a}, 2k = \left(\frac{2(p-a+1)}{a} + 1\right)q + \frac{q(a-2)-b}{a}.$$

Hence, $2S_q(k) - S_q(2k) = q - 1 \geq n$ and we are done.

Remark: This proof could be generalized to other cases. For example, if $a = 1, b > 0$ take $k = p - b, p > \max\{2b, n\}$. We have $2k = p + p - 2b$, hence $2S_p(k) - S_p(2k) = p - 1 \geq n$. Further, in the case of $a = 1, b < 0$ we can take $k = pq - b$, for some prime numbers p, q satisfying $p > \max\{2b, n\}, 2p > q > \frac{3}{2}p$. We then take $k = p^2 + (q-p)p - b, 2k = 3p^2 + (2q-3p)p - 2b$. Hence, $2S_p(k) - S_p(2k) = p - 1 \geq n$. Also $2k = q^2 + (2p-q)q - 2b$ and $2S_q(k) - S_q(2k) = q - 1 \geq n$.

Also solved by Dumitru Barac, Sibiu, Romania.

O546. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 6$. Find all possible values of the expression:

$$\left(\frac{a+b+c}{3} - a\right)^5 + \left(\frac{a+b+c}{3} - b\right)^5 + \left(\frac{a+b+c}{3} - c\right)^5.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author
Rewriting the expression gives

$$\frac{(a+b-2c)^5 + (b+c-2a)^5 + (c+a-2b)^5}{3^5}.$$

Using the symmetricity of the relations, WLOG assume $a \geq b \geq c$.

Now, consider the following 2 cases.
Case 1: If $c + a - 2b \leq 0$, then

$$(a+b-2c)^5 + (b+c-2a)^5 + (c+a-2b)^5 = (a+b-2c)^5 - (2a-b-c)^5 - (2b-c-a)^5 \geq 0$$

because it's clear that $(x+y)^5 - x^5 - y^5 \geq 0$ for $x, y > 0$.

Case 2: If $c + a - 2b \geq 0$, then applying Jensen's inequality on $x \mapsto x^5$ yields to

$$\frac{x^5 + y^5}{2} \geq \left(\frac{x+y}{2}\right)^5,$$

$$(a+b-2c)^5 + (c+a-2b)^5 - (2a-b-c)^5 \geq$$
$$\frac{(a+b-2c+c+a-2b)^5}{2^4} - (2a-b-c)^5 = -\frac{15}{16}(2a-b-c)^5.$$

Using Cauchy-Schwarz inequality results into

$$(2a-b-c)^2 \leq [2^2 + (-1)^2 + (-1)^2](a^2+b^2+c^2) = 36$$

and

$$2a - b - c \leq 6$$

Notice that

$$E(a,b,c) \geq -\frac{15}{16 \cdot 3^5}(2a-b-c)^5 \geq -\frac{15 \cdot 6^5}{2^4 \cdot 3^5} = -30.$$

Therefore, the minimum of the expression is -30 when $a = 2, b = -1, c = -1$. On the other side, notice that

$$E(a,b,c) = -E(-a,-b,-c) \leq 30$$

and the maximum of the expression is 30 when $a = -2, b = 1, c = 1$ and we are done.

Also solved by Marie-Nicole Gras, Le Bourg d'Oisans, France; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Taes Padhihary, Disha Delphi Public School, India.