Additive Functions on Domains

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Abstract

It is an exoteric fact that if an additive function $f:\mathbb{R}\to\mathbb{R}$ is bounded above (below) on an interval I, then there is a real number c such that $f(x)=cx$ for each real number x. But, what if we restrict the additivity of the function to some domain D, that is, a set of points (x,y) of Cartesian plane with some property? In this article, by defining a (quasi) extension function, we try to provide a detailed account of this problem.

1 Introduction

It is known that if an additive function $f:\mathbb{R}\to\mathbb{R}$ is bounded above (below) on an interval I, then there is a real number c such that $f(x)=cx$ for each real number x (see [1]). This result is a special case of a result obtained by *Darcozy* and *Losonczki* in [2] about additive functions on open connected sets. The latter result is unknown for the students who are preparing for mathematical competitions. In this article we will deeply investigate this result with two different approaches. We then discuss some problems from recent mathematical olympiads. We show that this result, amongst other things, will provide a richer toolbox to solve mathematical olympiad problems about functional equations. The rest of the article is divided into six parts. First, we work on additive functions on some intervals. Second, we study the bounded functions with some additional constraints on some intervals. Third, we shall provide the theoretical background needed for the main result. Fourth, we provide the main result based on *Darcozy* and *Losonczki* result (1966). Fifth, we study the bearings of our main result through solving some breathtaking problems along with trying to highlight the overlaps between the second and the fourth parts. Sixth, we finish this article by making some remarks.

2 Additive Functions and Intervals

We begin this section with some basic problems that can be converted to additive functions. First and foremost, we need to consider or prove the additivity or pseudo-additivity of the function. Then, we find some interval and prove that the function under study is (at least) a one-sided bounded function.

Problem 1. (*Hosszu*'s functional equation) Find all increasing functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y-xy)+f(xy)=f(x)+f(y)$$

for all real numbers x,y.

Solution. We shall prove that the function $f(x)$ is a shift of an additive function. Note that

$$\begin{aligned}&f(xy)+f(z)-f(xyz)+(f(x)+f(y)-f(xy))\\=\;&(f(y)+f(xz)-f(xyz))+(f(x)+f(z)-f(xz)).\end{aligned}$$

Hence,

$$f(xy+z-xyz)+f(x+y-xy)=f(y+xz-xyz)+f(x+z-xz).$$

Setting $zx=1$, we find that $f\left(xy+\frac{1}{x}-y\right)+f(x+y-xy)=f(1)+f\left(x+\frac{1}{x}-1\right)$. Defining $g(x)=f(x+1)-f(1)$, we have

$$g\left(xy+\frac{1}{x}-y-1\right)+g(x+y-xy-1)=g\left(x+\frac{1}{x}-2\right).$$

Consider now the system

$$a=xy+\frac{1}{x}-y-1,\quad b=x+y-xy-1=-(x-1)(y-1).$$

We find that $b = -(x-1)(y-1)$, $a+b+2 = \frac{1}{x} + x$. This system has real solutions (a,b) with $a+b>0$. Thus, for all $a,\ b,\ a+b>0$, we have $g(a)+g(b) = g(a+b)$. Now, considering a,b such that $a+b \le 0$, we have that there is a real number c such that $a+c,\ b+c>0$. Hence,

$$\begin{aligned} g(a+b)+2g(c) &= g(a+b)+g(2c) \\ &= g(a+b+2c) \\ &= g(a+c)+g(b+c) \\ &= g(a)+g(b)+2g(c). \end{aligned}$$

Hence, $g(x)$ is additive on the real line. Therefore, $f(x) = g(x-1)+f(1) = g(x)-g(1)+f(1)$. That is, $f(x)+f(y)-f(x+y) = f(1)-g(1)$. Thus, the function $f(x)+d$ is additive for some constant k. Since it is increasing, it follows that it is bounded below and above on some interval I. Hence, $f(x)+d = cx$ for some constant c. Thus, $f(x) = cx-d$. ■

The next problem is more interesting. It presents a criterion about additivity when the ratio of the variables is in a definite interval.

Problem 2. (Argentina Olympiad, 2010). Find all functions $f:\mathbb{N}\to\mathbb{R}$ satisfying

$$f(x+y) = f(x)+f(y)$$

for all natural numbers x,y such that $\frac{x}{y} \in (10^6 - 10^{-6}, 10^6 + 10^{-6})$.

Solution. We prove the problem in a more general case, that is, for $I=(a,b)$ such that I contains at least one integer and for all positive integers x,y such that $\frac{x}{y}\in I,\ f(x+y) = f(x)+f(y)$.We prove that $\frac{f(n)}{n}$ is constant.

Let n be a sufficiently large positive integer. We know that there is a positive integer m such that $\frac{m}{n+1}, \frac{m+1}{n} \in I$. Putting $x=m$, $y=n+1$ and then $x=m+1$, $y=n$, we find that

$$f(m)+f(n+1) = f(m+n+1) = f(m+1)+f(n).$$

Now, we require that $\frac{m}{n}, \frac{m+1}{n-1}\in I$. Since $\frac{m}{n+1} < \frac{m}{n} < \frac{m+1}{n} < \frac{m+1}{n-1}$, it suffices to choose m such that

$$a < \frac{m}{n+1}, \quad \frac{m+1}{n-1} < b.$$

That is, $a(n+1) < m < b(n-1)-1$ for all n large enough so that an integer m exists. Hence,

$$f(m)+f(n) = f(m+n) = f(m+1)+f(n-1).$$

Then, for all sufficiently large n we have

$$f(n+1)-f(n) = f(m+1)-f(m) = f(n)-f(n-1).$$

Therefore, $f(n+1)-f(n)$ is constant for all $n\ge k$. So,

$$f(n) = (n-k)\left(f(k+1)-f(k)\right)+f(k)$$

for all $n \ge k$.

Setting $A = f(k+1)-f(k)$, $B = f(k)$, then $f(n) = A(n-k)+B$, $n\ge k$. Take $\frac{r}{s}\in I$, where r,s are natural numbers.

Setting $x=rk$, $y=sk$, we have $x,y,\ x+y\ge k$ and thus

$$f(x) = A(x-k)+B,\ f(y) = A(y-k)+B,\ f(x+y) = A(x+y-k)+B.$$

Since $f(x+y) = f(x)+f(y)$, we find that

$$f(k) = B = kA = k(f(k+1)-f(k)).$$

That is, $\frac{f(k)}{k} = \frac{f(k+1)}{k+1} = C$ for some real number C. Hence, $f(k) = Ck$ implying that

$$f(n) = (n-k)(f(k+1) - f(k)) + f(k) = C(n-k) + Ck = Cn$$

for all $n \geq k$.

It remains to prove the statement for only finitely many positive integer less than k. Let t be the largest positive integer such that $f(t) \neq Ct$. Take $v \in I$. Setting $x = vt,\ y = t$, then $\frac{x}{y} \in I$. Hence,

$$f((v+1)t) = f(vt) + f(t).$$

If $v > 1$, then $(v+1)t > vt > t$. Hence, by the choice of t, we have

$$f((v+1)t) = C(v+1)t, \quad f(vt) = Cvt,$$

contradicting the choice of t. If $v = 1$, then $(v+1)t = 2t > t$. Hence, $f(2t) = 2Ct = 2f(t)$, again a contradiction. ∎

Remark. If we remove the condition that I contains at least one integer, then there are many other solutions. Consider for example, $i = (\frac{3}{2}, \frac{5}{3})$. Define $f(1) = f(2) = f(3) = f(4) = 2010$, $f(n) = n$ for all $n \geq 5$. You can see that it works properly!

Here is a more general case from the Chinese TST.

Problem 3. (Chinese TST 2005) Let α be positive real number. Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all positive integers k, msuch that $\alpha m = k < (\alpha+1)m$ we have

$$f(k+m) = f(k) + f(m).$$

Solution. First of all we prove following lemma.

Lemma. For n sufficiently large there exist a real number u such that

$$\alpha(n+1) = u < (\alpha+1)(n+1) \text{ and } \alpha n = u+1 < (\alpha+1)n$$

Proof. It suffices to prove that

$$\alpha n < \alpha(n+1) = u < u+1 < (\alpha+1)n < (\alpha+1)n+1$$

This is obvious for n large enough: $\alpha n = u < u+1 < (\alpha+1)(n-1)$. Indeed for this u, we have

$$(n-1)\alpha < \alpha(n+1) = u < u+1 < (\alpha+1)(n-1)$$

Now for the existence of such u, we must have $(\alpha+1)(n-1) - \alpha(n+1) = 2$, i.e. $n = 2\alpha+3$. Now, taking n larger than this quantity we are done. □

Now we have the following equation, which leads to the solution of the problem:

$$f(n-1) + f(u+1) = f(n+u) = f(n) + f(u).$$

Thus, $f(n) - f(n-1) = f(u+1) - f(u)$. We also have

$$f(n+1) + f(u) = f(n+u+1) = f(u+1) + f(n).$$

That is, $f(n+1) - f(n) = f(u+1) - f(u)$. Indeed for all $n \geq \lfloor 2\alpha+3 \rfloor$ the quantity $f(n+1) - f(n)$ remains constant (i.e., $f(n+1) - f(n) = f(n) - f(n-1)$).
Now set $n_0 = 1 + \lfloor 2\alpha+3 \rfloor$ (it is obvious that $n_0 = 3$). Thus for all $n = n_0 - 1$ we have

$$f(n+1) - f(n) = f(n_0) - f(n_0 - 1).$$

Then we can easily find that

$$f(n) = (n - n_0 + 1)(f(n_0) - f(n_0 - 1)) + f(n_0 - 1).$$

Set k, m such that $m = n_0$, $\alpha m = n_0$ and $\alpha m = k < (\alpha + 1)\, m$ which leads to $k = n_0$. Then,

$$
\begin{aligned}
& f(k+m) \\
= \;& f(k) + f(m) \\
= \;& (k + m - n_0 + 1)\,(f(n_0) - f(n_0 - 1)) + f(n_0 - 1) \\
= \;& (k - n_0 + 1)\,(f(n_0) - f(n_0 - 1)) + f(n_0 - 1) + (m - n_0 + 1)\,(f(n_0) - f(n_0 - 1)) + f(n_0 - 1)\,.
\end{aligned}
$$

Comparing the equalities we get that $(n_0 - 1)\, f(n_0) = n_0 f(n_0 - 1)$. Thus, we can see that $f(n_0) = an_0$ for some real number a. Then, for all $n = n_0 - 1$ we have $f(n) = an$. Now assume that there exists a positive integer n_1 such that $f(n_1) \neq an_1$. Now, assume that $\alpha > 1$. Then, there exists a positive integer k such that $\alpha n_1 = k < (\alpha + 1)n_1$. Then, we have $k > n_1$, $k + n_1 > n_1$. This implies that $f(k + n_1) = f(k) + f(n_1)$ or $f(n_1) = f(k + n_1) - f(k)$. Now, if we define n_1 as the minimal number for which the inequality holds, the problem is solved.
Now, if $\alpha = 1$, then $\alpha n_1 = n_1 < n_1(\alpha + 1)$. Then, $f(2n_1) = f(n_1) + f(n_1)$. Now, if we define n_1 as the maximal value for which the inequality holds, then $f(2n_1) = 2an_1$ and we are done. ■

We finish this section with a nice combination of a trigonometric problem with additive function.

Problem 4. Find all functions $f : [0, \infty) \to \mathbb{R}$ that are monotone on $[1, 2]$ and such that for all non-negative real numbers r and $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$, then

$$f(r \cos\theta) + f(r \sin\theta) = f(r).$$

Solution. Set $r = 0$. Then $f(0) = 0$. Set $\alpha = \frac{\pi}{4}$. Then $f(r) = 2f(\frac{r}{\sqrt{2}})$. Replacing r with $\frac{r}{\sqrt{2}}$, we find that $f\left(\frac{r}{\sqrt{2}}\right) = 2f(\frac{r}{2})$. Then, $f(r) = 4f(\frac{r}{4})$. For all integers k we have $f\left(r \cdot 2^k\right) = 2^{2k} f(r)$. Thus, we can find that $f(x)$ is monotone on $\left[2^k, 2^{k+1}\right]$ and indeed on the whole set of positive real numbers. Now, define $g : [0, \infty) \to \mathbb{R}$ by $g(x) = f(\sqrt{x})$. Then, $g(x)$ is also monotone on the set of positive real numbers. Therefore,

$$g(r^2 \cos^2\theta) + g(r^2 \sin^2\theta) = g(r^2) \quad .$$

Then, we can say that for $1 \leq \frac{u}{v} \leq 3$ we have $(u + v) = g(u) + g(v)$. By induction on n we find that $g(nx) = ng(x)$ for all $n \geq 1$, $x > 0$.

Indeed assume that the statement holds for all $n \leq k - 1$. For $n = k$, if $k = 2m$, then

$$g(kx) = g(mx) + g(mx) = 2mg(x) = kg(x).$$

If $k = 1 + 2m$, set $u = (m + 1)\, x$, $v = mx$ and $1 < \frac{u}{v} = \frac{m+1}{m} < 3$. We get

$$g(kx) = g((m + 1)\, x) + g(mx) = (m + 1)\, g(x) + mg(x) = (2m + 1)\, g(x) = kg(x).$$

Then, for all $r \in \mathbb{Q}^+$ we have $g(r) = cr$. Without loss of generality assume that $c \geq 0$. Then, for all positive real numbers x there are sequences of rational numbers x_n, y_n such that $0 < x_n < x < y_n$ and x_n, y_n approach to x as $n \to \infty$. Then, $cx_n = g(x_n) \leq g(x) \leq g(y_n) = cy_n$. Hence, $g(x) = cx$ and $f(x) = cx^2$. ■

3 Some Additional Constraints

In this section we try to explore the topic better. We first start with imposing some constraints on an additive function, that is we deal with the case of a function which is additive on and inside of a right triangle with vertices $(0, 1)$, $(1, 0)$ and the origin $(0, 0)$.
Problem 1. Let $f : [0, 1] \to \mathbb{R}$ such that for all $x, y,\ x + y \in [0, 1]$,

$$f(x + y) = f(x) + f(y)\,.$$

Does there exist an additive function $h : \mathbb{R} \to \mathbb{R}$ such that for all $x \in [0, 1]$,

$$f(x) = h(x)?$$

Is that function unique?

Solution. Define $g : [-1,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & x \in [0,1], \\ -f(-x) & x \in [-1,0]. \end{cases}$$

First, note that $g(-x) = -g(x)$ and $g(0) = 0$. We shall show that for all $x, y, x+y \in [-1,1]$ we have $g(x+y) = g(x) + g(y)$. Obviously, if $x, y > 0$, $x + y \in [0,1]$ we are done. Now, assume that $x, y < 0$, $-1 < x + y < 0$. Then,

$$g(x+y) = -f(-x-y) = -f(-x) - f(-y) = g(x) + g(y).$$

Finally, let $x < 0$, $y > 0$ we then have two different cases: $x+y > 0$ and $x+y < 0$. Assume the latter. Then,

$$g(x) = g(x+y-y) = g(x+y) + g(-y) = g(x+y) - g(y).$$

Now, consider the former. Then,

$$g(y) = g(x+y-x) = g(x+y) + g(-x) = g(x+y) - g(x).$$

The cases $x > 0, y < 0$, $x + y \in [-1,1]$ can be treated similarly. Now, we define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = ng\left(\frac{x}{n}\right)$, where n is a positive integer such that $\frac{x}{n} \in [-1,1]$. Note that if for example $\frac{x}{m} \in [-1,1]$, then $n\left(\frac{x}{mn}\right)$, $m(\frac{x}{mn}) \in [-1,1]$ and if t, $nt \in [-1,1]$. Then, $g(nt) = ng(t)$. Now,

$$mg\left(\frac{x}{m}\right) = mng\left(\frac{x}{mn}\right) = ng\left(\frac{x}{n}\right).$$

Take $n = 1$. We have that $h(x) = g(x)$ for all $x \in [-1,1]$. Finally, for arbitrary x, y, choose n so that $\frac{x}{n}, \frac{y}{n}, \frac{x+y}{n} \in [-1,1]$. Thus,

$$h(x+y) = ng\left(\frac{x+y}{n}\right) = n\left(g\left(\frac{x}{n}\right) + g\left(\frac{y}{n}\right)\right) = h(x) + h(y).$$

Furthermore, the function h is unique. Indeed, if there is another function, namely h_1, then, for arbitrary real numbers x and a positive integer n so that $\frac{x}{n} \in [-1,1]$, it follows that

$$h_1(x) = nh_1\left(\frac{x}{n}\right) = ng\left(\frac{x}{n}\right) = h(x).$$

■

With the next problem we impose a more tough constraint by removing the closure of the aforementioned domain.

Problem 2. Let $f : (0,1) \to \mathbb{R}$ such that for all $x, y, x+y \in (0,1)$

$$f(x+y) = f(x) + f(y).$$

Then, there is a function $g : [0,1] \to \mathbb{R}$ such that for all $x, y, x+y \in [0,1]$,

$$g(x+y) = g(x) + g(y)$$

and for all $t \in (0,1)$,

$$f(t) = g(t).$$

Solution. First note that for all $t, kt \in (0,1)$ where k is a positive integer, we have $f(kt) = kf(t)$. That is, for all integers $m, n > 1$, we have $nf\left(\frac{1}{n}\right) = mf\left(\frac{1}{m}\right)$. Indeed,

$$nf\left(\frac{1}{n}\right) = mnf\left(\frac{1}{mn}\right) = mf\left(\frac{1}{m}\right).$$

Define $g(x)$ on $[0,1]$ by

$$g(x)=\begin{cases} f(x) & x\in(0,1)\\ 0 & x=0\\ nf(\frac{1}{n}) & x=1.\end{cases}$$

We claim that $g(x+y)=g(x)+g(y)$ for each $x,y,x+y\in[0,1]$. This is true for all $x,y,x+y\in(0,1)$. Now, if $x=0$, $y\in(0,1)$, then $x+y\in(0,1)$. Hence,

$$g(x+y)=g(y)=g(0)+g(y)=g(x)+g(y).$$

For $x=0$, $y=1$, we have $g(x+y)=g(1)=g(1)+g(0)=g(x)+g(y)$.

Finally, if $x,y\in(0,1)$ and $x+y=1$, there is a positive integer n such that $\frac{x}{n},\frac{y}{n},\frac{x}{n}+\frac{y}{n}\in(0,1)$. Therefore, from $f\left(\frac{x}{n}+\frac{y}{n}\right)=f\left(\frac{x}{n}\right)+f(\frac{y}{n})$, we find that

$$nf\left(\frac{x}{n}+\frac{y}{n}\right)=nf\left(\frac{x}{n}\right)+nf\left(\frac{y}{n}\right).$$

In addition, $g(x+y)=g(1)=nf\left(\frac{1}{n}\right)=nf\left(\frac{x}{n}\right)+nf\left(\frac{y}{n}\right)=g(x)+g(y)$. ■

The next problem is a recombination of the aforementioned approaches. We also implement another approach that is equivalent to the first approach we provided here. This problem amongst other things can help the reader to implement what he has already learned.

Problem 3. Let $f:(-1,1)\to\mathbb{R}$ be a function such that

$$f(x+y)\left(f(x+y)-f(x)-f(y)\right)=0$$

for all $x,y,x+y\in(-1,1)$. Prove that

$$f(x+y)=f(x)+f(y).$$

Solution. At first we show that for each positive integer n,

$$f\left(\frac{x}{2^n}\right)=\frac{1}{2^n}f(x)$$

for all $x\in(-1,1)$. If $f(x)=0$, then it is easy to see that if $2^nx\in(-1,1)$, then $f(2^nx)=0$. This implies the desired relation. Assume now that $f(x)\neq 0$. Then, plugging $(x,y)=(\frac{x}{2},\frac{x}{2})$, we find that

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x)\neq 0.$$

Using the above fact repeatedly, we find that $f\left(\frac{x}{2^n}\right)=\frac{1}{2^n}f(x)$ for all $x\in(-1,1)$.
Now, for every real number t there is a non-negative integer n such that $x=\frac{t}{2^n}\in(-1,1)$. Now, define $g(t)=2^nf\left(\frac{t}{2^n}\right)$, $\frac{t}{2^n}\in(-1,\ 1)$. We can find that for each $x\in(-1,1)$, $f(x)=g(x)$. Furthermore, let x,y be real numbers. Again, there is a non-negative integer n such that $\frac{x}{2^n},\frac{y}{2^n},\frac{x+y}{2^n}\in(-1,\ 1)$. Therefore,

$$f\left(\frac{x+y}{2^n}\right)\left(f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)-f\left(\frac{x+y}{2^n}\right)\right)=0.$$

Therefore, $2^nf\left(\frac{x+y}{2^n}\right)\left(2^nf\left(\frac{x}{2^n}\right)+2^nf\left(\frac{y}{2^n}\right)-2^nf\left(\frac{x+y}{2^n}\right)\right)=0$. That is,

$$f(x+y)\left(f(x)+f(y)-f(x+y)\right)=0.$$

Hence, we can prove that the function $g(x)$ is indeed unique. That is, assume that there is another function, namely $h(x)$, such that $h(x)=f(x)$, $x\in(-1,1)$. Again, we can find that $h\left(\frac{x}{2^n}\right)=\frac{1}{2^n}h(x)$ for all x. Therefore, $h(x)=2^nh\left(\frac{x}{2^n}\right)=2^nf\left(\frac{x}{2^n}\right)=g(x)$. ■

Remark 3.1. In the general case, when the function $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$f(x+y)\left(f(x+y)-f(x)-f(y)\right)=0$$

if $f(x+y) \neq 0$, then $f(x+y) = f(x) + f(y)$. Define

$$S = \{x \in \mathbb{R} : f(x) = 0\}.$$

Moreover, define $S' = \mathbb{R} \setminus S$. Then, it is easy to see that $0 \in S$ and if $x, y \in S$, then $x+y \in S$. Now, suppose that there is a real number x such that $x \in S$ but $-x \in S'$. That is, $f(-x) \neq 0$. Hence,

$$f(-x) = f(-2x + x) = f(-2) + f(x) = f(-2x).$$

Hence, $f(-2x) \neq 0$. On the other hand,

$$f(-2x) = f(-x) + f(-x) = 2f(-x),$$

contradiction. Hence, if $x \in S$, then $-x \in S$. Furthermore, we can find that if $x \in S'$, then $-x \in S'$ and if $x \in S$ and $y \in S'$, then $x+y \in S'$. That is, if $x+y \in S$, then $y = (-x) + (x+y) \in S$, which is indeed a contradiction. Now, we can find that if $x \in S'$, then $2^n x \in S'$. Furthermore, $f(2x) = 2f(x)$, $f(-x) = -f(x)$. For the proof of the latter, assume that $x \in S'$. Then,

$$f(x) = f(2x - x) = f(2x) + f(-x) = 2f(x) + f(-x).$$

Finally, if $f(y) \neq 0$, then
$f(y) = f(-x+x+y) = f(-x) + f(x+y) = f(x+y) - f(x)$. The same is true for the case $f(x) = 0$.

Now, if $f(x) = f(y) = 0$, then $f(x+y) = 0 = f(x) + f(y)$.

4 Additive Functions and Domains

In this section we will try to provide a self-contained toolbox that will help the reader to go further. At the very beginning, we introduce the reader with some basic notation like domains, open sets, and connectedness.

Definition 4.1. A set E is **Open** if and only if for each $x \in E$ there is a neigborhood N such that $x \in N \subseteq E$. An open nen-emty set A is called **Connected** if it cannot be represented as the union of two disjoint non-empty open subsets. A set is **path-connected** if any two points can be connected with a path without exiting the set. One can prove that a set is connected if and only if it is path connected.

Definition 4.2. Any open connected set is called a *domain*.

The aim of our article is to prove the following fact:
Let the function f be additive on any open connected set D. Then, there is a unique additive function that is equal to this function.

The following lemma is a very important one that will open new windows for the future and shall bring a perfectly coherent view for studying additive functions with imposed conditions.

Lemma 4.3. Let f be an additive function on $H_r = \{(x,y) : x, y \geq 0, x+y < r\}$ or $C_r = \left\{(x,y), x^2+y^2 < r^2\right\}$. Then, there is a unique additive function that is equal to f on H_r or C_r.

Proof. The former case was considered before. Here we provide an alternative approach. Let us write $x = n\frac{r}{2} + z$, where n is an integer and $0 \leq z < \frac{r}{2}$. Now, define $g(x) = n\left(\frac{r}{2}\right) + f(z)$. Now, if $x = n\frac{r}{2} + z$, $y = m\frac{r}{2} + t$ and if $0 \leq z + t < \frac{r}{2}$, then

$$g(x+y) = g\left((n+m)\frac{r}{2} + z + t\right) = (m+n) f\left(\frac{r}{2}\right) + f(z+t) = g(z) + g(t).$$

Now, if $\frac{r}{2} \leq z + t < r$, write $z + t = \frac{r}{2} + v$ $0 \leq v < \frac{r}{2}$. Hence,

$$g(x+y) = g\left((n+m+1)\frac{r}{2} + v\right) = (n+m+1) f\left(\frac{r}{2}\right) + f(v) = nf\left(\frac{r}{2}\right) + mf\left(\frac{r}{2}\right) + f\left(\frac{r}{2}\right) + f(v).$$

Since $v+\frac{r}{2}<r$, then $f\left(\frac{r}{2}\right)+f\left(v\right)=f\left(v+\frac{r}{2}\right)=f\left(z+t\right)=f\left(z\right)+f(t)$. Therefore,

$$g\left(x+y\right)=nf\left(\frac{r}{2}\right)+mf\left(\frac{r}{2}\right)+f\left(z\right)+f\left(t\right)=g\left(y\right)+g\left(x\right).$$

Furthermore, this function is indeed unique. Assume on the contrary that there is another function, namely $h(x)$ with the above properties. Then, for each real number t there is an integer n such that $t=nx$, $0\leq x<\frac{r}{2}$. Since $h\left(x\right)=f(x)$ for all $0\leq x<\frac{r}{2}$, it follows that

$$g\left(t\right)=g\left(nx\right)=ng\left(x\right)=nf\left(x\right)=nh\left(x\right)=h\left(nx\right)=h\left(t\right).$$

Assume the latter. Since H_r is a subset of C_r and C_r is a subset of $H_{r\sqrt{2}}$, then, for non-negative x we have $x\in[0,r)$ and the function $g(x)$ defined above satisfies the statement of the problem. Define $D_f=\left(-r\sqrt{2},\ r\sqrt{2}\right)$. Hence, if $x\in[0,r)$, then the function g as preceding part satisfies the condition. Now, for $x\in\left(-r\sqrt{2},0\right)\cup[r,r\sqrt{2})$. It suffices to show that f and g are equal. Assume now that $x\in[r,r\sqrt{2})$. Then $\frac{x}{2}\in\left[\frac{r}{2},\frac{r\sqrt{2}}{2}\right)\subseteq[0,r)$. Thus, $(\frac{x}{2},\frac{x}{2})$ belongs to C_r, that is,

$$g\left(x\right)=g\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right)=f\left(\frac{x}{2}\right)+f\left(\frac{x}{2}\right)=f\left(\frac{x}{2}+\frac{x}{2}\right)=f\left(x\right).$$

Now, for $x\in\left(-r\sqrt{2},\ 0\right)$, we find that

$$g\left(x\right)=-g\left(-x\right)=-f\left(-x\right)=f\left(x\right).$$

□

The next definition would help us to deal with some other regions with even non-symmetric shapes.

Definition 4.4. Let D be a set of points (x,y) in the plane, we define D_x, D_y, D_{x+y} as follows

$$\begin{aligned}D_x&=\{x:\exists y\ni(x,y)\in D\},\\D_y&=\{y:\exists x\ni(x,y)\in D\},\\D_{x+y}&=\{x+y:(x,y)\in D\},\end{aligned}$$

We then say that the function $f(x)$ is additive on the set D if $f:D_f=D_x\cup D_y\cup D_{x+y}\to Im(f)$ and for all (x,y) in D, $f(x+y)=f(x)+f(y)$.

Definition 4.5. We define the *quasi extension* $g\left(x\right):\mathbb{R}\to\mathbb{R}$ of the function $f(x)$, which is additive on D as follows. Consider a point (u,v) on D. Then for all real numbers x,y such that $f(x)$ or $f(y)$ is meaningful, the following statements hold

(i) For each $x\in D_{x+y}:\ f\left(x\right)-f\left(u\right)-f\left(v\right)=g\left(x\right)-g\left(u\right)-g\left(v\right),$

(ii) For each $x\in D_x:\ f\left(x\right)-f\left(u\right)=g\left(x\right)-g\left(u\right),$

(iii) For each $y\in D_y:\ f\left(y\right)-f\left(v\right)=g\left(y\right)-g\left(v\right).$

Note that if $x=a+b\in D_{x+y}$, then $a\in D_x$ and $b\in D_y$.

It is easy to see that if we write the above equations for $a,b,a+b$ and subtract the corresponding equations, we can find similar equations where (u,v) is replaced by (a,b). Thus, if $(0,0)\in D$, the concept of quasi extension coincides with the concept of extension.

Lemma 4.6. If we replace C_r with $C_r\left(a,b\right)=\left\{(x,y),(x-a)^2+(y-b)^2<r^2\right\}$, then the quasi extension of f is unique.

Proof. Plugging $(x+a, y+b)$ instead of (x, y), then

$$f(x+y+a+b) = f(x+a) + f(y+b)$$

for all (x,y)in C_r. Now, setting $y=0$, we find that for $x \in (C_r)_x$, $f(x+a) = f(x+a+b) - f(b)$. Setting $x=0$, we find that for $y \in (C_r)_y$, $f(y+b) = f(y+a+b) - f(a)$. Hence, the function $g(x) = f(x+a+b) - f(a) - f(b)$ is an additive function. Hence, $g(x)$ is additive on C_r. Then there is only one additive extension of $g(x)$, namely $G(x)$. It can be shown that $G(x)$ is equal to the quasi extension of f. □

Next lemma is the central lemma of this article. Considering this lemma, one has a little risk of error and can remove a brake that has been already put on the extension of the statements on additive functions.

Lemma 4.7. Let $D = D_1 \bigcup D_2$ where D_1, D_2 are open sets with non-empty intersection. Assume that there is a quasi extension $G_i (i=1,2)$ of f on $D_i\,(i=1,2)$. Then $G_1 = G_2$ and hence f has only one quasi extension on D.

Proof. It is easy to find that $D_1 \cap D_2$ is an open non-empty set. Consider now $D_x^1 \cap D_x^2$ which contains open interval I. Since G_1, G_2 are quasi extensions of f on D_1, D_2, consider a real number a for which there is a real number b such that $(a,b) \in D_1 \cap D_2$. It follows that for each $x \in D_x^1$,

$$f(x) - f(a) = G_1(x) - G_1(a) = G_1(x-a).$$

Analogously, for each $x \in D_x^2$,

$$f(x) - f(a) = G_2(x) - G_2(a) = G_2(x-a).$$

Hence, it follows that for each $x \in I$, $G_1(x-a) = G_2(x-a)$. Now, assume that t is an arbitrary real number. Then, there is an integer n such that $\frac{t}{n}$ or nt lies inside the interval $I-a$. Assume the former. Then

$$G_1(t) = nG_1\left(\frac{t}{n}\right) = nG_2\left(\frac{t}{n}\right) = G_2(t).$$

Therefore, since G_1, G_2 are additive, we find that $G_1 = G_2$ □

5 Main Results

We hope to score enough genuine points that our reader will soon prepare to follow. This lemma helps to shed light on some aspects of additive functions that have hitherto been hidden from view. This result first appeared in [2].

Theorem 1. If $f(x)$ is an additive function on a domain D, then there is only one quasi extension of f.

Proof. Since every domain D might be represented as the union of open circular regions, that is, $D = \bigcup_{i=1}^{\infty} C_i$ such that for each i, $(\bigcup_{i=1}^{i} C_i) \cap C_{i+1} \neq \emptyset$, then for each C_i there is only one quasi-extension G_i. Hence, by the above fact, all the G_is are equal. □

Strategy: By now, whenever we faced a problem that is additive we can go through following guidelines:

(i) Proving that the function is additive on a domain.

(ii) Examining whether that set is an open connected or not.

(iii) Use the above theorem and define the quasi extension of the function.

(iv) Using some auxiliary facts about the function under study to deduce the quasi extension function is indeed of the form Cx.

6 Applications

In this section we shall use all we have learned to solve some relatively hard problems that have been recently proposed for mathematical competitions. Based on the detailed account of additive functions on domains, the reader can easily solve the problems.

Problem 1. Let $f:(0,1)\to\ \mathbb{R}$ be a function such that

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f\left(x\right)+\frac{1}{2}f\left(y\right).$$

for all real numbers x,y. Prove that there is a unique additive function $h:\mathbb{R}\to\ \mathbb{R}$ such that $f\left(x\right)=h\left(x-a\right)+b$ for some constants a,b.

Solution. Let us define $g\left(x\right)=f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)$, $g:(-\frac{1}{2},\frac{1}{2})\to\ \mathbb{R}$. We prove that $g(x)$ is an additive function on the set $D_1=\left\{(x,y):x,y\in\left(-\frac{1}{2},\frac{1}{2}\right),\ -x-\frac{1}{2}<y<-x+\frac{1}{2}\right\}$. Putting $(x+\frac{1}{2},y+\frac{1}{2})$ instead of (x,y), then for $x,y\in(-\frac{1}{2},\frac{1}{2})$ we find that

$$2f\left(\frac{x+y}{2}+\frac{1}{2}\right)=f\left(x+\frac{1}{2}\right)+f\left(y+\frac{1}{2}\right).$$

Setting $y=0$, we find that

$$2f\left(\frac{x}{2}+\frac{1}{2}\right)=f\left(x+\frac{1}{2}\right)+f\left(\frac{1}{2}\right).$$

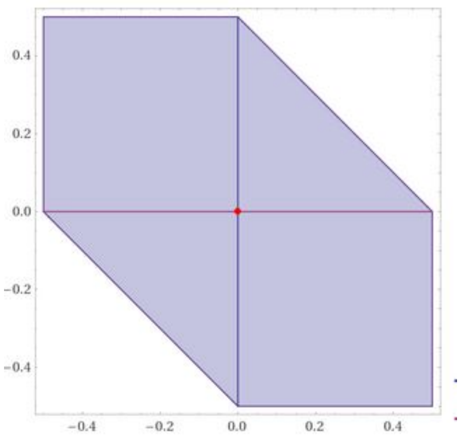
Hence, substituting it into the above equation, we get

$$f\left(x+y+\frac{1}{2}\right)+f\left(\frac{1}{2}\right)=f\left(x+\frac{1}{2}\right)+f\left(y+\frac{1}{2}\right).$$

Let us define$g\left(x\right)=f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)$, $g:\left(-\frac{1}{2},\frac{1}{2}\right)\to\ \mathbb{R}$,. We find that

$$g\left(x+y\right)=g\left(x\right)+g\left(y\right).$$

on the following hexagon, which can be regarded as set D, i.e., an open connected set.



Thus, by theorem 1, in the section 5, there is only one quasi extension of $g\left(x\right)$, namely $h\left(x\right)$. Hence, since $(0,0)$ is on the region, we find that $h\left(x\right)=g\left(x\right)$. ∎

Remark 6.1. If the range of the function $f(x)$ contains a bounded (above or below) interval, then $h\left(x\right)=cx$ and hence $f\left(x\right)=c\left(x-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)$.
The bearings of the theorem of the section 5 can be seen as an elegant solution of following problem from 2010 German TST.

Problem 2. (German TST 2010) Find all increasing functions $f:(0,1)\to\ \mathbb{R}$ such that

$$f(x+y-xy)+f\left(xy\right)=f\left(x\right)+f\left(y\right).$$

Solution. We define a function $g(x) = f\left(x+\frac{1}{2}\right) - f\left(\frac{1}{2}\right)$, $g : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$ for all (s,t) that belong to the following region

$$D = \left\{(s,t) : -\frac{1}{2} < t < 0,\ \frac{1}{2} + \frac{1}{2t-1} < s < \frac{1}{2} + t\right\}.$$

Define

$$F(x,y) = f(x) + f(y) - f(xy) = f(x+y-xy).$$

It is easy to check that for all $x, y, z \in (0,1)$ we have

$$F(xy,z) + F(x,y) = F(x,yz) + F(y,z) = f(x) + f(y) + f(z) - f(xyz).$$

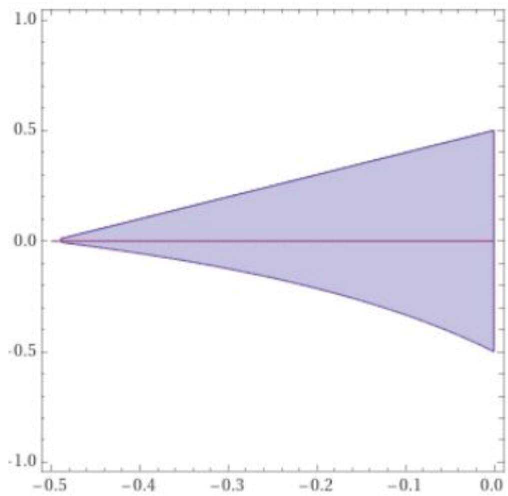
That is, $f(xy+z-xyz) + f(x+y-xy) = f(x+yz-xyz) + f(y+z-yz)$. Now, take the substitutions $t + \frac{1}{2} = xy + z - xyz$, $s + \frac{1}{2} = x + y - xy$, $\frac{1}{2} = y + z - yz$. We find that

$$f\left(t+\frac{1}{2}\right) + f\left(s+\frac{1}{2}\right) = f\left(s+t+\frac{1}{2}\right) + f\left(\frac{1}{2}\right).$$

Note that $z = \frac{\frac{1}{2}-y}{1-y}$, $0 < y < \frac{1}{2}$, $t = -\frac{y(1-x)}{2(1-y)}$, $s = x + y - xy - \frac{1}{2}$, $0 < x < 1$. At the boundaries, i. e, $x = 0, 1$, $y = 0, \frac{1}{2}$, we get the inequalities on s, t. The function $g(x) = f\left(x+\frac{1}{2}\right) - f\left(\frac{1}{2}\right)$, $g : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$ satisfies the equation

$$g(s) + g(t) = g(s+t),$$

on the following region, it is clear that points (s,t) of D formed an open connected set.



Thus, by the theorem 1, there is only one quasi extension of $g(x)$, namely $h(x)$. Hence, since $(0,0)$ is on the region, we find that $h(x) = g(x)$. Hence, h is strictly increasing on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and additive, yielding to the fact that $h(x) = Cx$. Therefore, $f(x) = f\left(\frac{1}{2}\right) + C\left(x - \frac{1}{2}\right)$. Hence, g is strictly increasing on $(0,1)$ and additive. Hence, $g(x) = Cx$. ■

The following marvelous problem can also be solved by this technique.

Problem 3. (Titu Andreescu and Nikolai Nikolov, USAMO 2018) Find all functions $f : (0,+\infty) \to (0,+\infty)$ such that for all positive real numbers x, y, z satisfying $xyz = 1$,

$$f\left(x+\frac{1}{y}\right) + f\left(y+\frac{1}{z}\right) + f\left(z+\frac{1}{x}\right) = 1.$$

Solution. It is easy to find that $f(x) \in (0,1)$. Setting $x = \frac{b}{c}$, $y = \frac{c}{a}$, $z = \frac{a}{b}$, $a, b, c > 0$. It follows that

$$f\left(\frac{b+c}{a}\right) + f\left(\frac{c+a}{b}\right) + f\left(\frac{a+b}{c}\right) = 1.$$

Plugging $g(x) = f(\frac{1}{x} - 1)$, we have $f(t) = g\left(\frac{1}{1+t}\right)$. Therefore,

$$g\left(\frac{a}{a+b+c}\right)+g\left(\frac{b}{a+b+c}\right)+g\left(\frac{c}{a+b+c}\right)=1.$$

Defining a function $g:(0,1)\to(0,1)$, then for all a,b,c such that $a+b+c=1$, then $g(a)+g(b)+g(c)=1$. Since $a+b<1$, then $g(a)+g(b)=1-g(1-a-b)=2g\left(\frac{a+b}{2}\right)$. Hence, based on problem 1 in the section 6 and remark 6.1, we obtain $g(x)=k\left(x-\frac{1}{2}\right)+g\left(\frac{1}{2}\right)$, where $-\frac{k}{2}+3g\left(\frac{1}{2}\right)=1$. That is,

$$g(x)=kx+\frac{1-k}{3},$$

where $-\frac{1}{2}\leq k\leq 1$. Hence, $f(x)=\frac{k}{1+x}+\frac{1-k}{3}$, $-\frac{1}{2}\leq k\leq 1$. ∎

By the last problem, we will use the same technique for some domains that are not open-connected but, based on the knowledge you gained, we count on your innovative power to produce a new proof.

Problem 4. (Korean Olympiad, 2020) Find all functions $f:\mathbb{Q}^+\to\mathbb{R}$ such that for all positive rational numbers x,y,z satisfying $x+y+z+1=4xyz$ the following equation holds:

$$f(x)+f(y)+f(z)=1.$$

Solution. Rewrite the equation $x+y+z+1=4xyz$ as $\frac{1}{1+2x}+\frac{1}{1+2y}+\frac{1}{1+2z}=1$. Defining $g(x)=f(\frac{1}{1+2x})$, then $g:\mathbb{Q}^+\bigcap(0,1)\to\mathbb{R}$ is such that $g(a)+g(b)+g(c)=1$ whenever $a,b,c\in\mathbb{Q}^+\bigcap(0,1)$ and $a+b+c=1$. Since $a+b<1$, then

$$g(a)+g(b)=1-g(1-a-b)=2g\left(\frac{a+b}{2}\right).$$

Let us define $h(x)=g\left(x+\frac{1}{2}\right)-g\left(\frac{1}{2}\right)$, $h:\mathbb{Q}\bigcap(-\frac{1}{2},\frac{1}{2})\to\mathbb{R}$. We prove that $h(x)$ is an additive function on the set $D_1=\mathbb{Q}\bigcap\left\{(x,y):x,y\in\left(-\frac{1}{2},\frac{1}{2}\right),\ -x-\frac{1}{2}<y<-x+\frac{1}{2}\right\}$.
Define $H:\mathbb{Q}^+\to\mathbb{R}$, $H(t)=2^nh\left(\frac{t}{2^n}\right)$, $\frac{t}{2^n}\in(-\frac{1}{2},\frac{1}{2})$. Then, we can find that for each $x\in\left(-\frac{1}{2},\frac{1}{2}\right)$, $h(x)=H(x)$. Let x,y be arbitrary positive rational numbers. Again, there is a non-negative integer n such that $\frac{x}{2^n},\frac{y}{2^n},\frac{x+y}{2^n}\in\left(-\frac{1}{2},\frac{1}{2}\right)$. Thus,

$$H(x+y)=2^nh\left(\frac{x+y}{2^n}\right)=2^nh\left(\frac{x}{2^n}\right)+2^nh\left(\frac{y}{2^n}\right)=H(x)+H(y).$$

Hence, $H(x)$ is an additive function and hence $H(x)=kx$. Hence, $h(x)=kx$ and therefore,

$$g(x)=k\left(x-\frac{1}{2}\right)+g\left(\frac{1}{2}\right).$$

Again, $g(x)=kx+\frac{1-k}{3}$, where $-\frac{1}{2}\leq k\leq 1$ and therefore,

$$f(x)=g\left(\frac{1-x}{2x}\right)=k\left(\frac{1-x}{2x}\right)+\frac{1-k}{3}$$

and $-\frac{1}{2}\leq k\leq 1$. ∎

7 Concluding Remarks

With this article we presented a new method on solving some challenging problems about additive functions. The linchpin of our argument was trying to find either an extension or a quasi extension function that is additive and is identical with our function under study at a certain domain. Then, based on the auxiliary criterion of that function, we try to characterize our (quasi) extension function and solve our functional equation.

Based on the main approaches followed so far, we tried to help the reader to understand one crucial point: sometimes extending the domain of the problem under scrutiny can help us to uncover some hidden facts.

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