Cubic Polynomials Revisited

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1 Abstract

Polynomials over the real field and their roots have been a vital area of Mathematical research and interest right from the early days of the subject. In this paper, we will shed light on the roots of a cubic polynomial $P(x)$, location of roots of the derivative of $P(x)$ and the relatively less known but beautiful Marden's Theorem (or the Siebeck Marden Theorem), which has been referred to as the *most marvellous theorem in mathematics* and some of its special cases and generalizations.

2 Keywords

Cubic Polynomials, Steiner Inellipse, Roots, Foci,Gauss Lucas Theorem, Triangle, Convex Hull, Affine equivalent, Isogonal Conjugates, Stagnation Point.

...as long as algebra and geometry have been studied separately, their progress have been slow and their uses limited; but when this two sciences have been united, they have lent mutual forces, and have marched together towards perfection.

Joseph Louis Lagrange

Polynomials and their derivatives have been the central area of studies in various branches of Mathematics. There are a number of well known results concerning the location of the roots of polynomials, namely the Rolle's Theorem, the Mean Value Theorem, Cauchy's Mean Value Theorem and so on. Here, we would mainly deal with the geometry of the roots of the derivative of a cubic polynomial. At the beautiful crossroad of algebra and geometry lies this enigmatic result, the Marden's Theorem. Named after Morris Marden, this result was proved about a century ago by Jorg Siebeck. Thus, it belongs to the list of things in mathematics, which are misattributed(Stigler's Law of Eponymy!). The theorem has a special place in mathematics, and it won Dan Kalman the 2009 Lestor R. Ford award of MAA(Mathematical Association of America). Several other mathematicians have also worked in this problem. We will state the theorem, and prove it using another famous theorem, the Gauss Lucas Theorem. Then we will discuss some generalizations and similar results. In the process, we will touch upon numerous concepts from algebra and geometry.

3 Some Preliminaries

Before jumping right into the formal proof, we would at first revise some important concepts and theorems.

- **Fundamental Theorem of Algebra -** One of the most well known and widely used theorem in Mathematics, right from High School level to Graduate studies. It states that a non-constant polynomial with complex coefficients having degree n cannot have more than n roots. This will help us to guarantee that the cubic **will** have 3 roots and the derivative *will* have 2 roots.
- **Convex Hull -** In geometry, a convex hull, or a convex closure of a set of points is the smallest convex set that contains all the points. In other words, it is the set of all *convex combination* of the points of a given shape.
- **Gauss Lucas Theorem**. This is an extremely vital theorem. The set of roots of a complex polynomial are a set of points in the complex plane. According to this theorem, all the roots of the derivative of the polynomial lie *within* the convex hull of the roots of the original polynomial. This theorem will be used by us to guarantee the fact that all the roots of the derivative of the cubic polynomial will lie inside the triangle formed by the roots of the cubic polynomial.

- **Steiner Inellipse**. Consider a $\triangle ABC$ let the midpoints of the sides BC, AC and AB be D, E and F respectively. The Steiner in ellipse, or the midpoint inellipse is the **unique** ellipse which is tangent to the sides at the midpoints.(Link to the proof is given in the Reference Section). It is the *largest* with respect to area among all possible inscribed ellipses of a triangle.

4 Statement of Theorem

Consider a cubic polynomial P(x), with three distinct, non-collinear zeroes in the complex plane. Let them be z_1, z_2 and z_3. By the given non collinearity condition, they will form a triangle. Then the roots of the derivative of the polynomial, $P'(x)$ will precisely be the **foci** of the ellipse, which is tangent to the 3 sides of the triangle formed by z_1, z_2 and z_3.

5 Proof

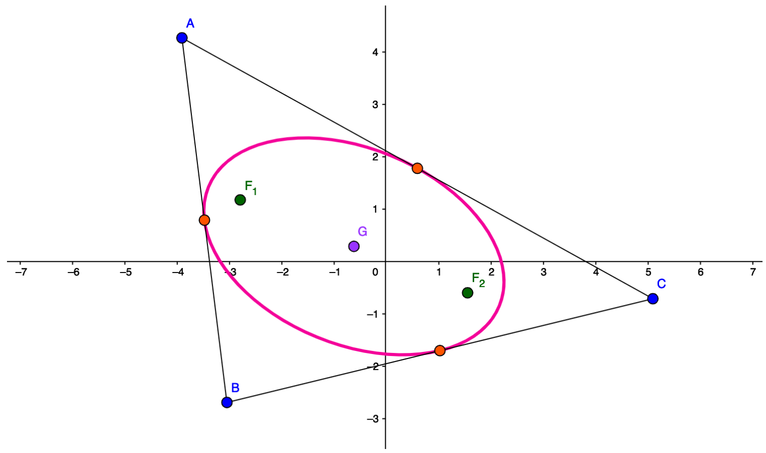

Figure 1: 9 Degree Polynomial Fit of the points

Any change of coordinates $z \mapsto z'$ can be described by the rotation $z \mapsto \epsilon z$ (with $\epsilon = 1$) followed by the translation $z = z + e$. In the new coordinates which we have used, the polynomial becomes $P(z') = P(e + \epsilon z)$ Let, $P(z') = Q(x)$. Now, as

$$\frac{dQ}{dz} = \epsilon \frac{dP}{dz'},$$

the roots of the first derivates of p and Q are **identical**. So, WLOG (without loss of generality), we can work with a coordinate system of our choice. Given an arbitrary triangle with Steiner Inellipse of semi axes lengths m and n, choose the origin O to be the centre of the Inellipse and the two new axes, O_x and O_y along the axes of the ellipse. Let F_1 and F_2 be the foci. Observe that they will have coordinates $(\pm\sqrt{m^2 - n^2}, 0)$. Let, $c = \sqrt{m^2 - n^2}$. Represent the roots of the polynomial equations, i.e the vertices of the triangle by $z_k = x_k + iy_k$ where $i = \sqrt{-1}$. Taking a linear transformation which stretches O_y by the factor $\frac{m}{n}$ will transform the ellipse into a **circle** and the vertices will be mapped to new points $\gamma_k = \alpha_k + i\beta_k$, such that

$$x_k = \alpha_k \quad ; \quad y_k = \left(\frac{m}{n}\right)\beta_k$$

Since the midpoints of the sides of the original triangle are mapped into the midpoints of the sides of the transformed triangle, which has an inscribed circle in place of the ellipse, the new triangle is **equilateral**. Hence,

$$\sum_{k=1}^{3} (\alpha_k + i\beta_k) = 0.$$

Hence, for the original triangle,

$$P(z) = z^3 + pz + q$$

and, γ_k must be the roots of $\gamma^3 = C$. So,

$$\sum \zeta_k \zeta_{k+1} = 0, \quad \sum (\zeta_k)^2 = 0$$

The last condition is enough to infer that $\alpha_k^2 = \beta_k^2$ and $\sum \alpha_k \beta_k = 0$. m, which was earlier the length of a semi major axis of an ellipse is now the **radius** of the incircle of the equilateral triangle. So

$$4m^2 = (2m)^2 = \alpha^2 + \beta^2$$

From this, we can conclude that

$$(\alpha_k)^2 = (\beta_k)^2 = \frac{1}{2} \cdot 3 \cdot 4a^2 = 6a^2$$

What follows is that $\sum (x_i)^2 = \frac{m^2}{n^2}$ and $\sum y_i^2 = 6a^2$. Finally, by the good old **Vieta's Theorem**, we have

$$p = \sum z_i z_{i+1}$$

$$v = 2 \sum z_i z_{i+1} = (\sum z_i)^2 - \sum (z_i)^2$$

As it is evident from the polynomial that the sum of roots do vanish, their squares will also vanish, so

$$(\sum z_i)^2 - \sum (z_i)^2 = -\sum (z_i)^2 = -\sum (x_i + iy_i)^2 = -\sum (x_i)^2 + \sum (y_i)^2 - 2 \sum x_i y_i$$

This is $6(n^2 - m^2) = -6c^2$. Thus, the derivative of the polynomial, $P'(z)$ is zero at $3z^2 = p = 3c^2$ i.e for $z = \pm c$. Although we broadly discuss only one proof over here, there are 2 more, and I would like to mention about them briefly about their idea.

1. First we have the work done by **Kalman** and **Bocher**. Both of their proofs of this theorem is based on the following result.

F_A is an exterior point to an ellipse E and X_1, X_2 are the two focal lines of the ellipse, then the angles made by the two tangents from A to E with AX_1 and, respectively, AX_2 are identical.

That is, if $A.B$ and C are the roots of the polynomial, then F_1 and F_2 are **Isogonal Conjugates** (Isogonal conjugates of a point with respect to a triangle are constructed by reflecting it about the angle bisector). Though the two authors follow different routes to arrive at this result, the rest of their work progresses more or less in the same way.

2. The proof by **D.Minda** and **S.Phelps** uses the algebraic properties of $\mathbb{C}$ more than the techniques discussed above.

6 Some Special Cases

If it happens that the triangle formed by the roots becomes equilateral, then the inscribed inellipse gets degenerated into a **circle**. Clearly, in that case, the derivative will have a double at the centre of the circle. In other words, the foci do not remain distinct! It is more interesting to note that the converse is true as well. That is, if the derivative has a double root, then the triangle formed by the roots must be nothing but equilateral! As a small conclusion(which basically follows directly from Gauss Lucas Theorem), I would like to mention that the roots of the double derivative of the polynomial i.e the roots of $P(x)$ would be average of the two focii , i.e the centre of the Steiner Inellipse, and the centroid of the original triangle. Another trivial but interesting conclusion would be that : if f_1 and f_2 are the roots of the derivative of the polynomial $P(x)$ and a, b and c are the roots of $P(x)$ itself, then

$$\frac{a+b+c}{3} = \frac{f_1 + f_2}{2}$$

7 Generalizations

To conclude this work, I would mention 2 deep generalizations of the theorem:
• Linfield, in 1920, developed a more general version of the theorem. It is applicable to polynomials of the form

$$f(x) = (x-u)^i(x-v)^j(x-w)^k$$

Here, (i + j + k) is allowed to have values exceeding 3, but the polynomial must have only 3 real roots, u, v, w. The theorem states that the roots of $f'(x)$ may be found at the multiple roots of the given polynomial (i.e, the roots whose exponent is greater than one) and at the foci of an ellipse whose points of tangency to the triangle divide its sides in the ratios $i:j, j:k$ and $k:i$. More specifically, his theorem can be used to find the roots of the derivative which are not the roots of the original polynomial.
• James Parish, in 2006, extended the study of geometric characterization of the critical points of a polynomial. What he proved follows:
Let $n \geq 3$, and let A be an n-gon with vertices $v_0, \ldots, v_n$, no three of which are *collinear*. The following statements are equivalent:

- There exists an ellipse tangent to the sides of G at their **midpoints**.
- A is **affine-equivalent** to A(ζ) for some primitive nth root of unity ζ .
- There is a primitive nth root of unity ζ and 3 complex constants a, b, c such that $\mid b \mid \neq \mid c \mid$ and for $k = 0, \cdots n-1$,

  $$v_k \ = \ a + b\zeta^k + c\zeta^{-k}$$

  where A(ζ) is the n-gon whose vertices are $1, \zeta, \cdots, \zeta^{n-1}$.
  • In the paper of Beniamin Bogosel, we come across another beautiful genralized result involving logarithms, which is worthy of being discussed. If c,d and e are the roots of a cubic polynomial, then consider a **new** function, namely :

  $$L(t) = \alpha \ln(t-c) + \beta \ln(t-d) + \gamma \ln(t-e),$$

  where α, β and γ are **positive** real numbers which add upto 1, i.e $\alpha + \beta + \gamma = 1$. THen, $L(t)$ will have 2 points where its derivatives vanish, i.e. the critical points. Let them be a_1 and a_2. Then, these 2 points are the **foci** of the inellipse that divides the the sides of the triangle formed by c,d and e into ratios $\beta : \gamma$, $\gamma : \alpha$ and $\alpha : \beta$, respectively. In fact, the **converse** is also true! For any ellipse inscribed in triangle cde, there exists a function of the form $L(t)$(as defined before) having its two critical points as the foci of the aforementioned inellipse.

8 References and Further Reading

Interested readers are encouraged to go through the following materials:

- https://proofwiki.org/wiki/Gauss-Lucas-Theorem
- Wikipedia article on Steiner Inellipse
- An Elementary Proof of Marden's Theorem, Dan Kalman
- http://forumgeom.fau.edu/FG2006volume6/FG200633.pdf
- Some propositions concerning the geometric representation of imaginaries, Maxime Bocher, Annals of Mathematics, 7(1-5):70-72, 1892/93
- Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Clarendon Press, Oxford, 2002
- Wikipedia Article on Isogonal Conjugates
- A Geometric Proof of the Siebeck Marden Theorem,Beniamin Bogosel, American Mathematical Monthly, Vol 124 No.5