J553. Solve in real numbers the equation

\[
\left( x^2 - 2\sqrt{2}x \right) \left( x^2 - 2 \right) = 2021.
\]

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Let \( x = (1 + y)/\sqrt{2} \), then the equation becomes

\[
0 = y^4 - 10y^2 - 8075 = (y^2 + 85)(y^2 - 95).
\]

Therefore, \( y = \pm \sqrt{95} \), so \( x = (1 \pm \sqrt{95})/\sqrt{2} \).

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Alex Bloom, Sega Math Academy; Nick Iliopoulos, AUTH, Greece; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Ashley Simone, SUNY Brockport; Daniel Văcaru, Pitești, Romania; Fred Frederickson, Utah Valley University, USA; Henry Ricardo, Westchester Area Math Circle, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Max Sachelarie, PA, USA; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Kanav Talwar, Delhi Public School, Faridabad, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA; David E. Manes, Oneonta, NY, USA; Titu Zvonaru, Comănești, Romania.
J554. Let $x, y, z$ be positive real numbers such that $x + y + z = xyz$. Prove that

$$
\frac{1}{1 + x^2} + \frac{1}{1 + y^2} + \frac{1}{1 + z^2} \geq \frac{1}{1 + xy} + \frac{1}{1 + yz} + \frac{1}{1 + zx}.
$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the given condition we have

$$
1 + x^2 = \frac{xyz}{x + y + z} + x^2 = \frac{xyz + x^2(x + y + z)}{x + y + z} = \frac{x(x + y)(x + z)}{xyz} = \frac{(x + y)(x + z)}{yz},
$$

and

$$
1 + yz = \frac{xyz}{x + y + z} + yz = \frac{xyz + yz(x + y + z)}{x + y + z} = \frac{yz(2x + y + z)}{xyz} = \frac{2x + y + z}{x}.
$$

Therefore the desired inequality may be rewritten as

$$
\sum_{cyc} \frac{yz}{(x + y)(x + z)} \geq \sum_{cyc} \frac{x}{2x + y + z}.
$$

This is equivalent to

$$
\sum_{cyc} \frac{yz(y + z)}{(x + y)(y + z)(z + x)} \geq \sum_{cyc} \frac{x}{2x + y + z},
$$

or

$$
\frac{(x + y)(y + z)(z + x) - 2xyz}{(x + y)(y + z)(z + x)} \geq \sum_{cyc} \frac{x}{2x + y + z},
$$

or

$$
1 \geq \frac{2xyz}{(x + y)(y + z)(z + x)} + \sum_{cyc} \frac{x}{2x + y + z}.
$$

This follows from the following two inequalities

$$
\frac{2xyz}{(x + y)(y + z)(z + x)} \leq \frac{1}{4}
$$

and

$$
\sum_{cyc} \frac{x}{2x + y + z} \leq \frac{3}{4}.
$$

The first inequality is true because

$$
\frac{2xyz}{(x + y)(y + z)(z + x)} \leq \frac{2xyz}{2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx}} = \frac{1}{4}.
$$

The second inequality is true because

$$
\sum_{cyc} \frac{x}{2x + y + z} \leq \sum_{cyc} \frac{x}{4} \left( \frac{1}{x + y} + \frac{1}{x + z} \right) = \frac{3}{4}.
$$

The proof is completed.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Alex Bloom, Sega Math Academy; Adnan Ali, NIT Silchar, Assam, India; Arighna Pan, Nabadwip Vidyasagar College, India; Corneliu Manescu-Avron, Ploiești, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Max Sachelarie, PA, USA; Nick Iliopoulos, AUTH, Greece; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Titu Zvonaru, Comănești, Romania.
J555. Let \( a, b, c \) be real numbers such that

\[
\frac{1}{a + 2 + \sqrt{a^2 + 8}} + \frac{1}{b + 2 + \sqrt{b^2 + 8}} + \frac{1}{c + 2 + \sqrt{c^2 + 8}} \leq \frac{1}{2}.
\]

Prove that \( a + b + c \geq 3 \). When does the equality occur?

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

First solution by the author

Let \( x = a + \sqrt{a^2 + 8} > 0 \), implying \( a = \frac{x}{2} - \frac{4}{x} \). Similarly, \( b = \frac{y}{2} - \frac{4}{y} \) and \( c = \frac{z}{2} - \frac{4}{z} \). Then

\[
\frac{1}{2} - \frac{1}{x + 2} \geq \frac{1}{y + 2} + \frac{1}{z + 2} \geq \frac{4}{y + 2 + z + 2},
\]

so \( y + z + 4 \geq 8 + \frac{16}{x} \). Summing up with the two analogues inequalities yields

\[
4 \left( \frac{x}{2} - \frac{4}{x} \right) + 4 \left( \frac{y}{2} - \frac{4}{y} \right) + 4 \left( \frac{z}{2} - \frac{4}{z} \right) \geq 12.
\]

Hence the conclusion.
Second solution by Marius Stanean, Zalau, Romania

Because \( a + 2 + \sqrt{a^2 + 8} > 2 \), let \( a + 2 + \sqrt{a^2 + 8} = 4x + 2 \), \( b + 2 + \sqrt{b^2 + 8} = 4y + 2 \), \( c + 2 + \sqrt{c^2 + 8} = 4z + 2 \), where \( x, y, z > 0 \). Therefore
\[
a = 2x - \frac{1}{x}, \quad b = 2y - \frac{1}{y}, \quad c = 2z - \frac{1}{z}.
\]
The given condition becomes
\[
\frac{1}{2x+1} + \frac{1}{2y+1} + \frac{1}{2z+1} \leq 1,
\]
or
\[
x + y + z + 1 \leq 4xyz.
\]
But \( 27xyz \leq (x + y + z)^3 \), so \( 27(x + y + z + 1) \leq 4(x + y + z)^3 \) which means \( x + y + z \geq 3 \). We need to show that
\[
2(x + y + z) - \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 3.
\]
We have
\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{xy + yz + zx}{xyz} \leq \frac{4(x + y + z)^2}{3(x + y + z + 1)},
\]
so it remains to prove that
\[
2(x + y + z) - \frac{4(x + y + z)^2}{3(x + y + z + 1)} \geq 3,
\]
or if we denote \( t = x + y + z \geq 3 \),
\[
\frac{(t - 3)(2t + 3)}{3(t + 1)} \geq 0,
\]
true.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Adnan Ali, NIT Silchar, Assam, India.
J556. Let $ABC$ be a triangle with circumcircle $\Gamma$ and circumcenter $O$. The tangents in $B$ and $C$ to $\Gamma$ intersect in $D$ and $AD$ intersects $\Gamma$ in $E$. The parallel through $A$ to $BC$ intersects $\Gamma$ in $F$. Prove that $EF$ bisects side $BC$.

Proposed by Mihaela Berindeanu, Bucharest, România

First solution by the author

If $BD$ and $CD$ are tangents lines to $\Gamma \Rightarrow AD$ is the symmedian from $A$ in $\triangle ABC$.

We will build the mediator of the $BC$ segment, named $d$, $d \cap BC = \{M\}$, $M \in BC$, $BM \equiv MC$.

Notations: $AM \cap \Gamma = \{J\}$, $d \cap \Gamma = \{I\}$, $FM \cap \Gamma = \{K\}$.

I will prove that $AK$ is the symmedian from $A$ in $\triangle ABC$, so $K \in AD \Rightarrow E \equiv K$.

$BC \parallel AF \Rightarrow BCAF$ is an isosceles trapezoid. Line $d$ divides the $BC$ into two equal parts $\Rightarrow BI = IC$.

d is mediator for $BC$ and $AF$ also, $\Rightarrow d$ is $BCAF$ symmetry axis, so $\triangle ABC = \triangle FBC \Rightarrow BI = IC \Rightarrow AI$ is $\angle BAC$ bisector.

Due to the symmetry, $\angle BFM \equiv \angle CAM$ or $\angle JAC \equiv \angle KFB \Rightarrow \triangle ABC$ is isosceles.

$\begin{align*}
\angle KB = \angle JC \\
\angle IB = \angle IC \\
\Rightarrow KI = IJ \text{ and } KC = BJ \Rightarrow \angle BAJ \equiv \angle KAC.
\end{align*}$

So, $AM$ and $AK$ are isogonal lines

$\begin{align*}
AM, AK = \text{ isogonal lines} \\
AM = \text{ the median from } A \text{ in } \triangle ABC \Rightarrow AK = \text{ symmedian } \Rightarrow E \equiv K.
\end{align*}$

Conclusion: $EF$ passes through the middle of the side $BC$
Let $M$ be the midpoint of $BC$. Since $AF \parallel CB$, $AFBC$ is an isosceles trapezoid, so $\angle BMF = \angle CMA$. Suppose that $AD$ intersects $BC$ at $P$. Since $AD$ is the $A$-symmedian of $\triangle ABC$, we know that $P, D$ divide $A, E$ harmonically. Also, $DM \perp MP$. By the right-angle-and-bisector lemma (see E. Chen, Euclidean Geometry in Math. Olympiads, MAA, 2016, p. 177), $PM$ bisects $\angle AME$. Therefore, $\angle EMP = \angle FMB$, that is, $E, M, F$ are collinear.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Kanav Talwar, Delhi Public School, Faridabad, India; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploiești, Romania; Max Sachelarie, PA, USA; Nick Iliopoulos, AUTH, Greece; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.
Let $ABCD$ be a parallelogram with $AB \neq AD$ and $\angle BAD > 90^\circ$. We denote by $M, N, P$ the orthogonal projections of $A$ on $BC, CD, BD$, respectively, and let $O$ be the intersection of diagonals $AC$ and $BD$. Prove that points $M, N, O, P$ lie on a circle.

Proposed by Mihai Miculița, Oradea, România

Solution by Polyahedra, Polk State College, USA

Notice that $ANCM$ is cyclic with center $O$, and $ADNP$ is also cyclic. Thus, $AO = MO$, so

$$\angle NMO = 90^\circ - \angle OMA - \angle CMN = 90^\circ - \angle MAO - \angle CAN = \angle BAM = \angle NAD = \angle NPD,$$

completing the proof.

Also solved by Kamran Mehdiyev, Baku, Azerbaijan; Theo Kouvelis, Broward College, Davie, FL, USA; Kanav Talwar, Delhi Public School, Faridabad, India; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploiești, Romania; Max Sachelarie, PA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Titu Zvonaru, Comănești, Romania.
J558. Let $ABCDE$ be a convex pentagon with $BC = CD$, $DE = EA$ and $\angle BCD + \angle DEA = 180^\circ$. Knowing that $\angle BCD = \alpha$ and $EC = a$, determine the area of the pentagon.

Proposed by Waldemar Pompe, Warsaw, Poland

First solution by the author

Rotate triangle $EDC$ around point $C$ and angle $\alpha$. Then $D$ is mapped to $B$ and $E$ is taken to some point $F$. Since $\angle BCD + \angle DEA = 180^\circ$, it follows that segments $AE$ and $BF$ are parallel. They are also of equal length. Therefore the midpoints of segments $AB$ and $EF$ coincide; let’s label this common midpoint by $M$. Then triangles $AEM$ and $BFM$ are congruent. Hence the area of pentagon $ABCDE$ is equal to the area of triangle $ECF$, which is $\frac{1}{2}a^2\sin \alpha$. 
Second solution by Polyahedra, Polk State College, USA

Let $F$ be the reflection of $B$ about $C$. We have $\triangle DFC \sim \triangle DAE$ with spiral center $D$. Let $G$ be the intersection of $AF$ and $CE$, then $G$ is on the circumcircles $(DFC)$ and $(DAE)$ as well. Suppose the line through $A$ and parallel to $BC$ intersects $CD$ at $H$. Then $\angle AHD = \angle DCF = \angle AED$, so $H$ is on the circumcircle $(DAE)$.

Therefore, $\angle AHE = \angle ADE = \angle CFD$, thus $EH \parallel DF$. Consequently, $[DFE] = [DFH]$, so $[CFED] = [CFH] = [BCA]$. Hence, $[ABCDE] = [ACFE] = \frac{1}{2}AF \cdot EC \sin \angle AGE$. Since $AF/EC = DF/DC = 2 \cos \frac{\alpha}{2}$ and $\angle AGE = \frac{\alpha}{2}$, we obtain the answer $\frac{1}{2}a^2 \sin \alpha$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Max Sachelarie, PA, USA.
S553. Solve in real numbers the equation
\[(x^3 - 3x)^2 + (x^2 - 2)^2 = 4.\]

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*First solution by the author*

Clearly, $x = 0$ is a double root and so the equation can have at most four nonzero solutions. We will first seek nonzero solutions in the interval $(-2, 2]$. To this end, let $x = 2 \cos t$, where $t \in [0, \pi] \setminus \{\pi/2\}$.

Because
\[x^3 - 3x = 2(4 \cos^3 t - 3 \cos t) = 2 \cos 3t\]
and
\[x^2 - 2 = 2(2 \cos^2 t - 1) = 2 \cos 2t\]
the equation becomes $4 \cos^2 3t + 4 \cos^2 2t = 4$ which can be rewritten as
\[2(1 + \cos 6t + 1 + \cos 4t) = 4.\]

This reduces to $\cos 6t + \cos 4t = 0$, that is $2 \cos 5t \cos t = 0$, yielding $t = \pi/10, 3\pi/10, 7\pi/10, 9\pi/10$. We get four more distinct solutions, and thus we have found all the solutions. In conclusion, the solutions are $x = 0, 2 \cos \frac{\pi}{10}, 2 \cos \frac{3\pi}{10}, 2 \cos \frac{7\pi}{10}, 2 \cos \frac{9\pi}{10}$.
Second solution by Henry Ricardo, Westchester Area Math Circle

Expanding the given equation, we get

\[ x^6 - 5x^4 + 5x^2 = x^2(x^4 - 5x^2 + 5) = 0. \]

Thus \( x = 0 \) is a solution of multiplicity 2. Then, setting \( X = x^2 \), we find that the solutions of \( X^2 - 5X + 5 = 0 \) are \( X = \left(5 \pm \sqrt{5}\right)/2 \), so that

\[ x = \pm \sqrt{X} = \pm \frac{\sqrt{5} \pm \sqrt{5}}{\sqrt{2}} = \pm \frac{\sqrt{10} \pm 2\sqrt{5}}{2}. \]

Therefore, the solutions of our original equation are

\[ x = 0, 0, \frac{\sqrt{10} + 2\sqrt{5}}{2}, -\frac{\sqrt{10} + 2\sqrt{5}}{2}, \frac{\sqrt{10} - 2\sqrt{5}}{2}, -\frac{\sqrt{10} - 2\sqrt{5}}{2}. \]

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Alex Bloom, Sega Math Academy; Kanav Talwar, Delhi Public School, Faridabad, India; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ewripides P. Nastou, 6th High School, Nea Smyrni, Greece; Daniel Văcaru, Pitești, Romania; Ashley Simone, SUNY Brockport, USA; Fred Frederickson, Utah Valley University, USA; G. C. Greubel, Newport News, VA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Max Sachlerarie, PA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nick Iliopoulos, AUTH, Greece; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Le Hoang Bao, Tien Giang, Vietnam; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.
S554. Let $a, b, c, x, y, z$ be positive real numbers such that

$$\frac{x}{b^2 + c^2 + x} + \frac{y}{c^2 + a^2 + y} + \frac{z}{a^2 + b^2 + z} \geq 1.$$ 

Prove that

$$ab + bc + ca \leq x + y + z.$$ 

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the given condition and using Cauchy-Schwarz inequality we have

$$1 \leq \sum_{\text{cyc}} \frac{x}{b^2 + c^2 + x} = \sum_{\text{cyc}} \frac{x(2 + \frac{a^2}{x})}{(b^2 + c^2 + x)(1 + 1 + \frac{a^2}{x})} \leq \sum_{\text{cyc}} \frac{2x + a^2}{(b + c + a)^2} = \frac{2(x + y + z) + a^2 + b^2 + c^2}{(a + b + c)^2}.$$ 

It follows that

$$(a + b + c)^2 \leq 2(x + y + z) + a^2 + b^2 + c^2$$

which is equivalent to

$$ab + bc + ca \leq x + y + z$$

as desired.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Arkady Alt, San Jose, CA, USA.
S555. Let $ABC$ be a scalene triangle. We construct externally to $\triangle ABC$ the isosceles triangles $XAB$, $YAC$ and $ZBC$ such that: $\angle AXB = \angle AYC = 90^\circ$ and $\angle ZBC = \angle ZCB = \angle BAC$. Knowing that $BY$, $CX$ and $AZ$ are concurrent, find $\angle BAC$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by the author

Notations: $R = \triangle ABC$ circumradius, $CX \cap AB = \{X'\}$, $BY \cap AC = \{Y'\}$, $AZ \cap BC = \{Z'\}$.

From $\angle (ZBC) = \angle (ZCB) = \angle (BAC)$ results that $BZ$ and $CZ$ are tangents to $ABC$ circumcircle.

So, $AZ$ is symmedian in $\triangle ABC \Rightarrow \frac{BZ'}{Z'C} = \frac{AB^2}{AC^2}$.

\[
\frac{Y'A}{Y'C} = \frac{\sigma (AYY')}{\sigma (CYY')} = \frac{AY \cdot YY' \cdot \sin (AYB) \cdot \frac{1}{2}}{YC \cdot YY' \cdot \sin (CYB) \cdot \frac{1}{2}} \text{ with } \frac{AY}{YC} = \frac{Y'A}{Y'C} \Rightarrow \frac{Y'A}{Y'C} = \frac{\sin (AYB)}{\sin (BYC)}
\]

According to Sine Rule:

\[
\begin{align*}
\triangle ABY : \quad \frac{\sin (AYB)}{AB} &= \frac{\sin (A + 45^\circ)}{AC} \\
\triangle BCY : \quad \frac{\sin (BYC)}{BC} &= \frac{\sin (B + 45^\circ)}{BY} \quad \Rightarrow \quad \frac{\sin (AYB)}{\sin (BYC)} = \frac{AB \cdot \sin (A + 45^\circ)}{BC \cdot \sin (C + 45^\circ)}
\end{align*}
\]

\[
\frac{AX'}{X'B} = \frac{AC \cdot \sin (A + 45^\circ)}{BC \cdot \sin (B + 45^\circ)}
\]
\[ AZ \cap BY \cap CX = \{ W \} \Rightarrow \text{according to Ceva’s theorem} \]

\[
\frac{BY'}{Y'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AX'}{X'B} = 1 \Rightarrow \frac{AB^2}{AC^2} \cdot \frac{BC \cdot \sin (C + 45^\circ)}{AB \cdot \sin (A + 45^\circ)} \cdot \frac{AC \cdot \sin (A + 45^\circ)}{BC \cdot \sin (B + 45^\circ)} = 1 \Rightarrow
\]

\[
AB \cdot \sin (C + 45^\circ) = AC \cdot \sin (B + 45^\circ) \quad (1)
\]

\[
AB = 2R \cdot \sin C \quad (2)
\]

\[
AC = 2R \cdot \sin B \quad (3)
\]

(1) with (2) and (3) \(2R \sin C \cdot \sin (C + 45^\circ) = 2R \sin B \cdot \sin (B + 45^\circ)\) \Rightarrow

\[
\sin C \sin (C + 45^\circ) = \sin B \sin (B + 45^\circ) \Leftrightarrow \cos 45^\circ - \cos (2C + 45^\circ) = \cos 45^\circ - \cos (2B + 45^\circ) \Rightarrow
\]

\[
\cos (2C + 45^\circ) = \cos (2B + 45^\circ)
\]

\[ABC\] is non-isosceles triangle \(\Rightarrow 2 \angle B + 45^\circ + 2 \angle C + 45^\circ = 360^\circ \Rightarrow \angle B + \angle C = 135^\circ\)

So,

\[
\angle A = 180^\circ - 135^\circ = 45^\circ
\]

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Spyros Kallias, Volos, Greece; Kanav Talwar, Delhi Public School, Faridabad, India; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam; Titu Zvonaru, Comănești, Romania; Adnan Ali, NIT Silchar, Assam, India; Max Sachelarie, PA, USA.
S556. Let $a, b, c$ be positive real numbers such that $a + b \leq 3c$. Find the maximum possible value of 
\[
\left( \frac{a}{6b + c} + \frac{a}{b + 6c} \right) \left( \frac{b}{6c + a} + \frac{b}{c + 6a} \right).
\]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

The maximum is $\frac{49}{400}$.

We have 
\[
\frac{a}{6b + c} + \frac{a}{b + 6c} = \frac{7a(b + c)}{6b^2 + 37bc + 6c^2}
\]
and 
\[
6b^2 + 37bc + 6c^2 = 6(b + c)^2 + 25bc \geq 10(b + c)\sqrt{6bc},
\]
implying
\[
\frac{a}{6b + c} + \frac{a}{b + 6c} \leq \frac{7a}{10\sqrt{6bc}}.
\]
Similarly,
\[
\frac{b}{6c + a} + \frac{b}{c + 6a} \leq \frac{7b}{10\sqrt{6ca}}.
\]
Hence
\[
\left( \frac{a}{6b + c} + \frac{a}{b + 6c} \right) \left( \frac{b}{6c + a} + \frac{b}{c + 6a} \right) \leq \left( \frac{49\sqrt{ab}}{100(6c)} \right)^2 \leq \frac{49}{400}.
\]
as $2\sqrt{ab} \leq a + b \leq 3c$.

For the equality case we must have $a = b$, $6(b+c)^2 = 25bc$, and $6(c+a)^2 = 25ca$, implying $(3b-2c)(2b-3c) = 0$ and $(3c-2a)(2c-3a) = 0$, which, together with the condition $a + b \leq 3c$, yields $a = b = 3c/2$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.
S557. Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{8a^2}{b + c + 2} + \frac{8b^2}{c + a + 2} + \frac{8c^2}{a + b + 2} + 33 \geq 13(a + b + c).$$

*Proposed by Marius Stănean, Zalău, România*

**Solution by the author**

The inequality can be rewrite as follows:

$$\sum_{cyc} \left( \frac{4a^2}{b + c + 2} - \frac{b - c}{2} \right) \geq \frac{9}{2} (a + b + c) - \frac{33}{2},$$

$$\sum_{cyc} \frac{4a^2 - (b + c)^2}{b + c + 2} - \sum_{cyc} \frac{2(b + c)}{b + c + 2} \geq \frac{9}{2} (a + b + c) - \frac{33}{2}.$$

By the AM-GM Inequality, we have

$$2(b + c) \leq \frac{(b + c + 2)^2}{4},$$

so it suffices to prove that

$$\sum_{cyc} \frac{4a^2 - (b + c)^2}{b + c + 2} - \sum_{cyc} \frac{b + c + 2}{4} \geq \frac{9}{2} (a + b + c) - \frac{33}{2},$$

that is

$$\sum_{cyc} \frac{4a^2 - (b + c)^2}{b + c + 2} \geq 5(a + b + c - 3),$$

$$\sum_{cyc} \frac{(a - b + a - c)(2a + b + c)}{b + c + 2} \geq 5(a + b + c - 3),$$

$$\sum_{cyc} (a - b) \left( \frac{2a + b + c}{b + c + 2} - \frac{2b + c + a}{c + a + 2} \right) \geq 5(a + b + c - 3),$$

$$\sum_{cyc} \frac{2(a + b + c + 1)(a - b)^2}{(b + c + 2)(c + a + 2)} \geq 5 \cdot \frac{(a + b + c)^2 - 9}{a + b + c + 3},$$

$$\sum_{cyc} \frac{(a - b)^2}{(b + c + 2)(c + a + 2)} \geq \frac{5}{4} \cdot \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{(a + b + c + 1)(a + b + c + 3)},$$

$$\sum_{cyc} \left( \frac{1}{(b + c + 2)(c + a + 2)} - \frac{5}{4(a + b + c + 1)(a + b + c + 3)} \right) (a - b)^2 \geq 0,$$

$$\sum_{cyc} S_c (a - b)^2 \geq 0.$$
Without loss of generality, we may assume that \( a \geq b \geq c \). Certainly, \( S_c \geq S_b \geq S_a \). Let’s study the sign of \( S_b \). Since

\[
\begin{align*}
4(a + b + c + 1)(a + b + c + 3) - 5(a + b + 2)(b + c + 2) \\
= 4(a^2 + c^2) - b^2 + 3(ab + bc + ca) + 6(a + c) - 4b - 8 \\
= a^2 - b^2 + 3a^2 + 4c^2 + 4(a - b) + 2a + 6c + 1 \geq 0,
\end{align*}
\]

we deduce that \( S_b \geq 0 \). Therefore

\[
\sum_{cyc} S_a(b - c)^2 = S_a(b - c)^2 + S_b(a - b + c)^2 + S_c(a - b)^2 \\
\geq S_a(b - c)^2 + S_b(a - b)^2 + S_b(b - c)^2 + S_c(a - b)^2 \\
= (S_a + S_b)(b - c)^2 + (S_b + S_c)(a - b)^2.
\]

Hence, it remains to prove that \( S_a + S_b \geq 0 \), i.e.

\[
\frac{1}{a + b + 2} \left( \frac{1}{c + a + 2} + \frac{1}{b + c + 2} \right) \geq \frac{5}{2(a + b + c + 1)(a + b + c + 3)}.
\]

By the Cauchy-Schwarz Inequality, we have

\[
\frac{1}{a + b + 2} \left( \frac{1}{c + a + 2} + \frac{1}{b + c + 2} \right) \geq \frac{4}{(a + b + 2)(a + b + 2c + 4)} \geq \frac{5}{2(a + b + c + 1)(a + b + c + 3)}.
\]

The last inequalities are equivalent to

\[
(a + b + c + 2)(a + b + c + 4) \geq (a + b + 2)(a + b + 2c + 4) \iff c(c + 2) \geq 0,
\]

and

\[
8(a + b + c + 1)(a + b + c + 3) \geq 5(a + b + c + 2)(a + b + c + 4) \iff (a + b + c - 2)(3a + 3b + 3c + 8) \geq 0
\]

which is clearly true because \( a + b + c \geq \sqrt{3(ab + bc + ca)} = 3 \).

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.
S558. Let $ABC$ be a scalene triangle. Let $A_1$ be the symmetric of $A$ with respect to $N$ — the center of the nine-point circle of $\triangle ABC$. If $A_1$ lies on the circumcircle of $\triangle ABC$, then calculate $\angle BAC$. 

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution by the author

Let $O, I, H, G, M$ be the center of circumcircle, center of incircle, orthocenter, centroid of $\triangle ABC$ and midpoint of $BC$ respectively.

It is well known that $N$ lies on the Euler Line $HGO$ and $HN = NO; HG = 2GO; AG = 2GM$.

$AN = NA_1, HN = NO$ hence $AHA_1O$ is a parallelogram. It follows that $AH \parallel OA_1$. But $AH \perp BC$ so $OA_1 \perp BC$, or $A_1$ lies on the perpendicular bisector $OM$ of $BC$. Let $OM$ intersect the circumcircle $\Gamma$ of $\triangle ABC$ at points $D \in BC$ and $E \in BAC$ (see Figure 1). The point $A_1 \in \Gamma$ hence either $A_1 \equiv D$ or $A_1 \equiv E$.

$N$ is the midpoint of the chord $AA_1$, hence $ON$ is the perpendicular bisector of $AA_1$. It follows that $\triangle AHN \cong \triangle AON \cong \triangle A_1ON \cong \triangle A_1HN$, $AH = AO = A_1O = A_1H$.

$\triangle AHG \sim \triangle MOG$ so $AH : OM = AG : GM = 2 : 1$.

$$OM = \frac{1}{2}OA_1 = MA_1; \ OA_1 \perp BC \ \text{so} \ A_1 \ \text{is the symmetric point of} \ O \ \text{with respect to} \ M. \ \text{It follows that} \ BA_1 = BO = OA_1 = OC = CA_1 = R, \ i.e. \ \text{the triangles} \ BOA_1 \ \text{and} \ COA_1 \ \text{are equilateral triangles and} \ \\
\angle BOA_1 = \angle A_1OC = 60^\circ; \ \angle BOC = \angle BA_1C = 120^\circ.$$

Case 1. $A_1 \equiv D$

Figura 1: $A_1 \equiv D$

The points $A_1, N$ lie on the angle bisector $AI$.

$$\angle BAC = \frac{1}{2} \angle BOC = 60^\circ \ (4)$$
Case 2. $A_1 \equiv E$

The points $E, N$ lie on the external angle bisector $I_CAI_B$ of $\angle BAC$. The quadrilateral $BAEC$ is a cyclic quadrilateral.

\[
\angle BAC = \angle BEC = \angle BA_1C = 120^\circ \tag{5}
\]

In conclusion, from (1) and (2) : $\angle BAC = 60^\circ$ or $\angle BAC = 120^\circ$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Kamran Mehdiyev, Baku, Azerbaijan; Spyros Kallias, Volos, Greece; Kanav Talwar, Delhi Public School, Faridabad, India; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam; Adnan Ali, NIT Silchar, Assam, India; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Max Sachelarie, PA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Titu Zvonaru, Comănești, Romania.
Undergraduate problems

U553. Let $A$ be an $n \times n$ matrix such that $A^4 = I_n$. Prove that $A^2 + (A + I_n)^2$ and $A^2 + (A - I_n)^2$ are invertible.

Proposed by Adrian Andreescu, University of Texas at Dallas

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
First,

$$A^2 + (A + I_n)^2 = 2A^2 + 2A + I_n$$

and

$$(2A^2 + 2A + I_n)(A^2 - A + \frac{1}{2} I_n) = 2A^4 + \frac{1}{2} I_n = \frac{5}{2} I_n,$$

so $A^2 + (A + I_n)^2$ is invertible with

$$(A^2 + (A + I_n)^2)^{-1} = \frac{2}{5} A^2 - \frac{2}{5} A + \frac{1}{5} I_n.$$

Next,

$$A^2 + (A - I_n)^2 = 2A^2 - 2A + I_n$$

and

$$(2A^2 - 2A + I_n)(A^2 + A + \frac{1}{2} I_n) = 2A^4 + \frac{1}{2} I_n = \frac{5}{2} I_n,$$

so $A^2 + (A - I_n)^2$ is invertible with

$$(A^2 + (A - I_n)^2)^{-1} = \frac{2}{5} A^2 + \frac{2}{5} A + \frac{1}{5} I_n.$$
Evaluate

\[ \int_a^b \left\{ \frac{x^2 + 1}{x^2 - x + 1} \right\} \, dx \]

in terms of \( a \) and \( b \), where \( a < 0 < b \) and \( \{t\} \) denotes the fractional part of \( t \).

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

We have

\[
\begin{align*}
\int_a^b \left\{ \frac{x^2 + 1}{x^2 - x + 1} \right\} \, dx &= \int_a^b \left\{ \frac{x^2 + 1}{x^2 - x + 1} \right\} \, dx + \int_0^1 \left\{ \frac{x^2 + 1}{x^2 - x + 1} \right\} \, dx + \int_1^b \left\{ \frac{x^2 + 1}{x^2 - x + 1} \right\} \, dx \\
&= \int_a^b \frac{x^2 + 1}{x^2 - x + 1} \, dx + \int_0^1 \left( \frac{x^2 + 1}{x^2 - x + 1} - 1 \right) \, dx + \int_1^b \left( \frac{x^2 + 1}{x^2 - x + 1} - 1 \right) \, dx \\
&= \int_a^b \frac{x^2 + 1}{x^2 - x + 1} \, dx + \int_0^b \frac{x}{x^2 - x + 1} \, dx \\
&= \int_a^b \left( 1 + \frac{x}{x^2 - x + 1} \right) \, dx + \int_0^b \frac{x}{x^2 - x + 1} \, dx \\
&= \left. x \right|_a^b + \int_a^b \frac{x}{x^2 - x + 1} \, dx \\
&= -a + \frac{1}{2} \int_a^b \left( \frac{2x - 1}{x^2 - x + 1} + \frac{1}{x^2 - x + 1} \right) \, dx \\
&= -a + \frac{1}{2} \ln(x^2 - x + 1) \bigg|_a^b + \frac{1}{2} \int_a^b \frac{dx}{x^2 - x + 1} =
\end{align*}
\]
\[-a + \frac{1}{2} \ln \frac{b^2 - b + 1}{a^2 - a + 1} + \frac{1}{2} \int_a^b \frac{dx}{(x - \frac{1}{2})^2 + \frac{3}{4}}\]

\[-a + \frac{1}{2} \ln \frac{b^2 - b + 1}{a^2 - a + 1} + \frac{2}{3} \int_a^b dx \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}\]

\[-a + \frac{1}{2} \ln \frac{b^2 - b + 1}{a^2 - a + 1} + \frac{1}{\sqrt{3}} \arctan \left(\frac{2x-1}{\sqrt{3}}\right)\bigg|_a^b\]

\[-a + \frac{1}{2} \ln \frac{b^2 - b + 1}{a^2 - a + 1} + \frac{1}{\sqrt{3}} \left(\arctan \left(\frac{2b-1}{\sqrt{3}}\right) - \arctan \left(\frac{2a-1}{\sqrt{3}}\right)\right)\].

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Daniel Văcaru, Pitești, Romania; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploiești, Romania; S.Chandrasekhar, Chennai, India; Henry Ricardo, Westchester Area Math Circle, USA; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania.
U555. Let $f : [0, +\infty) \to [0, \infty)$ be a differentiable function such that $f(x)e^{f(x)} = x$, for all $x \geq 0$. Evaluate 

$$\int_0^e f(x) \, dx.$$

Proposed by Prithvijit Chatraborty, Kolkata, India

**Solution by Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA**

Let $g(x) = xe^x$. Since $g'(x) = (x+1)e^x > 0$, and $g(0) = 0$, $g$ is increasing, hence one-to-one and $g^{-1} : [0, \infty) \to [0, \infty)$ exists. Then,

$$f(x)e^{f(x)} = x \iff g(f(x)) = x \iff f(x) = g^{-1}(x),$$

i.e. $f$ is the inverse function of $g$. Consequently, introducing the substitution $x = g(t)$ we have

$$\int_0^e f(x) \, dx = \int_0^e g^{-1}(x) \, dx = \int_0^1 g^{-1}(g(t))g'(t) \, dt = \int_0^1 t(t+1)e^t \, dt = \int_0^1 (t^2 + t)e^t \, dt = (t^2 - t + 1)e^t \bigg|_0^1 = e - 1,$$

where the last integration was done by integrating by parts ($u = t^2 + t$, $dv = e^t \, dt$).

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploiești, Romania; Fred Frederickson, Utah Valley University, USA; Henry Ricardo, Westchester Area Math Circle, USA; G. C. Greubel, Newport News, VA, USA; Ibrahim Suleiman, New York University Abu Dhabi, United Arab Emirates; Moubinool Omarjee, Lycée Henri IV, Paris, France; Arkady Alt, San Jose, CA, USA.
U556. Find the volume of the solid obtained by rotating a unit cube about an axis connecting opposite vertices.

First solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Orient the axis of rotation along the $y$-axis with the lower vertex of the cube at the origin and the upper vertex at $(0, \sqrt{3})$. By symmetry, the volume of the solid obtained by rotating a unit cube about this axis can be obtained by computing the volume associated with rotating the lower half of the cube and then multiplying by two. For $0 \leq y \leq \sqrt{3}/3$, the square of the radius of the disk that results from rotation is given by $2y^2$, while for $\sqrt{3}/3 \leq y \leq \sqrt{3}/2$, the square of the radius of the disk that results from rotation is given by $2(y^2 - \sqrt{3}y + 1)$ – see below. The volume of the solid is then

$$ V = 2\pi \left( \int_0^{\sqrt{3}/3} 2y^2 \, dy + \int_{\sqrt{3}/3}^{\sqrt{3}/2} 2(y^2 - \sqrt{3}y + 1) \, dy \right) $$

$$ = 2\pi \left( \frac{2}{3} y^3 \bigg|_0^{\sqrt{3}/3} + \left( \frac{2}{3} y^3 - \sqrt{3}y^2 + 2y \right) \bigg|_{\sqrt{3}/3}^{\sqrt{3}/2} \right) $$

$$ = 2\pi \left( \frac{2\sqrt{3}}{27} + \frac{\sqrt{3}}{2} - \frac{11\sqrt{3}}{27} \right) = \frac{\pi\sqrt{3}}{3}. $$

In the diagram below, suppose the point along the indicated diagonal (which corresponds to the axis of rotation) that is nearest to the point $(a,0,0)$ is the point $(b,b,b)$. Then the vector $(b-a,b,b)$ must be orthogonal to the vector $(1,1,1)$. Thus,

$$ (b-a,b,b) \cdot (1,1,1) = 0, \quad \text{or} \quad 3b = a. $$

The square of the distance from $(b,b,b)$ to $(3b,0,0)$ is $6b^2$. As $0 \leq a \leq 1$, $0 \leq b \leq 1/3$. Moreover, $y = \sqrt{3}b$, so, for $0 \leq y \leq \sqrt{3}/3$, the square of the radius of the disk that results from rotation is given by $2y^2$. Next, consider the point $(1,0,a)$. Suppose the point along the indicated diagonal that is nearest to the point $(1,0,a)$ is the point $(b,b,b)$. Then the vector $(b-1,b-b-a)$ must be orthogonal to the vector $(1,1,1)$. Thus,

$$ (b-1,b-b-a) \cdot (1,1,1) = 0, \quad \text{or} \quad 3b-1 = a. $$

The square of the distance from $(b,b,b)$ to $(1,0,3b-1)$ is

$$ (b-1)^2 + b^2 + (1-2b)^2 = 6b^2 - 6b + 2 = 2(3b^2 - 3b + 1). $$

For $0 \leq a \leq 1/2$, which is sufficient to consider based on symmetry, $1/3 \leq b \leq 1/2$. Thus, for $\sqrt{3}/3 \leq y \leq \sqrt{3}/2$, the square of the radius of the disk that results from rotation is given by $2(y^2 - \sqrt{3}y + 1)$. 

\[ \text{Diagram showing the cube and the axis of rotation.} \]
Second solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Let $O$ be the center, $AB$ be an edge, and $AC$ be a diameter of the cube. Let $m$ be the distance from $O$ to the midpoint of $AB$. It is easy to see that $r = OA = \sqrt{3}/2$ and $m = 1/\sqrt{2}$. Let $h$ be the distance from $B$ to the line $AC$, then $hr = m$, thus $h = m/r = \sqrt{2}/3$. Let $S$ be the resulting solid of rotating the cube about the axis $AC$. Clearly, $S$ has two circular cones with vertices $A$ and $C$, and their total volume is

$$2 \left( \frac{1}{3} \pi h^2 \sqrt{1-h^2} \right) = \frac{4\pi}{9\sqrt{3}}.$$

The middle part of $S$ is a hyperboloid, since its surface is doubly ruled by the three pairs of edges of the cube. Set up a coordinate system with $O$ as the origin, $A = (r, 0)$, and $B = (\sqrt{r^2 - h^2}, h)$. Considering the hyperbola $y^2/m^2 - x^2/b^2 = 1$ passing through $B$, we get $h^2/m^2 - (r^2 - h^2)/b^2 = 1$, thus $b = 1/2$. Since the hyperboloid is the same as rotating the hyperbola about the $x$-axis, its volume equals

$$2 \int_0^{\sqrt{r^2-h^2}} \pi m^2 \left(1 + \frac{x^2}{b^2}\right) dx = \frac{5\pi}{9\sqrt{3}}.$$

Therefore, the volume of $S$ is $\pi/\sqrt{3} = 1.81\ldots$. Notice that this is $2/3$ of the sphere of the same diameter. For a video of such a spinning cube, see https://www.youtube.com/watch?v=wsMah8e6bYA.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Corneliu Mănescu-Avram, Ploiești, Romania.
U557. Evaluate

\[ \int_2^3 x e^x (\ln x + 1) \, dx. \]

Proposed by Toyesh Prakash Sharma, St. C.F. Andrews School, Agra, India

Solution by G. C. Greubel, Newport News, VA, USA

Consider integration-by-parts on the integrals

\[ I_1 = \int x e^x \, dx \]
\[ I_2 = \int e^x x \ln x \, dx \]

yields

\[ \int x e^x \, dx = (x - 1) e^x \]

and

\[
\int e^x x \ln x \, dx = x \ln x e^x - \int (1 + \ln x) e^x \, dx \\
= (x \ln x - 1) e^x - \ln x e^x - \int e^x \frac{dx}{x} \\
= ((x - 1) \ln x - 1) e^x - \text{Ei}(x).
\]

Now,

\[ \int_a^b x e^x (\ln x + 1) \, dx = ((b - 1) \ln b - 1) e^b - ((a - 1) \ln a - 1) e^a + \text{Ei}(a) - \text{Ei}(b) \]

and setting \( a = 2 \) and \( b = 3 \) gives the desired result.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adnan Ali, NIT Silchar, Assam, India; Le Hoang Bao, Tien Giang, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; Henry Ricardo, Westchester Area Math Circle, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France.
U558. For every polynomial $P(x) = c_0 + c_1 x + ... + c_n x^n$ define its reciprocal $\tilde{P}$ by $\tilde{P} = c_0 x^n + c_1 x^{n-1} + ... + c_n$. Let $f(x) = a_r x^{d_r} + ... + a_0 x^{d_0}$ be a polynomial with integer coefficients and $n = d_r > d_{r-1} > ... > d_0 = 0$. Let $g(x) = b_s x^{e_s} + ... + b_0$ be a polynomial with positive integer coefficients and $n = e_s > e_{s-1} > ... > e_0 = 0$. Let $\tilde{f}(x)$ be defined as the reciprocal polynomial of $f(x)$.

Prove that if $f(x) \tilde{f}(x) = g(x) \tilde{g}(x)$ and $a_0 = a_1 = ... = a_r = 1$, then $r = s$ and $b_0 = b_1 = ... = b_s = 1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

The polynomials $f(x) \tilde{f}(x), g(x) \tilde{g}(x)$ are polynomials of degree at most $2n$. The coefficient of $x^n$ in the former is equal to $a_0^2 + ... + a_r^2$ and this coefficient in the later polynomial is $b_0^2 + ... + b_s^2$. Then we can find that $a_0^2 + ... + a_r^2 = b_0^2 + ... + b_s^2$. Further, since $\tilde{f}(1) = f(1)$ and $g(1) = \tilde{g}(1)$ we find that $f(1)^2 = g(1)^2$. That is,

\[
\left( \sum_{j=0}^{s} b_j \right)^2 \leq \left( \sum_{j=0}^{s} b_j^2 \right) = \left( \sum_{j=0}^{r} a_j^2 \right) = \left( \sum_{j=0}^{r} a_j \right)^2 = \left( \sum_{j=0}^{s} b_j \right)^2
\]

We deduce that the equality holds in the above inequality. Since $b_i > 0$ for each $i$, we find that $b_i = 1$. Thus, $r = s$. 
Olympiad problems

O553. Let $ABC$ be a triangle with $AB = AC$ and let $M$ be the midpoint of $BC$. Circle $\omega$ is tangent to $BC$ at $M$ and lies outside of triangle $ABC$. Circle $\Omega$ passes through $A$, is internally tangent to $\omega$, and its center lies on $AM$. Circle $\gamma$ is internally tangent to circle $\Omega$, touches segment $BC$ and the extension of line $AC$. Through $A$ tangents to $\omega$ and $\gamma$ are drawn intersecting segment $MC$ at points $K$ and $L$, respectively. Prove that the inradius of triangle $ABL$ is twice the inradius of triangle $AKC$.

Proposed by Waldemar Pompe, Warzaw, Poland

Solution by the author
Assume the incircle of triangle $ABL$ touches segment $BC$ at $X$, and the incircle of triangle $AKC$ touches $BC$ at $Y$. Since $\angle ABC = \angle ACB$, it suffices to show that $BX = 2CY$.

Let $AP$ and $AQ$ be the tangent segments to circles $\omega$ and $\gamma$, respectively. Assume that $\gamma$ touches $BC$ at $S$ and let $\delta$ be a circle symmetric to $\gamma$ with respect to line $AM$. Let $T$ be a touching point of $\delta$ and $BC$. Finally, let $E$ and $F$ be the touching points of $\delta$ and $\gamma$ with $AB$ and $AC$, respectively.

An inversion with center $A$ that sends line $BC$ to circle $\Omega$ maps circle $\gamma$ to $\gamma$ and $\omega$ to $\omega$. It follows that the inversion maps $P$ to $P$ and $Q$ to $Q$. Therefore, $AP = AQ = AE = AF$.

Now, observe that

$$BX = \frac{1}{2}(AE + TS - AQ) = \frac{1}{2}TS, \quad CY = \frac{1}{2}(AF + SM - AP) = \frac{1}{2}SM.$$

Since $SM = \frac{1}{2}TS$, it follows that $BX = 2CY$, which completes the proof.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA.
O554. Let $a, b, c, d$ be real numbers such that $|a|, |b|, |c|, |d| \geq 1$ and
\[a + b + c + d + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 0.\]

Prove that
\[a + b + c + d \leq 2\sqrt{2}.
\]

Proposed by Marius Stănean, Zalău, România

\[\text{Solution by the author}\]
Due to symmetry, without loss of generality, we may assume that $a \geq b \geq c \geq d$. It is clear that $a \geq 1$ and $d \leq -1$.

If we denote $x = \frac{1}{2}(a + \frac{1}{a})$, $y = \frac{1}{2}(b + \frac{1}{b})$, $z = \frac{1}{2}(c + \frac{1}{c})$, $t = \frac{1}{2}(d + \frac{1}{d})$, then $x + y + z + t = 0$, $x \geq y \geq z \geq t$ and $x, y, z, t \in (-\infty, -1] \cup [1, \infty)$. Also, we have
\[\frac{1}{4}(a - \frac{1}{a})^2 = x^2 - 1 \implies \frac{1}{2}(a - \frac{1}{a}) = \pm \sqrt{x^2 - 1},\]
and similarly for $b, c, d$. Since $d \leq -1$ and $a \geq 1$ it follows that $\frac{1}{2}(d - \frac{1}{d}) = -\sqrt{t^2 - 1}$, $\frac{1}{2}(a - \frac{1}{a}) = \sqrt{x^2 - 1}$.

Hence,
\[a + b + c + d = \frac{1}{2}\left(a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c} + d - \frac{1}{d}\right)\]
\[= \sqrt{x^2 - 1} \pm \sqrt{y^2 - 1} \pm \sqrt{z^2 - 1} - \sqrt{t^2 - 1}.\]

We have three cases:

Case 1. If $z \geq 1$, then we need to show that
\[\sqrt{x^2 - 1} + \sqrt{y^2 - 1} + \sqrt{z^2 - 1} - \sqrt{t^2 - 1} \leq 2\sqrt{2}.\]
Consider the function $f$ defined as $f(x) = \sqrt{x^2 - 1}$ on $[1, \infty)$. Since
\[f''(x) = -\frac{1}{(x^2 - 1)^{3/2}},\]
it follows that $f$ is concave on the interval $[1, \infty)$. We have
\[(x, y, z) > \left(x + y + z \frac{3}{3}, x + y + z \frac{3}{3}, x + y + z \frac{3}{3}\right),\]
so by Karamata’s Inequality,
\[\sqrt{x^2 - 1} + \sqrt{y^2 - 1} + \sqrt{z^2 - 1} \leq 3\sqrt{\left(x + y + z \frac{3}{3}\right)^2 - 1}\]
\[\sqrt{t^2 - 9} \leq \sqrt{t^2 - 1},\]
that is
\[\sqrt{x^2 - 1} + \sqrt{y^2 - 1} + \sqrt{z^2 - 1} - \sqrt{t^2 - 1} \leq 0 < 2\sqrt{2}.\]
Case 2. If $y \geq 1$, $z \leq -1$, then we need to show that
\[
\sqrt{x^2 - 1} + \sqrt{y^2 - 1} - \sqrt{z^2 - 1} - \sqrt{t^2 - 1} \leq 2\sqrt{2}.
\]
Let $s = x + y = -z - t \geq 2$. We have
\[
(x, y) > \left( \frac{x + y}{2}, \frac{x + y}{2} \right), \quad (-t - z - 1, 1) > (-t, -z),
\]
so by Karamata’s Inequality,
\[
\sqrt{x^2 - 1} + \sqrt{y^2 - 1} \leq 2\sqrt{\left( \frac{x + y}{2} \right)^2 - 1} = \sqrt{s^2 - 4},
\]
\[
\sqrt{z^2 - 1} + \sqrt{t^2 - 1} \geq \sqrt{(-t - z - 1)^2 - 1} = \sqrt{s^2 - 2s}.
\]
It remains to show that
\[
\sqrt{s^2 - 4} - \sqrt{s^2 - 2s} \leq 2\sqrt{2},
\]
or
\[
\sqrt{s - 2} \left( \sqrt{s + 2} - \sqrt{s} \right) \leq 2\sqrt{2},
\]
or
\[
\sqrt{s - 2} \leq \sqrt{2(s + 2)} + \sqrt{2s},
\]
which is clearly true.

Case 3. If $y \leq -1$, then we need to show that
\[
\sqrt{x^2 - 1} - \sqrt{y^2 - 1} - \sqrt{z^2 - 1} - \sqrt{t^2 - 1} \leq 2\sqrt{2}.
\]
We have
\[
(-y - z - t - 2, 1, 1) > (-t, -z, -y),
\]
so by Karamata’s Inequality,
\[
\sqrt{y^2 - 1} + \sqrt{z^2 - 1} + \sqrt{t^2 - 1} \geq \sqrt{(-y - z - t - 2)^2 - 1} = \sqrt{(x - 2)^2 - 1} = \sqrt{x^2 - 4x + 3}.
\]
Hence, we need to prove that
\[
\sqrt{x^2 - 1} \leq \sqrt{x^2 - 4x + 3} + 2\sqrt{2}.
\]
Squaring both sides yields the equivalent inequality
\[
x - 3 \leq 2(x - 1)(x - 3),
\]
which is true because $x = -y - z - t \geq 3$ and $2(x - 1) > (x - 3)$. The equality holds when $x = 3$, $y = z = t = -1$ which means $a = 3 + 2\sqrt{2}$, $b = c = d = -1$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA.
Let $ABCD$ be a square of side length $1$. Point $X$ lies on the smaller arc $DA$ of the circumcircle of square $ABCD$. Let $r_1, r_2, r_3, r_4$ be the inradii of triangles $XDA, XAB, XBC, XCD$, respectively. Determine all possible values of

$$\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}$$

as $X$ varies on the smaller arc $DA$ of the circumcircle of square $ABCD$.

*Proposed by Waldemar Pompe, Warzaw, Poland*

*Solution by the author*

Let $XAB$ be a triangle with $\angle AXB = \alpha$ and let $A^*, B^*$ be the images of points $A, B$, respectively, under inversion with center $X$ and radius 1. If $r$ is the inradius of triangle $XAB$, then

$$\frac{\sin \alpha}{r} = XA^* + XB^* + A^*B^* .$$

Indeed, if $r^*$ is the inradius of triangle $XA^*B^*$, then

$$\frac{r^*}{r} = \frac{XB^* \cdotXA^*}{XA \cdot XA^*} = \frac{2[XA^*B^*]}{\sin \alpha} = \frac{(XA^* + XB^* + A^*B^*)r^*}{\sin \alpha},$$

which is the claimed formula. Similarly, we show that if $h$ is the altitude of triangle $XAB$ taken from $X$, then

$$\frac{\sin \alpha}{h} = A^*B^* .$$

To solve the problem, consider the inversion with center $X$ and radius 1. Then the images $A^*, B^*, C^*, D^*$ of points $A, B, C, D$ respectively, lie in a line in that order. Then the above formulas give

$$\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} = \frac{2B^*C^*}{\sin 45^\circ} = \frac{2}{h},$$

where $h$ is a distance from point $X$ to line $BC$. As $h$ ranges from 1 to $\frac{1}{2}(\sqrt{2}+1)$, $2/h$ ranges from $4/(\sqrt{2}+1)$ to 2.

*Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Max Sachelarie, PA, USA.*
O556. Let $a, b, c$ be the sides of a triangle $ABC$. Prove that
\[
(a^2 - bc) \cos \frac{B - C}{2} + (b^2 - ca) \cos \frac{C - A}{2} + (c^2 - ab) \cos \frac{A - B}{2} \geq 0.
\]

Proposed by Marius Stănean, Zalău, România

Solution by the author
Since $\cos \frac{B - C}{2} = \frac{b + c}{a} \sin \frac{A}{2}$ the inequality becomes
\[
\sum_{cyc} \left[ a^2(b + c) - bc(b + c) \right] \frac{\sin \frac{A}{2}}{a} \geq 0.
\]
(1)

Without loss of generality, assume that $a \geq b \geq c$ which implies $A \geq B \geq C$. Then, we have
\[
\frac{1}{\cos \frac{A}{2}} \geq \frac{1}{\cos \frac{B}{2}} \geq \frac{1}{\cos \frac{C}{2}},
\]
that is
\[
\frac{\sin \frac{A}{2}}{a} \geq \frac{\sin \frac{B}{2}}{b} \geq \frac{\sin \frac{C}{2}}{c}.
\]

Also, we have
\[
a^2(b + c) \geq b^2(c + a) \geq c^2(a + b)
\]
and
\[-bc(b + c) \geq -ac(a + c) \geq -ab(a + b),
\]
so
\[
a^2(b + c) - bc(b + c) \geq b^2(c + a) - ac(a + c) \geq c^2(a + b) - ab(a + b).
\]

These being established, applying Chebyshev’s Inequality, we get
\[
3 \sum_{cyc} \left[ a^2(b + c) - bc(b + c) \right] \frac{\sin \frac{A}{2}}{a} \geq \sum_{cyc} \left[ a^2(b + c) - bc(b + c) \right] \sum_{cyc} \frac{\sin \frac{A}{2}}{a}
\]
\[
= 0 \cdot \sum_{cyc} \frac{\sin \frac{A}{2}}{a} = 0
\]
as desired.

Observation: Here https://artofproblemsolving.com/community/c6h1142667p5374755AoPS we have the following inequality:

Let $S$ be the area of triangle $ABC$ with length-side $a, b, c$. Prove that
\[
ab \cos \frac{A - B}{2} + bc \cos \frac{B - C}{2} + ca \cos \frac{C - A}{2} \geq 4\sqrt{3}S.
\]

From the above problem we deduce that
\[
c^2 \cos \frac{A - B}{2} + a^2 \cos \frac{B - C}{2} + b^2 \cos \frac{C - A}{2} \geq 4\sqrt{3}S
\]
which is an equivalent form of the problem S530 from Mathematical Reflections no 5-2020.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Max Sachelarie, PA, USA; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Arkady Alt, San Jose, CA, USA.
O557. Evaluate
\[
\sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n}{2k+1}.
\]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author
Using the fact that
\[
\frac{1}{a+1} \binom{b}{a} = \frac{1}{b+1} \binom{b+1}{a+1}
\]
and De Moivre’s formula we have
\[
\sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n}{2k+1} = \sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n+1}{2k+1} = \frac{1}{i(n+1)} \sum_{k=0}^{\frac{n}{2}} i^{2k+1} \binom{n+1}{2k+1} \quad \text{(where } i^2 = -1)\\
= \frac{1}{i(n+1)} \cdot \frac{(1 + i)^{n+1} - (1 - i)^{n+1}}{2}\\
= \frac{(\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}))^{n+1} - (\sqrt{2}(\cos \frac{-\pi}{4} + i \sin \frac{-\pi}{4}))^{n+1}}{2i(n+1)}\\
= \frac{(\sqrt{2})^{n+1} \left( \cos \frac{(n+1)\pi}{4} + i \sin \frac{(n+1)\pi}{4} \right) - (\sqrt{2})^{n+1} \left( \cos \frac{-(n+1)\pi}{4} + i \sin \frac{-(n+1)\pi}{4} \right)}{2i(n+1)}\\
= \frac{(\sqrt{2})^{n+1} \sin \frac{(n+1)\pi}{4}}{n+1}.
\]

Also solved by Spyros Kallias, Volos, Greece; Kanav Talwar, Delhi Public School, Faridabad, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adnan Ali, NIT Silchar, Assam, India; G. C. Greuel, Newport News, VA, USA; Mouhinool Omarjee, Lycée Henri IV, Paris, France; Henry Ricardo, Westchester Area Math Circle, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Le Hoang Bao, Tien Giang, Vietnam; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam.
Let \( \{x\} \) be the fractional part of the real number \( x \). Prove that for all positive integers \( n \) there are pairwise distinct rational numbers \( x_1, ..., x_n > n \) such that \( \{x_i x_j\} \in \left(\frac{1}{2}, \frac{5}{6}\right) \) for \( 1 \leq i, j \leq n \).

Proposed by Titu Andreescu, Dallas, USA and Navid Safaei, Tehran, Iran

Solution by the authors

Start by choosing pairwise different and large enough prime numbers \( p_1, ..., p_n \) congruent to 1 modulo 8. For instance, we can take \( p_i \) an arbitrary prime divisor of \( 2^{2i-1} + 1 \) for some sufficiently large \( k \). Next, choose integers \( m_i \in (0,3p_i) \) such that \( 2m_i \equiv 3 \pmod{p_i} \) and \( m_i \equiv 1 \pmod{3} \). This is possible by the Chinese remainder theorem, since \( p_i > 3 \). For any \( i \) the congruence \( 3x^2 \equiv m_i \pmod{p_i^2} \) has solutions: by Hensel’s lemma (which is of course elementary in this case) it suffices to argue modulo \( p_i \). But then it suffices to show that the congruence \( 2x^2 \equiv 1 \pmod{p_i} \) has solutions, which is true since \( p_i \equiv 1 \pmod{8} \). Note that we can prove this very easily in our context: if \( p \) divides \( 2^{2i} + 1 \), then \( 2^{2i-2} - 2^{i-2} \) is a solution of the congruence \( z^2 \equiv 2 \pmod{p} \). Now, another application of the Chinese remainder theorem yields a positive integer \( N \) such that \( 3N^2 \equiv 1 \pmod{p_i^2} \) for all \( i \). We will choose \( N > p_1...p_n \cdot n \).

With the above data, consider \( x_i = \frac{N}{p_i} \). Then \( x_i > n \) and we will show that \( \{x_i x_j\} \in \left(\frac{1}{2}, \frac{5}{6}\right) \) for all \( i, j \).

Suppose first that \( i \neq j \). Note that \( 2N^2 \equiv 1 \pmod{p_i} \) for all \( i \), thus \( 2N^2 = 1 + (2k+1)p_ip_j \) for some positive integer \( k \). Then
\[
\{x_i x_j\} = \left\{ \frac{2N^2}{2p_ip_j} \right\} = \left\{ \frac{1 + p_ip_j + 2kp_ip_j}{2p_ip_j} \right\} = \left\{ \frac{1 + p_ip_j}{2p_ip_j} - \frac{1}{2} \right\} = \left\{ \frac{1}{2}, \frac{5}{6} \right\}.
\]

Suppose now that \( i = j \) and write \( 3N^2 = m_i + k_ip_i^2 \). Taking this relation modulo 3 yields \( 0 \equiv 1 + k_i \pmod{3} \), thus \( 3N^2 = m_i + (3l_i + 2)p_i^2 \) for some integer \( l_i \). As above, this implies
\[
\{x_i^2\} = \left\{ \frac{3N^2}{3p_i^2} \right\} = \left\{ \frac{m_i + 2p_i^2}{3p_i^2} \right\} = \left\{ \frac{2}{3} + \frac{m_i}{3p_i^2} \right\}.
\]

Since \( 0 < m_i < 3p_i \) and \( p_i \) is sufficiently large, the last expression belongs to \( \left(\frac{1}{2}, \frac{5}{6}\right) \). The result follows.

Also solved by Kanav Talwar, Delhi Public School, Faridabad, India; Adnan Ali, NIT Silchar, Assam, India.