Junior problems

J553. Solve in real numbers the equation

$$\left(x^2 - 2\sqrt{2}x\right)\left(x^2 - 2\right) = 2021.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA
Let $x = (1+y)/\sqrt{2}$, then the equation becomes $0 = y^4 - 10y^2 - 8075 = (y^2+85)(y^2-95)$.

Therefore, $y = \pm\sqrt{95}$, so $x = (1 \pm \sqrt{95})/\sqrt{2}$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Alex Bloom, Sega Math Academy; Nick Iliopoulos, AUTH, Greece; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Ashley Simone, SUNY Brockport; Daniel Văcaru, Pitești, Romania; Fred Frederickson, Utah Valley University, USA; Henry Ricardo, Westchester Area Math Circle, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Max Sachelarie, PA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Kanav Talwar, Delhi Public School, Faridabad, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA; David E. Manes, Oneonta, NY, USA; Titu Zvonaru, Comănești, Romania.

J554. Let x, y, z be positive real numbers such that $x+y+z=xyz$. Prove that

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+z^2}\geq\frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+zx}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the given condition we have

$$1+x^2=\frac{xyz}{x+y+z}+x^2=\frac{xyz+x^2(x+y+z)}{x+y+z}=\frac{x(x+y)(x+z)}{xyz}=\frac{(x+y)(x+z)}{yz},$$

and

$$1+yz=\frac{xyz}{x+y+z}+yz=\frac{xyz+yz(x+y+z)}{x+y+z}=\frac{yz(2x+y+z)}{xyz}=\frac{2x+y+z}{x}.$$

Therefore the desired inequality may be rewritten as

$$\sum_{\text{cyc}}\frac{yz}{(x+y)(x+z)}\geq\sum_{\text{cyc}}\frac{x}{2x+y+z}.$$

This is equivalent to

$$\sum_{\text{cyc}}\frac{yz(y+z)}{(x+y)(y+z)(z+x)}\geq\sum_{\text{cyc}}\frac{x}{2x+y+z},$$

or

$$\frac{(x+y)(y+z)(z+x)-2xyz}{(x+y)(y+z)(z+x)}\geq\sum_{\text{cyc}}\frac{x}{2x+y+z},$$

or

$$1\geq\frac{2xyz}{(x+y)(y+z)(z+x)}+\sum_{\text{cyc}}\frac{x}{2x+y+z}.$$

This follows from the following two inequalities

$$\frac{2xyz}{(x+y)(y+z)(z+x)}\leq\frac{1}{4}$$

and

$$\sum_{\text{cyc}}\frac{x}{2x+y+z}\leq\frac{3}{4}.$$

The first inequality is true because

$$\frac{2xyz}{(x+y)(y+z)(z+x)}\leq\frac{2xyz}{2\sqrt{xy}\cdot2\sqrt{yz}\cdot2\sqrt{zx}}=\frac{1}{4}.$$

The second inequality is true because

$$\sum_{\text{cyc}}\frac{x}{2x+y+z}\leq\sum_{\text{cyc}}\frac{x}{4}\left(\frac{1}{x+y}+\frac{1}{x+z}\right)=\frac{3}{4}.$$

The proof is completed.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Alex Bloom, Sega Math Academy; Adnan Ali, NIT Silchar, Assam, India; Arighna Pan, Nabadwip Vidyasagar College, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Max Sachelarie, PA, USA; Nick Iliopoulos, AUTH, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

J555. Let a, b, c be real numbers such that

$$\frac{1}{a+2+\sqrt{a^2+8}}+\frac{1}{b+2+\sqrt{b^2+8}}+\frac{1}{c+2+\sqrt{c^2+8}}\leq\frac{1}{2}.$$

Prove that $a+b+c\geq 3$. When does the equality occur?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Let $x=a+\sqrt{a^2+8}>0$, implying $a=\frac{x}{2}-\frac{4}{x}$. Similarly, $b=\frac{y}{2}-\frac{4}{y}$ and $c=\frac{z}{2}-\frac{4}{z}$. Then

$$\frac{1}{2}-\frac{1}{x+2}>=\frac{1}{y+2}+\frac{1}{z+2}\geq\frac{4}{y+2+z+2},$$

so $y+z+4\geq 8+\frac{16}{x}$. Summing up with the two analogues inequalities yields

$$4\left(\frac{x}{2}-\frac{4}{x}\right)+4\left(\frac{y}{2}-\frac{4}{y}\right)+4\left(\frac{z}{2}-\frac{4}{z}\right)\geq 12.$$

Hence the conclusion.

Second solution by Marius Stanean, Zalau, Romania
Because $a+2+\sqrt{a^2+8}>2$, let $a+2+\sqrt{a^2+8}=4x+2$, $b+2+\sqrt{b^2+8}=4y+2$, $c+2+\sqrt{c^2+8}=4z+2$, where $x,y,z>0$. Therefore

$$a=2x-\frac{1}{x},\ b=2y-\frac{1}{y},\ c=2z-\frac{1}{z}.$$

The given condition becomes

$$\frac{1}{2x+1}+\frac{1}{2y+1}+\frac{1}{2z+1}\le 1,$$

or

$$x+y+z+1\le 4xyz.$$

But $27xyz\le(x+y+z)^3$, so $27(x+y+z+1)\le 4(x+y+z)^3$ which means $x+y+z\ge 3$. We need to show that

$$2(x+y+z)-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge 3.$$

We have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{xy+yz+zx}{xyz}\le\frac{4(x+y+z)^2}{3(x+y+z+1)},$$

so it remains to prove that

$$2(x+y+z)-\frac{4(x+y+z)^2}{3(x+y+z+1)}\ge 3,$$

or if we denote $t=x+y+z\ge 3$,

$$\frac{(t-3)(2t+3)}{3(t+1)}\ge 0,$$

true.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania ; Adnan Ali, NIT Silchar, Assam, India.

J556. Let ABC be a triangle with circumcircle Γ and circumcenter O. The tangents in B and C to Γ intersect in D and AD intersects Γ in E. The parallel through A to BC intersects Γ in F. Prove that EF bisects side BC.

Proposed by Mihaela Berindeanu, Bucharest, România

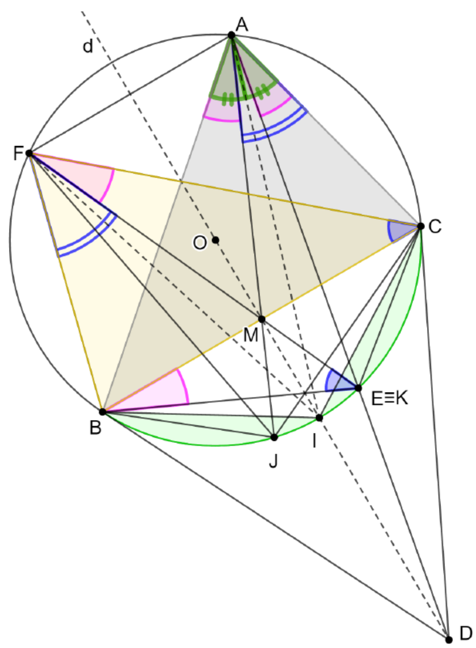
First solution by the author

If BD and CD are tangents lines to $\Gamma \Rightarrow AD$ is the symmedian from A in $\triangle ABC$.

We will build the mediator of the BC segment, named d, $d \cap BC = \{M\}$, $M \in BC$, $BM \equiv MC$.

Notations: $AM \cap \Gamma = \{J\}$, $d \cap \Gamma = \{I\}$, $FM \cap \Gamma = \{K\}$.

I will prove that AK is the symmedian from A in $\triangle ABC$, so $K \in AD \Rightarrow E \equiv K$.



$BC \parallel AF \Rightarrow BCAF$ is an isosceles trapezoid. Line d divides the $\overset{\frown}{BC}$ into two equal parts $\Rightarrow \overset{\frown}{BI} \equiv \overset{\frown}{IC}$.

d is mediator for BC and AF also, $\Rightarrow d$ is $BCAF$ symmetry axis, so $\triangle ABC = \triangle FBC \Rightarrow \overset{\frown}{BI} = \overset{\frown}{IC} \Rightarrow AI$ is $\angle BAC$ bisector.

Due to the symmetry, $\measuredangle BFM \equiv \measuredangle CAM$ or $\measuredangle JAC \equiv \measuredangle KFB \Rightarrow \overset{\frown}{JC} = \overset{\frown}{KB}$

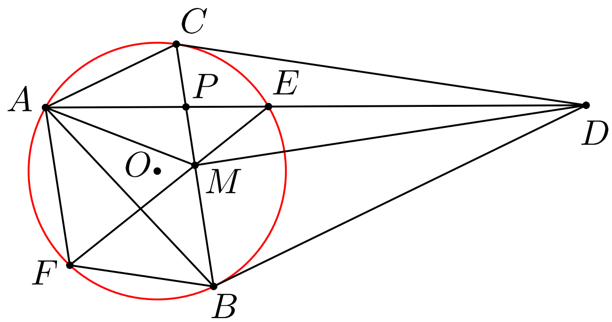
$$\left| \begin{array}{l} \overset{\frown}{KB} = \overset{\frown}{JC} \\ \overset{\frown}{IB} = \overset{\frown}{IC} \end{array} \right. \Rightarrow \overset{\frown}{KI} = \overset{\frown}{IJ} \text{ and } \overset{\frown}{KC} \equiv \overset{\frown}{BJ} \Rightarrow \measuredangle BAJ \equiv \measuredangle KAC.$$

So, AM and AK are isogonal lines

$$\left| \begin{array}{l} AM,\ AK = \text{ isogonal lines} \\ AM = \text{ the median from } A \text{ in } \triangle ABC \end{array} \right. \Rightarrow AK = \text{ symmedian } \Rightarrow E \equiv K.$$

Conclusion: EF passes through the middle of the side BC

Second solution by Polyahedra, Polk State College, USA



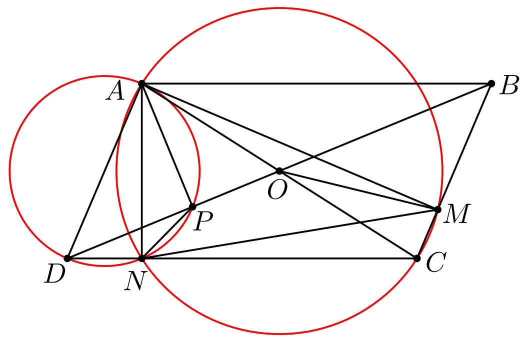
Let M be the midpoint of BC. Since $AF \parallel CB$, $AFBC$ is an isosceles trapezoid, so $\angle BMF = \angle CMA$. Suppose that AD intersects BC at P. Since AD is the A-symmedian of $\triangle ABC$, we know that P, D divide A, E harmonically. Also, $DM \perp MP$. By the right-angle-and-bisector lemma (see E. Chen, *Euclidean Geometry in Math. Olympiads*, MAA, 2016, p. 177), PM bisects $\angle AME$. Therefore, $\angle EMP = \angle FMB$, that is, E, M, F are collinear.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Kanav Talwar, Delhi Public School, Faridabad, India; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Max Sachelarie, PA, USA; Nick Iliopoulos, AUTH, Greece; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

J557. Let $ABCD$ be a parallelogram with $AB \neq AD$ and $\angle BAD > 90^\circ$. We denote by M, N, P the orthogonal projections of A on BC, CD, BD, respectively, and let O be the intersection of diagonals AC and BD. Prove that points M, N, O, P lie on a circle.

Proposed by Mihai Miculiţa, Oradea, România

Solution by Polyahedra, Polk State College, USA



Notice that $ANCM$ is cyclic with center O, and $ADNP$ is also cyclic. Thus, $AO = MO$, so

$$\angle NMO = 90^\circ - \angle OMA - \angle CMN = 90^\circ - \angle MAO - \angle CAN = \angle BAM = \angle NAD = \angle NPD,$$

completing the proof.

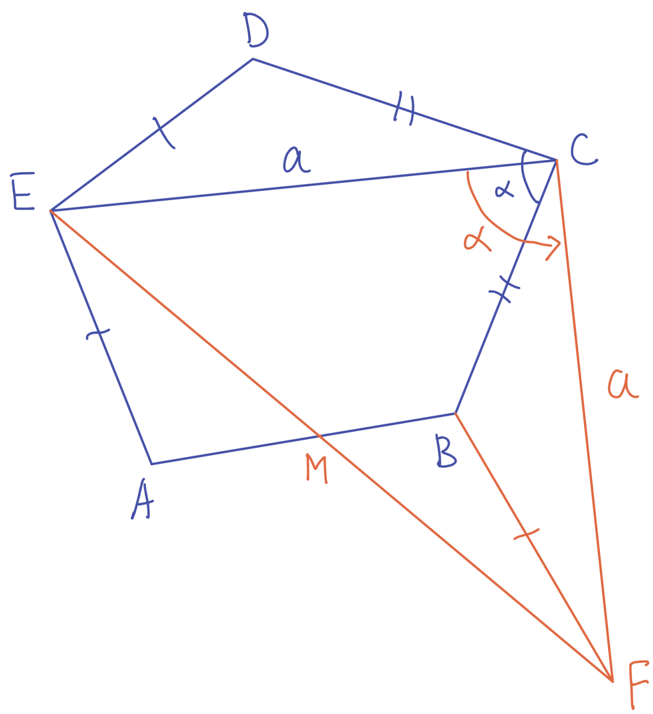
Also solved by Kamran Mehdiyev, Baku, Azerbaijan; Theo Koupelis, Broward College, Davie, FL, USA; Kanav Talwar, Delhi Public School, Faridabad, India; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Max Sachelarie, PA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Titu Zvonaru, Comănești, Romania.

J558. Let $ABCDE$ be a convex pentagon with $BC = CD$, $DE = EA$ and $\angle BCD + \angle DEA = 180°$. Knowing that $\angle BCD = \alpha$ and $EC = a$, determine the area of the pentagon.

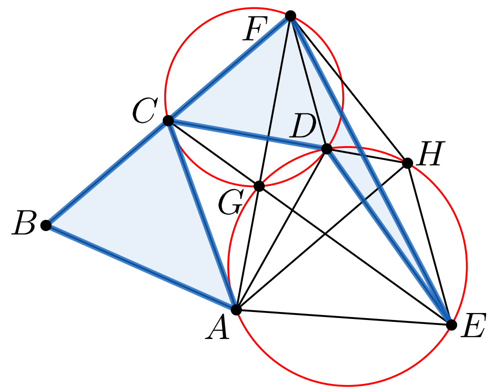
Proposed by Waldemar Pompe, Warsaw, Poland

First solution by the author

Rotate triangle EDC around point C and angle α. Then D is mapped to B and E is taken to some point F. Since $\angle BCD + \angle DEA = 180°$, it follows that segments AE and BF are parallel. They are also of equal length. Therefore the midpoints of segments AB and EF coincide; let's label this common midpoint by M. Then triangles AEM and BFM are congruent. Hence the area of pentagon $ABCDE$ is equal to the area of triangle ECF, which is $\frac{1}{2}a^2 \sin\alpha$.



Second solution by Polyahedra, Polk State College, USA



Let F be the reflection of B about C. We have $\triangle DFC \sim \triangle DAE$ with spiral center D. Let G be the intersection of AF and CE, then G is on the circumcircles (DFC) and (DAE) as well. Suppose the line through A and parallel to BC intersects CD at H. Then $\angle AHD = \angle DCF = \angle AED$, so H is on the circumcircle (DAE).

Therefore, $\angle AHE = \angle ADE = \angle CFD$, thus $EH \parallel DF$. Consequently, $[DFE] = [DFH]$, so $[CFED] = [CFH] = [BCA]$. Hence, $[ABCDE] = [ACFE] = \frac{1}{2}AF \cdot EC \sin \angle AGE$. Since $AF/EC = DF/DC = 2\cos\frac{\alpha}{2}$ and $\angle AGE = \frac{\alpha}{2}$, we obtain the answer $\frac{1}{2}a^2 \sin\alpha$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Max Sachelarie, PA, USA.

Senior problems

S553. Solve in real numbers the equation

$$(x^3 - 3x)^2 + (x^2 - 2)^2 = 4.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Clearly, $x = 0$ is a double root and so the equation can have at most four nonzero solutions. We will first seek nonzero solutions in the interval $(-2, 2]$. To this end, let $x = 2\cos t$, where $t \in [0, \pi] \setminus \{\pi/2\}$.

Because

$$x^3 - 3x = 2(4\cos^3 t - 3\cos t) = 2\cos 3t$$

and

$$x^2 - 2 = 2(2\cos^2 t - 1) = 2\cos 2t$$

the equation becomes $4\cos^2 3t + 4\cos^2 2t = 4$ which can be rewritten as

$$2(1 + \cos 6t + 1 + \cos 4t) = 4.$$

This reduces to $\cos 6t + \cos 4t = 0$, that is $2\cos 5t \cos t = 0$, yielding $t = \pi/10, 3\pi/10, 7\pi/10, 9\pi/10$. We get four more distinct solutions, and thus we have found all the solutions. In conclusion, the solutions are $x = 0, 2\cos\frac{\pi}{10}, 2\cos\frac{3\pi}{10}, 2\cos\frac{7\pi}{10}, 2\cos\frac{9\pi}{10}$

Second solution by Henry Ricardo, Westchester Area Math Circle
Expanding the given equation, we get

$$x^6 - 5x^4 + 5x^2 = x^2(x^4 - 5x^2 + 5) = 0.$$

Thus $x = 0$ is a solution of multiplicity 2. Then, setting $X = x^2$, we find that the solutions of $X^2 - 5X + 5 = 0$ are $X = \left(5 \pm \sqrt{5}\right)/2$, so that

$$x = \pm\sqrt{X} = \pm\frac{\sqrt{5 \pm \sqrt{5}}}{\sqrt{2}} = \pm\frac{\sqrt{10 \pm 2\sqrt{5}}}{2}.$$

Therefore, the solutions of our original equation are

$$x = 0,\ 0,\ \frac{\sqrt{10+2\sqrt{5}}}{2},\ -\frac{\sqrt{10+2\sqrt{5}}}{2},\ \frac{\sqrt{10-2\sqrt{5}}}{2},\ -\frac{\sqrt{10-2\sqrt{5}}}{2}.$$

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Alex Bloom, Sega Math Academy; Kanav Talwar, Delhi Public School, Faridabad, India; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Adnan Ali, NIT Silchar, Assam, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Evripides P. Nastou, 6th High School, Nea Smyrni, Greece; Daniel Văcaru, Pitești, Romania; Ashley Simone, SUNY Brockport, USA; Fred Frederickson, Utah Valley University, USA; G. C. Greubel, Newport News, VA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Max Sachelarie, PA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nick Iliopoulos, AUTH, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Le Hoang Bao, Tien Giang, Vietnam; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

S554. Let a, b, c, x, y, z be positive real numbers such that

$$\frac{x}{b^2+c^2+x} + \frac{y}{c^2+a^2+y} + \frac{z}{a^2+b^2+z} \ge 1.$$

Prove that

$$ab + bc + ca \le x + y + z.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the given condition and using Cauchy-Schwarz inequality we have

$$\begin{aligned} 1 &\le \sum_{\text{cyc}} \frac{x}{b^2+c^2+x} = \sum_{\text{cyc}} \frac{x(2+\frac{a^2}{x})}{(b^2+c^2+x)(1+1+\frac{a^2}{x})} \\ &\le \sum_{\text{cyc}} \frac{2x+a^2}{(b+c+a)^2} \\ &= \frac{2(x+y+z)+a^2+b^2+c^2}{(a+b+c)^2}. \end{aligned}$$

It follows that

$$(a+b+c)^2 \le 2(x+y+z) + a^2 + b^2 + c^2$$

which is equivalent to

$$ab + bc + ca \le x + y + z$$

as desired.

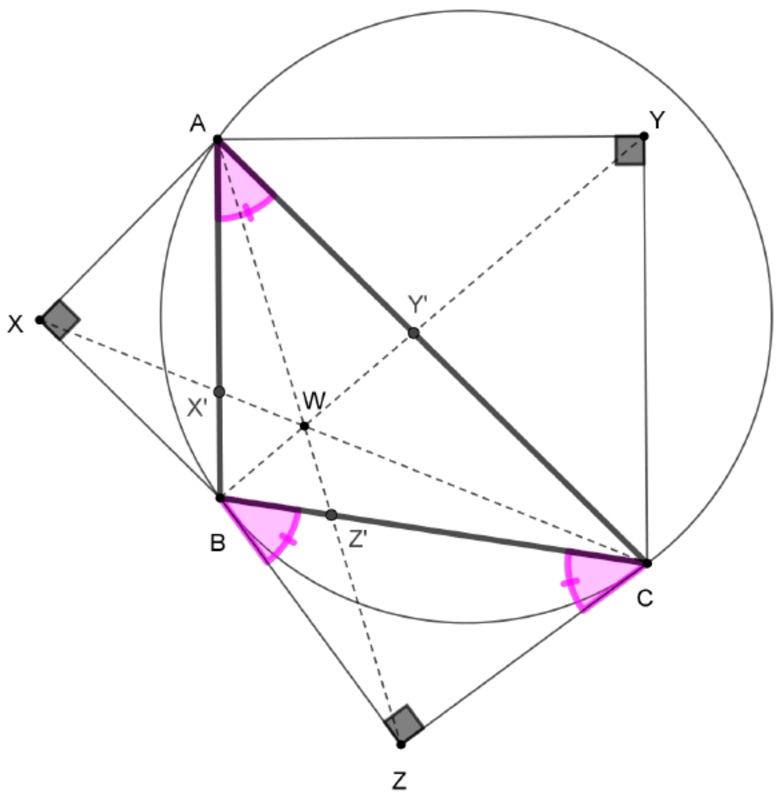
Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Arkady Alt, San Jose, CA, USA.

S555. Let ABC be a scalene triangle. We construct externally to $\triangle ABC$ the isosceles triangles XAB, YAC and ZBC such that: $\angle AXB = \angle AYC = 90^\circ$ and $\angle ZBC = \angle ZCB = \angle BAC$. Knowing that BY, CX and AZ are concurrent, find $\angle BAC$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by the author

Notations: $R = \triangle ABC$ circumradius, $CX \cap AB = \{X'\}$, $BY \cap AC = \{Y'\}$, $AZ \cap BC = \{Z'\}$.



From $\measuredangle(ZBC) = \measuredangle(ZCB) = \measuredangle(BAC)$ results that BZ and CZ are tangents to ABC circumcircle.

So, AZ is symmedian in $\triangle ABC \Rightarrow \dfrac{BZ'}{Z'C} = \dfrac{AB^2}{AC^2}$.

$$\frac{Y'A}{Y'C} = \frac{\sigma(AYY')}{\sigma(CYY')} = \frac{AY \cdot YY' \cdot \sin(AYB) \cdot \frac{1}{2}}{YC \cdot YY' \cdot \sin(CYB) \cdot \frac{1}{2}} \text{ with } AY = YC \Rightarrow \frac{Y'A}{Y'C} = \frac{\sin(AYB)}{\sin(BYC)}$$

According to Sine Rule:

$$\left.\begin{array}{l} \triangle ABY: \dfrac{\sin(AYB)}{AB} = \dfrac{\sin(A+45^\circ)}{BY} \\ \triangle BCY: \dfrac{\sin(BYC)}{BC} = \dfrac{\sin(C+45^\circ)}{BY} \end{array}\right\} \Rightarrow \frac{\sin(AYB)}{\sin(BYC)} = \frac{AB \cdot \sin(A+45^\circ)}{BC \cdot \sin(C+45^\circ)}$$

$$\frac{AX'}{X'B} = \frac{AC \cdot \sin(A+45^\circ)}{BC \cdot \sin(B+45^\circ)}$$

$AZ \cap BY \cap CX = \{W\} \Rightarrow$ according to Ceva's theorem

$$\Rightarrow \frac{BY'}{Y'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AX'}{X'B} = 1 \Rightarrow \frac{AB^2}{AC^2} \cdot \frac{BC \cdot \sin(C + 45^\circ)}{AB \cdot \sin(A + 45^\circ)} \cdot \frac{AC \cdot \sin(A + 45^\circ)}{BC \cdot \sin(B + 45^\circ)} = 1 \Rightarrow$$

$$AB \cdot \sin(C + 45^\circ) = AC \cdot \sin(B + 45^\circ) \quad (1)$$

$$AB = 2R \cdot \sin C \quad (2)$$

$$AC = 2R \cdot \sin B \quad (3)$$

(1) with (2) and (3) $\Rightarrow 2R \sin C \cdot \sin(C + 45^\circ) = 2R \sin B \cdot \sin(B + 45^\circ) \Rightarrow$

$$\sin C \sin(C + 45^\circ) = \sin B \sin(B + 45^\circ) \Leftrightarrow \cos 45^\circ - \cos(2C + 45^\circ) = \cos 45^\circ - \cos(2B + 45^\circ) \Rightarrow$$

$$\cos(2C + 45^\circ) = \cos(2B + 45^\circ)$$

ABC is non-isosceles triangle $\Rightarrow 2\measuredangle B + 45^\circ + 2\measuredangle C + 45^\circ = 360^\circ \Rightarrow \measuredangle B + \measuredangle C = 135^\circ$
So,

$$\measuredangle A = 180^\circ - 135^\circ = 45^\circ$$

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Spyros Kallias, Volos, Greece; Kanav Talwar, Delhi Public School, Faridabad, India; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam; Titu Zvonaru, Comănești, Romania; Adnan Ali, NIT Silchar, Assam, India; Max Sachelarie, PA, USA.

S556. Let a, b, c be positive real numbers such that $a + b \leq 3c$. Find the maximum possible value of

$$\left(\frac{a}{6b+c}+\frac{a}{b+6c}\right)\left(\frac{b}{6c+a}+\frac{b}{c+6a}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

The maximum is $\frac{49}{400}$.

We have

$$\frac{a}{6b+c}+\frac{a}{b+6c}=\frac{7a(b+c)}{6b^2+37bc+6c^2}$$

and

$$6b^2+37bc+6c^2=6(b+c)^2+25bc\geq 10(b+c)\sqrt{6bc},$$

implying

$$\frac{a}{6b+c}+\frac{a}{b+6c}\leq\frac{7a}{10\sqrt{6bc}}.$$

Similarly,

$$\frac{b}{6c+a}+\frac{b}{c+6a}\leq\frac{7b}{10\sqrt{6ca}}.$$

Hence

$$\left(\frac{a}{6b+c}+\frac{a}{b+6c}\right)\left(\frac{b}{6c+a}+\frac{b}{c+6a}\right)\leq\frac{49\sqrt{ab}}{100(6c)}\leq\frac{49}{400}.$$

as $2\sqrt{ab}\leq a+b\leq 3c$.

For the equality case we must have $a=b$, $6(b+c)^2=25bc$, and $6(c+a)^2=25ca$, implying $(3b-2c)(2b-3c)=0$ and $(3c-2a)(2c-3a)=0$, which, together with the condition $a+b\leq 3c$, yields $a=b=3c/2$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

S557. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{8a^2}{b+c+2} + \frac{8b^2}{c+a+2} + \frac{8c^2}{a+b+2} + 33 \geq 13(a+b+c).$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

The inequality can be rewrite as follows:

$$\sum_{cyc} \left(\frac{4a^2}{b+c+2} - b - c \right) \geq \frac{9}{2}(a+b+c) - \frac{33}{2},$$

$$\sum_{cyc} \frac{4a^2 - (b+c)^2}{b+c+2} - \sum_{cyc} \frac{2(b+c)}{b+c+2} \geq \frac{9}{2}(a+b+c) - \frac{33}{2}.$$

By the AM-GM Inequality, we have

$$2(b+c) \leq \frac{(b+c+2)^2}{4},$$

so it suffices to prove that

$$\sum_{cyc} \frac{4a^2 - (b+c)^2}{b+c+2} - \sum_{cyc} \frac{b+c+2}{4} \geq \frac{9}{2}(a+b+c) - \frac{33}{2},$$

that is

$$\sum_{cyc} \frac{4a^2 - (b+c)^2}{b+c+2} \geq 5(a+b+c-3),$$

$$\sum_{cyc} \frac{(a-b+a-c)(2a+b+c)}{b+c+2} \geq 5(a+b+c-3),$$

$$\sum_{cyc} (a-b) \left(\frac{2a+b+c}{b+c+2} - \frac{2b+c+a}{c+a+2} \right) \geq 5(a+b+c-3),$$

$$\sum_{cyc} \frac{2(a+b+c+1)(a-b)^2}{(b+c+2)(c+a+2)} \geq 5(a+b+c-3),$$

$$\sum_{cyc} \frac{2(a+b+c+1)(a-b)^2}{(b+c+2)(c+a+2)} \geq 5 \cdot \frac{(a+b+c)^2 - 9}{a+b+c+3},$$

$$\sum_{cyc} \frac{(a-b)^2}{(b+c+2)(c+a+2)} \geq \frac{5}{4} \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a+b+c+1)(a+b+c+3)},$$

$$\sum_{cyc} \left(\frac{1}{(b+c+2)(c+a+2)} - \frac{5}{4(a+b+c+1)(a+b+c+3)} \right) (a-b)^2 \geq 0,$$

$$\sum_{cyc} S_c (a-b)^2 \geq 0.$$

Without loss of generality, we may assume that $a \geq b \geq c$. Certainly, $S_c \geq S_b \geq S_a$. Let's study the sign of S_b. Since

$$\begin{aligned}
&4(a+b+c+1)(a+b+c+3)-5(a+b+2)(b+c+2)\\
=&4(a^2+c^2)-b^2+3(ab+bc+ca)+6(a+c)-4b-8\\
=&a^2-b^2+3a^2+4c^2+4(a-b)+2a+6c+1 \geq 0,
\end{aligned}$$

we deduce that $S_b \geq 0$. Therefore

$$\begin{aligned}
\sum_{cyc} S_a(b-c)^2 =& S_a(b-c)^2+S_b(a-b+b-c)^2+S_c(a-b)^2\\
\geq& S_a(b-c)^2+S_b(a-b)^2+S_b(b-c)^2+S_c(a-b)^2\\
=&(S_a+S_b)(b-c)^2+(S_b+S_c)(a-b)^2.
\end{aligned}$$

Hence, it remains to prove that $S_a+S_b \geq 0$, i.e.

$$\frac{1}{a+b+2}\left(\frac{1}{c+a+2}+\frac{1}{b+c+2}\right) \geq \frac{5}{2(a+b+c+1)(a+b+c+3)}.$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}
\frac{1}{a+b+2}\left(\frac{1}{c+a+2}+\frac{1}{b+c+2}\right) \geq& \frac{4}{(a+b+2)(a+b+2c+4)}\\
\geq& \frac{4}{(a+b+c+2)(a+b+c+4)}\\
\geq& \frac{5}{2(a+b+c+1)(a+b+c+3)}.
\end{aligned}$$

The last inequalities are equivalent to

$$(a+b+c+2)(a+b+c+4) \geq (a+b+2)(a+b+2c+4) \iff c(c+2) \geq 0,$$

and

$$8(a+b+c+1)(a+b+c+3) \geq 5(a+b+c+2)(a+b+c+4) \iff (a+b+c-2)(3a+3b+3c+8) \geq 0$$

which is clearly true because $a+b+c \geq \sqrt{3(ab+bc+ca)} = 3$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

S558. Let ABC be a scalene triangle. Let A_1 be the symmetric of A with respect to N — the center of the nine-point circle of $\triangle ABC$. If A_1 lies on the circumcircle of $\triangle ABC$, then calculate $\angle BAC$.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution by the author

Let O, I, H, G, M be the center of circumcircle, center of incircle, orthocenter, centroid of $\triangle ABC$ and midpoint of BC respectively.

It is well known that N lies on the Euler Line HGO and $HN = NO$; $HG = 2GO$; $AG = 2GM$.

$AN = NA_1, HN = NO$ hence AHA_1O is a parallelogram. It follows that $AH \parallel OA_1$. But $AH \perp BC$ so $OA_1 \perp BC$, or A_1 lies on the perpendicular bisector OM of BC. Let OM intersect the circumcircle Γ of $\triangle ABC$ at points $D \in \overset{\frown}{BC}$ and $E \in \overset{\frown}{BAC}$ (see Figure 1). The point $A_1 \in \Gamma$ hence either $A_1 \equiv D$ or $A_1 \equiv E$.

N is the midpoint of the chord AA_1, hence ON is the perpendicular bisector of AA_1. It follows that $\triangle AHN \cong \triangle AON \cong \triangle A_1ON \cong \triangle A_1HN$, $AH = AO = A_1O = A_1H$.

$\triangle AHG \sim \triangle MOG$ so $AH : OM = AG : GM = 2 : 1$.

$$\frac{OM}{OA_1} = \frac{OM}{AH} = \frac{1}{2}$$

$OM = \frac{1}{2}OA_1 = MA_1$; $OA_1 \perp BC$ so A_1 is the symmetric point of O with respect to M. It follows that $BA_1 = BO = OA_1 = OC = CA_1 = R$, i.e. the triangles BOA_1 and COA_1 are equilateral triangles and $\angle BOA_1 = \angle A_1OC = 60°$; $\angle BOC = \angle BA_1C = 120°$.

Case 1. $A_1 \equiv D$

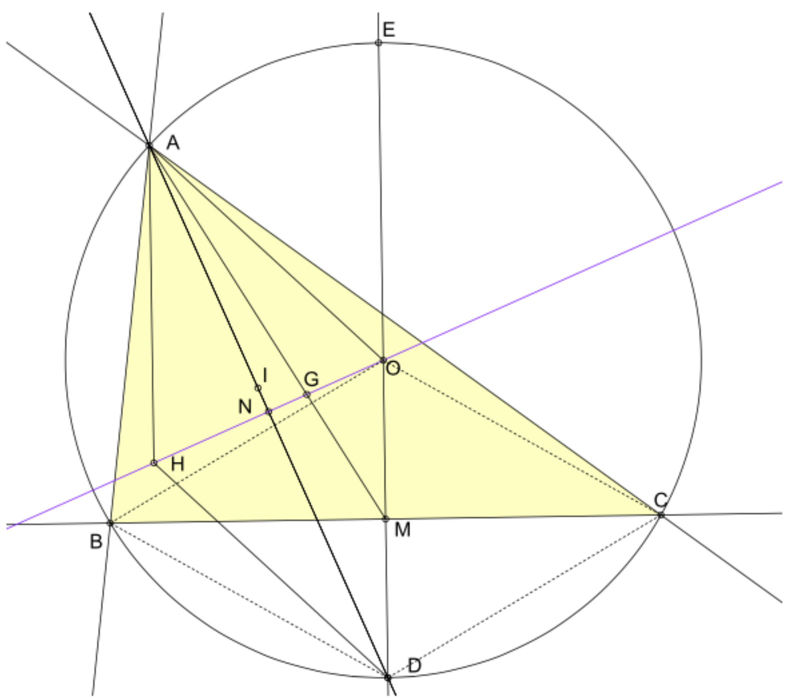


Figura 1: $A_1 \equiv D$

The points A_1, N lie on the angle bisector AI.

$$\angle BAC = \frac{1}{2}\angle BOC = 60° \tag{4}$$

Case 2. $A_1 \equiv E$

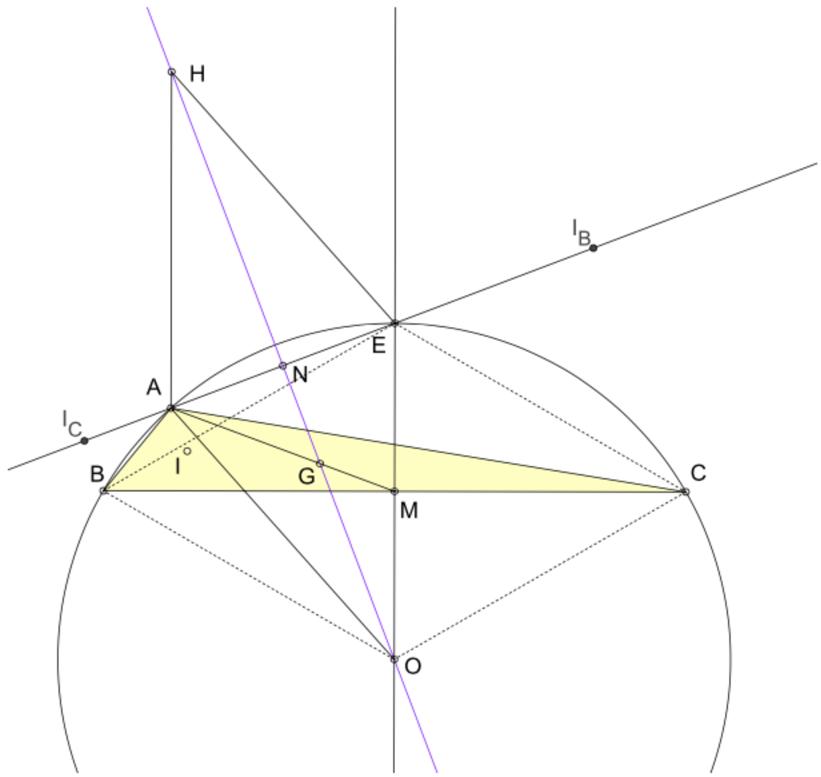


Figura 2: $A_1 \equiv E$

The points E, N lie on the external angle bisector $I_C A I_B$ of $\angle BAC$. The quadrilateral $BAEC$ is a cyclic quadrilateral.

$$\angle BAC = \angle BEC = \angle BA_1C = 120^\circ \tag{5}$$

In conclusion, from (1) and (2) : $\angle BAC = 60^\circ$ or $\angle BAC = 120^\circ$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Kamran Mehdiyev, Baku, Azerbaijan; Spyros Kallias, Volos, Greece; Kanav Talwar, Delhi Public School, Faridabad, India; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam; Adnan Ali, NIT Silchar, Assam, India; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Max Sachelarie, PA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Titu Zvonaru, Comănești, Romania.

Undergraduate problems

U553. Let A be an $n\times n$ matrix such that $A^4 = \mathrm{I}_n$. Prove that $A^2+(A+\mathrm{I}_n)^2$ and $A^2+(A-\mathrm{I}_n)^2$ are invertible.

Proposed by Adrian Andreescu, University of Texas at Dallas

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

First,

$$A^2 + (A + I_n)^2 = 2A^2 + 2A + I_n$$

and

$$(2A^2 + 2A + I_n)(A^2 - A + \frac{1}{2}I_n) = 2A^4 + \frac{1}{2}I_n = \frac{5}{2}I_n,$$

so $A^2 + (A + I_n)^2$ is invertible with

$$\left(A^2 + (A + I_n)^2\right)^{-1} = \frac{2}{5}A^2 - \frac{2}{5}A + \frac{1}{5}I_n.$$

Next,

$$A^2 + (A - I_n)^2 = 2A^2 - 2A + I_n$$

and

$$(2A^2 - 2A + I_n)(A^2 + A + \frac{1}{2}I_n) = 2A^4 + \frac{1}{2}I_n = \frac{5}{2}I_n,$$

so $A^2 + (A - I_n)^2$ is invertible with

$$\left(A^2 + (A - I_n)^2\right)^{-1} = \frac{2}{5}A^2 + \frac{2}{5}A + \frac{1}{5}I_n.$$

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Daniel Văcaru, Pitești, Romania; Adnan Ali, NIT Silchar, Assam, India; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Fred Frederickson, Utah Valley University, USA; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, USA; Ibrahim Suleiman, New York University Abu Dhabi, United Arab Emirates; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France.

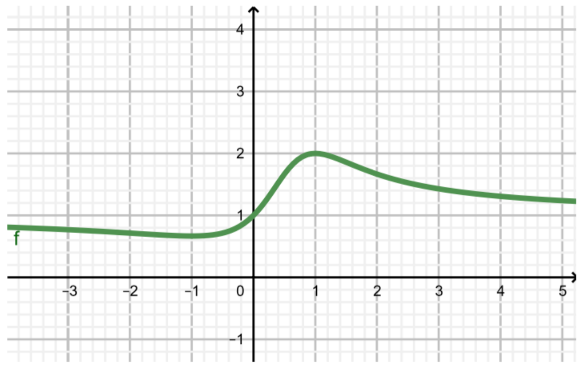
U554. Evaluate

$$\int_a^b \left\{ \frac{x^2+1}{x^2-x+1} \right\} dx$$

in terms of a and b, where $a < 0 < b$ and $\{t\}$ denotes the fractional part of t.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author



We have

$$\int_a^b \left\{ \frac{x^2+1}{x^2-x+1} \right\} \mathrm{d}x = \int_a^0 \left\{ \frac{x^2+1}{x^2-x+1} \right\} \mathrm{d}x + \int_0^1 \left\{ \frac{x^2+1}{x^2-x+1} \right\} \mathrm{d}x + \int_1^b \left\{ \frac{x^2+1}{x^2-x+1} \right\} \mathrm{d}x$$

$$= \int_a^0 \frac{x^2+1}{x^2-x+1} \mathrm{d}x + \int_0^1 \left(\frac{x^2+1}{x^2-x+1} - 1 \right) \mathrm{d}x + \int_1^b \left(\frac{x^2+1}{x^2-x+1} - 1 \right) \mathrm{d}x$$

$$= \int_a^0 \frac{x^2+1}{x^2-x+1} \mathrm{d}x + \int_0^b \frac{x}{x^2-x+1} \mathrm{d}x$$

$$= \int_a^0 \left(1 + \frac{x}{x^2-x+1} \right) \mathrm{d}x + \int_0^b \frac{x}{x^2-x+1} \mathrm{d}x$$

$$= x \Big|_a^0 + \int_a^b \frac{x}{x^2-x+1} \mathrm{d}x$$

$$= -a + \frac{1}{2} \int_a^b \left(\frac{2x-1}{x^2-x+1} + \frac{1}{x^2-x+1} \right) \mathrm{d}x$$

$$= -a + \frac{1}{2} \ln(x^2-x+1) \Big|_a^b + \frac{1}{2} \int_a^b \frac{\mathrm{d}x}{x^2-x+1} =$$

$$
\begin{aligned}
&= -a + \frac{1}{2}\ln\frac{b^2-b+1}{a^2-a+1} + \frac{1}{2}\int_a^b \frac{\mathrm{d}x}{(x-\frac{1}{2})^2+\frac{3}{4}} \\
&= -a + \frac{1}{2}\ln\frac{b^2-b+1}{a^2-a+1} + \frac{2}{3}\int_a^b \frac{\mathrm{d}x}{(\frac{2x-1}{\sqrt{3}})^2+1} \\
&= -a + \frac{1}{2}\ln\frac{b^2-b+1}{a^2-a+1} + \frac{1}{\sqrt{3}}\arctan(\frac{2x-1}{\sqrt{3}})\Big|_a^b \\
&= -a + \frac{1}{2}\ln\frac{b^2-b+1}{a^2-a+1} + \frac{1}{\sqrt{3}}\left(\arctan(\frac{2b-1}{\sqrt{3}}) - \arctan(\frac{2a-1}{\sqrt{3}})\right).
\end{aligned}
$$

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Daniel Văcaru, Pitești, Romania; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploieşti, Romania; S.Chandrasekhar, Chennai, India; Henry Ricardo, Westchester Area Math Circle, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U555. Let $f : [0, +\infty) \to [0, \infty)$ be a differentiable function such that $f(x)e^{f(x)} = x$, for all $x \geq 0$. Evaluate

$$\int_0^e f(x)\, dx.$$

Proposed by Prithvijit Chatraborty, Kolkata, India

Solution by Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA

Let $g(x) = xe^x$. Since $g'(x) = (x+1)e^x > 0$, and $g(0) = 0$, g is increasing, hence one-to-one and $g^{-1} : [0, \infty) \to [0, \infty)$ exists. Then,

$$f(x)e^{f(x)} = x \quad \Longleftrightarrow \quad g(f(x)) = x \quad \Longleftrightarrow \quad f(x) = g^{-1}(x),$$

i.e. f is the inverse function of g. Consequently, introducing the substitution $x = g(t)$ we have

$$\begin{aligned} \int_0^e f(x)\, dx &= \int_0^e g^{-1}(x)\, dx = \int_0^1 g^{-1}(g(t))g'(t)\, dt \\ &= \int_0^1 t(t+1)\, e^t\, dt = \int_0^1 (t^2 + t)e^t\, dt \\ &= (t^2 - t + 1)\, e^t \Big|_0^1 = e - 1, \end{aligned}$$

where the last integration was done by integrating by parts
$(u = t^2 + t,\ dv = e^t\, dt)$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adnan Ali, NIT Silchar, Assam, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Fred Frederickson, Utah Valley University, USA; Henry Ricardo, Westchester Area Math Circle, USA; G. C. Greubel, Newport News, VA, USA; Ibrahim Suleiman, New York University Abu Dhabi, United Arab Emirates; Moubinool Omarjee, Lycée Henri IV, Paris, France; Arkady Alt, San Jose, CA, USA.

U556. Find the volume of the solid obtained by rotating a unit cube about an axis connecting opposite vertices.

Proposed by Li Zhou, Polk State College

First solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Orient the axis of rotation along the y-axis with the lower vertex of the cube at the origin and the upper vertex at $(0, \sqrt{3})$. By symmetry, the volume of the solid obtained by rotating a unit cube about this axis can be obtained by computing the volume associated with rotating the lower half of the cube and then multiplying by two. For $0 \le y \le \sqrt{3}/3$, the square of the radius of the disk that results from rotation is given by $2y^2$, while for $\sqrt{3}/3 \le y \le \sqrt{3}/2$, the square of the radius of the disk that results from rotation is given by $2(y^2 - \sqrt{3}y + 1)$ – see below. The volume of the solid is then

$$\begin{aligned} V &= 2\pi\left(\int_0^{\sqrt{3}/3} 2y^2\,dy + \int_{\sqrt{3}/3}^{\sqrt{3}/2} 2(y^2 - \sqrt{3}y + 1)\,dy\right) \\ &= 2\pi\left(\frac{2}{3}y^3\Big|_0^{\sqrt{3}/3} + \left(\frac{2}{3}y^3 - \sqrt{3}y^2 + 2y\right)\Big|_{\sqrt{3}/3}^{\sqrt{3}/2}\right) \\ &= 2\pi\left(\frac{2\sqrt{3}}{27} + \frac{\sqrt{3}}{2} - \frac{11\sqrt{3}}{27}\right) = \frac{\pi\sqrt{3}}{3}. \end{aligned}$$

In the diagram below, suppose the point along the indicated diagonal (which corresponds to the axis of rotation) that is nearest to the point $(a, 0, 0)$ is the point (b, b, b). Then the vector $\langle b - a, b, b\rangle$ must be orthogonal to the vector $\langle 1, 1, 1\rangle$. Thus,

$$\langle b - a, b, b\rangle \cdot \langle 1, 1, 1\rangle = 0, \quad \text{or} \quad 3b = a.$$

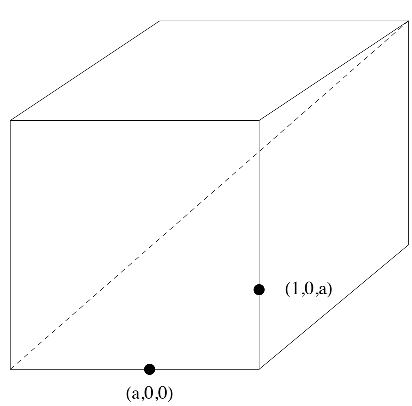
The square of the distance from (b, b, b) to $(3b, 0, 0)$ is $6b^2$. As $0 \le a \le 1$, $0 \le b \le 1/3$. Moreover, $y = \sqrt{3}b$, so, for $0 \le y \le \sqrt{3}/3$, the square of the radius of the disk that results from rotation is given by $2y^2$. Next, consider the point $(1, 0, a)$. Suppose the point along the indicated diagonal that is nearest to the point $(1, 0, a)$ is the point (b, b, b). Then the vector $\langle b - 1, b, b - a\rangle$ must be orthogonal to the vector $\langle 1, 1, 1\rangle$. Thus,

$$\langle b - 1, b, b - a\rangle \cdot \langle 1, 1, 1\rangle = 0, \quad \text{or} \quad 3b - 1 = a.$$

The square of the distance from (b, b, b) to $(1, 0, 3b - 1)$ is

$$(b - 1)^2 + b^2 + (1 - 2b)^2 = 6b^2 - 6b + 2 = 2(3b^2 - 3b + 1).$$

For $0 \le a \le 1/2$, which is sufficient to consider based on symmetry, $1/3 \le b \le 1/2$. Thus, for $\sqrt{3}/3 \le y \le \sqrt{3}/2$, the square of the radius of the disk that results from rotation is given by $2(y^2 - \sqrt{3}y + 1)$.



Second solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Let O be the center, AB be an edge, and AC be a diameter of the cube. Let m be the distance from O to the midpoint of AB. It is easy to see that $r = OA = \sqrt{3}/2$ and $m = 1/\sqrt{2}$. Let h be the distance from B to the line AC, then $hr = m$, thus $h = m/r = \sqrt{2/3}$. Let S be the resulting solid of rotating the cube about the axis AC. Clearly, S has two circular cones with vertices A and C, and their total volume is

$$2\left(\frac{1}{3}\pi h^2\sqrt{1-h^2}\right) = \frac{4\pi}{9\sqrt{3}}.$$

The middle part of S is a hyperboloid, since its surface is doubly ruled by the three pairs of edges of the cube. Set up a coordinate system with O as the origin, $A = (r, 0)$, and $B = (\sqrt{r^2 - h^2}, h)$. Considering the the hyperbola $y^2/m^2 - x^2/b^2 = 1$ passing through B, we get $h^2/m^2 - (r^2 - h^2)/b^2 = 1$, thus $b = 1/2$. Since the hyperpoloid is the same as rotating the hyperbola about the x-axis, its volume equals

$$2\int_0^{\sqrt{r^2-h^2}} \pi m^2\left(1 + \frac{x^2}{b^2}\right) dx = \frac{5\pi}{9\sqrt{3}}.$$

Therefore, the volume of S is $\pi/\sqrt{3} = 1.81\ldots$. Notice that this is $2/3$ of the sphere of the same diameter. For a video of such a spinning cube, see `https://www.youtube.com/watch?v=wsMah8e6bYA`.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania.

U557. Evaluate

$$\int_2^3 x\, e^x (\ln x + 1)\, dx.$$

Proposed by Toyesh Prakash Sharma, St. C.F. Andrews School, Agra, India

Solution by G. C. Greubel, Newport News, VA, USA
Consider integration-by-parts on the integrals

$$I_1 = \int x\, e^x\, dx$$
$$I_2 = \int e^x\, x\, \ln x\, dx$$

yields

$$\int x\, e^x\, dx = (x-1)\, e^x$$

and

$$\begin{aligned}\int e^x\, x\, \ln x\, dx &= x\, \ln x\, e^x - \int (1+\ln x)\, e^x\, dx \\ &= (x\, \ln x - 1)\, e^x - \ln x\, e^x - \int e^x \frac{dx}{x} \\ &= ((x-1)\, \ln x - 1)\, e^x - \mathrm{Ei}(x).\end{aligned}$$

Now,

$$\int_a^b x\, e^x (\ln x + 1)\, dx = ((b-1)\, \ln b - 1)\, e^b - ((a-1)\, \ln a - 1)\, e^a + \mathrm{Ei}(a) - \mathrm{Ei}(b)$$

and setting $a = 2$ and $b = 3$ gives the desired result.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adnan Ali, NIT Silchar, Assam, India; Le Hoang Bao, Tien Giang, Vietnam; Corneliu Mănescu-Avram, Ploieşti, Romania; Henry Ricardo, Westchester Area Math Circle, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France.

U558. For every polynomial $P(x) = c_0 + c_1 x + ... + c_n x^n$ define its reciprocal $\tilde{P}$ by $\tilde{P} = c_0 x^n + c_1 x^{n-1} + ... + c_n$. Let $f(x) = a_r x^{d_r} + ... + a_0 x^{d_0}$ be a polynomial with integer coefficients and $n = d_r > d_{r-1} > ... > d_0 = 0$. Let $g(x) = b_s x^{e_s} + ... + b_0$ be a polynomial with positive integer coefficients and $n = e_s > e_{s-1} > ... > e_0 = 0$. Prove that if $f(x)\tilde{f}(x) = g(x)\tilde{g}(x)$ and $a_0 = a_1 = ... = a_r = 1$, then $r = s$ and $b_0 = b_1 = ... = b_s = 1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

The polynomials $f(x)\tilde{f}(x), g(x)\tilde{g}(x)$ are polynomials of degree at most $2n$. The coefficient of x^n in the former is equal to $a_0^2 + ... + a_r^2$ and this coefficient in the later polynomial is $b_0^2 + ... + b_s^2$. Then we can find that $a_0^2 + ... + a_r^2 = b_0^2 + ... + b_s^2$ Further, since $\tilde{f}(1) = f(1)$ and $g(1) = \tilde{g}(1)$ we find that $f(1)^2 = g(1)^2$. That is,

$$\left(\sum_{j=0}^{s} b_j\right)^2 \leq \left(\sum_{j=0}^{s} b_j^2\right)^2 = \left(\sum_{j=0}^{r} a_j^2\right)^2 = \left(\sum_{j=0}^{r} a_j\right)^2 = \left(\sum_{j=0}^{s} b_j\right)^2$$

We deduce that the equality holds in the above inequality. Since $b_i > 0$ for each i, we find that $b_i = 1$. Thus, $r = s$.

O553. Let ABC be a triangle with $AB = AC$ and let M be the midpoint of BC. Circle ω is tangent to BC at M and lies outside of triangle ABC. Circle Ω passes through A, is internally tangent to ω, and its center lies on AM. Circle γ is internally tangent to circle Ω, touches segment BC and the extention of line AC. Through A tangents to ω and γ are drawn intersecting segment MC at points K and L, respectively. Prove that the inradius of triangle ABL is twice the inradius of triangle AKC.

Proposed by Waldemar Pompe, Warzaw, Poland

Solution by the author

Assume the incircle of triangle ABL touches segment BC at X, and the incircle of triangle AKC touches BC at Y. Since $\angle ABC = \angle ACB$, it suffices to show that $BX = 2CY$.

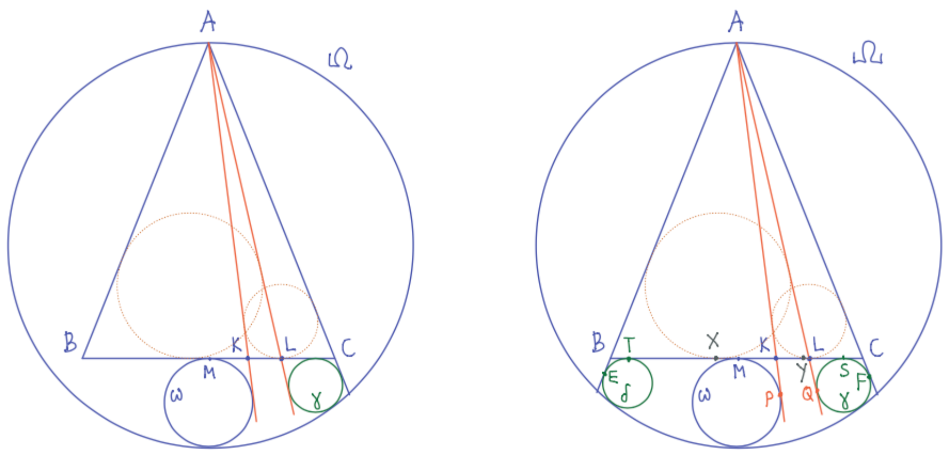
Let AP and AQ be the tangent segments to circles ω and γ, respectively. Assume that γ touches BC at S and let δ be a circle symmetric to γ with respect to line AM. Let T be a touching point of δ and BC. Finally, let E and F be the touching points of δ and γ with AB and AC, respectively.

An inversion with center A that sends line BC to circle Ω maps circle γ to γ and ω to ω. It follows that the inversion maps P to P and Q to Q. Therefore, $AP = AQ = AE = AF$.

Now, observe that

$$BX = \frac{1}{2}(AE + TS - AQ) = \frac{1}{2}TS, \quad CY = \frac{1}{2}(AF + SM - AP) = \frac{1}{2}SM.$$

Since $SM = \frac{1}{2}TS$, it follows that $BX = 2CY$, which completes the proof.



Also solved by Theo Koupelis, Broward College, Davie, FL, USA.

O554. Let a, b, c, d be real numbers such that $|a|, |b|, |c|, |d| \ge 1$ and

$$a+b+c+d+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0.$$

Prove that

$$a+b+c+d \le 2\sqrt{2}.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Due to symmmetry, without loss of generality, we may assume that $a \ge b \ge c \ge d$. It is clear that $a \ge 1$ and $d \le -1$.

If we denote $x=\frac{1}{2}\left(a+\frac{1}{a}\right)$, $y=\frac{1}{2}\left(b+\frac{1}{b}\right)$, $z=\frac{1}{2}\left(c+\frac{1}{c}\right)$, $t=\frac{1}{2}\left(d+\frac{1}{d}\right)$, then $x+y+z+t=0$, $x\ge y\ge z\ge t$ and $x,y,z,t \in (-\infty,-1]\cup[1,\infty)$. Also, we have

$$\frac{1}{4}\left(a-\frac{1}{a}\right)^2 = x^2-1 \Longrightarrow \frac{1}{2}\left(a-\frac{1}{a}\right) = \pm\sqrt{x^2-1},$$

and similarly for b,c,d. Since $d\le -1$ and $a \ge 1$ it follows that $\frac{1}{2}\left(d-\frac{1}{d}\right) = -\sqrt{t^2-1}$, $\frac{1}{2}\left(a-\frac{1}{a}\right)=\sqrt{x^2-1}$. Hence,

$$\begin{aligned} a+b+c+d =& \frac{1}{2}\left(a-\frac{1}{a}+b-\frac{1}{b}+c-\frac{1}{c}+d-\frac{1}{d}\right) \\ =& \sqrt{x^2-1}\pm\sqrt{y^2-1}\pm\sqrt{z^2-1}-\sqrt{t^2-1}. \end{aligned}$$

We have three cases:

Case 1. If $z \ge 1$, then we need to show that

$$\sqrt{x^2-1}+\sqrt{y^2-1}+\sqrt{z^2-1}-\sqrt{t^2-1}\le 2\sqrt{2}.$$

Consider the function f defined as $f(x)=\sqrt{x^2-1}$ on $[1,\infty)$. Since

$$f''(x) = -\frac{1}{(x^2-1)^{3/2}}$$

it follows that f is concave on the interval $[1,\infty)$. We have

$$(x,y,z) \succ \left(\frac{x+y+z}{3},\frac{x+y+z}{3},\frac{x+y+z}{3}\right),$$

so by Karamata's Inequality,

$$\begin{aligned} \sqrt{x^2-1}+\sqrt{y^2-1}+\sqrt{z^2-1} \le& 3\sqrt{\left(\frac{x+y+z}{3}\right)^2-1} \\ & \sqrt{t^2-9}\le\sqrt{t^2-1}, \end{aligned}$$

that is

$$\sqrt{x^2-1}+\sqrt{y^2-1}+\sqrt{z^2-1}-\sqrt{t^2-1}\le 0 < 2\sqrt{2}.$$

Case 2. If $y \geq 1$, $z \leq -1$, then we need to show that

$$\sqrt{x^2-1}+\sqrt{y^2-1}-\sqrt{z^2-1}-\sqrt{t^2-1} \leq 2\sqrt{2}.$$

Let $s = x + y = -z - t \geq 2$. We have

$$(x, y) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}\right), \quad (-t-z-1, 1) \succ (-t, -z),$$

so by Karamata's Inequality,

$$\sqrt{x^2-1}+\sqrt{y^2-1} \leq 2\sqrt{\left(\frac{x+y}{2}\right)^2-1} = \sqrt{s^2-4},$$

$$\sqrt{z^2-1}+\sqrt{t^2-1} \geq \sqrt{(-t-z-1)^2-1} = \sqrt{s^2-2s}.$$

It remains to show that

$$\sqrt{s^2-4}-\sqrt{s^2-2s} \leq 2\sqrt{2},$$

or

$$\sqrt{s-2}\left(\sqrt{s+2}-\sqrt{s}\right) \leq 2\sqrt{2},$$

or

$$\sqrt{s-2} \leq \sqrt{2(s+2)}+\sqrt{2s},$$

which is clearly true.

Case 3. If $y \leq -1$, then we need to show that

$$\sqrt{x^2-1}-\sqrt{y^2-1}-\sqrt{z^2-1}-\sqrt{t^2-1} \leq 2\sqrt{2}.$$

We have

$$(-y-z-t-2, 1, 1) \succ (-t, -z, -y),$$

so by Karamata's Inequality,

$$\begin{aligned}\sqrt{y^2-1}+\sqrt{z^2-1}+\sqrt{t^2-1} \geq& \sqrt{(-y-z-t-2)^2-1} \\ =& \sqrt{(x-2)^2-1} = \sqrt{x^2-4x+3}.\end{aligned}$$

Hence, we need to prove that

$$\sqrt{x^2-1} \leq \sqrt{x^2-4x+3}+2\sqrt{2}.$$

Squaring both sides yields the equivalent inequality

$$x-3 \leq \sqrt{2(x-1)(x-3)},$$

which is true because $x = -y - z - t \geq 3$ and $2(x-1) > (x-3)$. The equality holds when $x = 3$, $y = z = t = -1$ which means $a = 3 + 2\sqrt{2}$, $b = c = d = -1$.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA.

O555. Let $ABCD$ be a square of side length 1. Point X lies on the smaller arc DA of the circumcircle of square $ABCD$. Let r_1, r_2, r_3, r_4 be the inradii of triangles XDA, XAB, XBC, XCD, respectively. Determine all possible values of

$$\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}$$

as X varies on the smaller arc DA of the circumcircle of square $ABCD$.

Proposed by Waldemar Pompe, Warzaw, Poland

Solution by the author

Let XAB be a triangle with $\angle AXB = \alpha$ and let A^*, B^* be the images of points A, B, respectively, under inversion with center X and radius 1. If r is the inradius of triangle XAB, then

$$\frac{\sin\alpha}{r} = XA^* + XB^* + A^*B^* .$$

Indeed, if r^* is the inradius of triangle XA^*B^*, then

$$\frac{r^*}{r} = \frac{XB^* \cdot XA^*}{XA \cdot XA^*} = \frac{2[XA^*B^*]}{\sin\alpha} = \frac{(XA^* + XB^* + A^*B^*)r^*}{\sin\alpha},$$

which is the claimed formula. Similarly, we show that if h is the altitude of triangle XAB taken from X, then

$$\frac{\sin\alpha}{h} = A^*B^* .$$

To solve the problem, consider the inversion with center X and radius 1. Then the images A^*, B^*, C^*, D^* of points A, B, C, D respectively, lie in a line in that order. Then the above formulas give

$$\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} = \frac{2B^*C^*}{\sin 45^\circ} = \frac{2}{h},$$

where h is a distance from point X to line BC. As h ranges from 1 to $\frac{1}{2}(\sqrt{2}+1)$, $2/h$ ranges from $4/(\sqrt{2}+1)$ to 2.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Max Sachelarie, PA, USA.

O556. Let a, b, c be the sides of a triangle ABC. Prove that

$$\left(a^2 - bc\right)\cos\frac{B-C}{2} + \left(b^2 - ca\right)\cos\frac{C-A}{2} + \left(c^2 - ab\right)\cos\frac{A-B}{2} \geq 0.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Since $\cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2}$ the inequality becomes

$$\sum_{cyc}\left[a^2(b+c) - bc(b+c)\right]\frac{\sin\frac{A}{2}}{a} \geq 0. \tag{1}$$

Without loss of generality, assume that $a \geq b \geq c$ which implies $A \geq B \geq C$. Then, we have

$$\frac{1}{\cos\frac{A}{2}} \geq \frac{1}{\cos\frac{B}{2}} \geq \frac{1}{\cos\frac{C}{2}},$$

that is

$$\frac{\sin\frac{A}{2}}{a} \geq \frac{\sin\frac{B}{2}}{b} \geq \frac{\sin\frac{C}{2}}{c}.$$

Also, we have

$$a^2(b+c) \geq b^2(c+a) \geq c^2(a+b)$$

and

$$-bc(b+c) \geq -ac(a+c) \geq -ab(a+b),$$

so

$$a^2(b+c) - bc(b+c) \geq b^2(c+a) - ac(a+c) \geq c^2(a+b) - ab(a+b).$$

These being established, applying Chebyshev's Inequality, we get

$$3\sum_{cyc}\left[a^2(b+c) - bc(b+c)\right]\frac{\sin\frac{A}{2}}{a} \geq \sum_{cyc}\left[a^2(b+c) - bc(b+c)\right]\sum_{cyc}\frac{\sin\frac{A}{2}}{a}$$
$$= 0 \cdot \sum_{cyc}\frac{\sin\frac{A}{2}}{a} = 0$$

as desired.

Observation: Here `https://artofproblemsolving.com/community/c6h1142667p5374755` AoPS we have the following inequality:

Let S be the area of triangle ABC with length-side a, b, c . Prove that

$$ab\cos\frac{A-B}{2} + bc\cos\frac{B-C}{2} + ca\cos\frac{C-A}{2} \geq 4\sqrt{3}S.$$

From the above problem we deduce that

$$c^2\cos\frac{A-B}{2} + a^2\cos\frac{B-C}{2} + b^2\cos\frac{C-A}{2} \geq 4\sqrt{3}S$$

which is an equivalent form of the problem S530 from Mathematical Reflections no 5-2020.

Also solved by Theo Koupelis, Broward College, Davie, FL, USA; Adnan Ali, NIT Silchar, Assam, India; Max Sachelarie, PA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

O557. Evaluate

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{2k+1}\binom{n}{2k}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

Using the fact that

$$\frac{1}{a+1}\binom{b}{a} = \frac{1}{b+1}\binom{b+1}{a+1}$$

and De Moivre's formula we have

$$\begin{aligned}
\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{2k+1}\binom{n}{2k} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n+1}\binom{n+1}{2k+1} \\
&= \frac{1}{i(n+1)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} i^{2k+1}\binom{n+1}{2k+1} \quad (\text{where } i^2 = -1) \\
&= \frac{1}{i(n+1)} \cdot \frac{(1+i)^{n+1} - (1-i)^{n+1}}{2} \\
&= \frac{\left(\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})\right)^{n+1} - \left(\sqrt{2}(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4})\right)^{n+1}}{2i(n+1)} \\
&= \frac{(\sqrt{2})^{n+1}\left(\cos\frac{(n+1)\pi}{4} + i\sin\frac{(n+1)\pi}{4}\right) - (\sqrt{2})^{n+1}\left(\cos\frac{-(n+1)\pi}{4} + i\sin\frac{-(n+1)\pi}{4}\right)}{2i(n+1)} \\
&= \frac{(\sqrt{2})^{n+1}\sin\frac{(n+1)\pi}{4}}{n+1}.
\end{aligned}$$

Also solved by Spyros Kallias, Volos, Greece; Kanav Talwar, Delhi Public School, Faridabad, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adnan Ali, NIT Silchar, Assam, India; G. C. Greubel, Newport News, VA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Henry Ricardo, Westchester Area Math Circle, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Le Hoang Bao, Tien Giang, Vietnam; Nguyen Huy Hoang, Le Quy Don High School, Binh Dinh Province, Vietnam.

O558. Let $\{x\}$ be the fractional part of the real number x. Prove that for all positive integers n there are pairwise distinct rational numbers $x_1, ..., x_n > n$ such that $\{x_i x_j\} \in (\frac{1}{2}, \frac{5}{6})$ for $1 \le i, j \le n$.

Proposed by Titu Andreescu, Dallas, USA and Navid Safaei, Tehran, Iran

Solution by the authors

Start by choosing pairwise different and large enough prime numbers $p_1, ..., p_n$ congruent to 1 modulo 8. For instance, we can take p_i an arbitrary prime divisor of $2^{2^{i+k}} + 1$ for some sufficiently large k. Next, choose integers $m_i \in (0, 3p_i)$ such that $2m_i \equiv 3 \pmod{p_i}$ and $m_i \equiv 1 \pmod 3$. This is possible by the Chinese remainder theorem, since $p_i > 3$. For any i the congruence $3x^2 \equiv m_i \pmod{p_i^2}$ has solutions: by Hensel's lemma (which is of course elementary in this case) it suffices to argue modulo p_i. But then it suffices to show that the congruence $2x^2 \equiv 1 \pmod{p_i}$ has solutions, which is true since $p_i \equiv 1 \pmod 8$. Note that we can prove this very easily in our context: if p divides $2^{2^a} + 1$, then $2^{2^{a-2}} + 2^{-2^{a-2}}$ is a solution of the congruence $z^2 \equiv 2 \pmod p$. Now, another application of the Chinese remainder theorem yields a positive integer N such that $3N^2 \equiv 1 \pmod{p_i^2}$ for all i. We will choose $N > p_1 ... p_n \cdot n$.

With the above data, consider $x_i = \frac{N}{p_i}$. Then $x_i > n$ and we will show that $\{x_i x_j\} \in (\frac{1}{2}, \frac{5}{6})$ for all i, j. Suppose first that $i \ne j$. Note that $2N^2 \equiv 1 \pmod{p_i}$ for all i, thus $2N^2 = 1 + (2k+1)p_i p_j$ for some positive integer k. Then

$$\{x_i x_j\} = \{\frac{2N^2}{2p_i p_j}\} = \{\frac{1 + p_i p_j + 2k p_i p_j}{2p_i p_j}\} = \frac{1 + p_i p_j}{2p_i p_j} = \frac{1}{2} + \frac{1}{2p_i p_j} \in (\frac{1}{2}, \frac{5}{6}).$$

Suppose now that $i = j$ and write $3N^2 = m_i + k_i p_i^2$. Taking this relation modulo 3 yields $0 \equiv 1 + k_i \pmod 3$, thus $3N^2 = m_i + (3l_i + 2)p_i^2$ for some integer l_i. As above, this implies

$$\{x_i^2\} = \{\frac{3N^2}{3p_i^2}\} = \frac{m_i + 2p_i^2}{3p_i^2} = \frac{2}{3} + \frac{m_i}{3p_i^2}.$$

Since $0 < m_i < 3p_i$ and p_i is sufficiently large, the last expression belongs to $(\frac{1}{2}, \frac{5}{6})$. The result follows.

Also solved by Kanav Talwar, Delhi Public School, Faridabad, India; Adnan Ali, NIT Silchar, Assam, India.