Junior Problems

J559. Let

$$a_n = 1 - \frac{2n^2}{1+\sqrt{1+4n^4}}\ , \quad n = 1, 2, 3, ...$$

Prove that $\sqrt{a_1} + 2\sqrt{a_2} + \cdots + 20\sqrt{a_{20}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J560. Let a, b, c be positive real numbers. Prove that

$$\frac{2}{a^2} + \frac{5}{b^2} + \frac{45}{c^2} > \frac{16}{(a+b)^2} + \frac{24}{(b+c)^2} + \frac{48}{(c+a)^2}.$$

Proposed by Kartik Vedula, James S. Rickards High School, Tallahassee, USA

J561. Solve in nonzero real numbers the system of equations:

$$x - \frac{1}{x} + \frac{2}{y} = y - \frac{1}{y} + \frac{2}{z} = z - \frac{1}{z} + \frac{2}{x} = 0.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J562. Let ABC be a triangle and let D, E, F be points on sides BC, CA, AB, respectively, such that AD, BE, CF are concurrent in X. Assume that the ratios $\frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB}$ are in the interval $\left[\frac{1}{5}, 5\right]$ and that $\frac{BD}{DC} + \frac{CE}{EA} + \frac{AF}{FB} = \frac{31}{5}$. Evaluate

$$\frac{AX}{XD} + \frac{BX}{XE} + \frac{CX}{XF}.$$

Proposed by Mohammad Imran, India

J563. Let $a, b, c \geq 0$ be real numbers such that $ab + bc + ca = a + b + c > 0$. Prove that

$$1 \leq \frac{1}{1+2a} + \frac{1}{1+2b} + \frac{1}{1+2c} \leq \frac{7}{5}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J564. Find all complex numbers z such that for each real number a and each positive integer n

$$(\cos a + z \sin a)^n = \cos na + z \sin na$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Senior Problems

S559. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n \leq n$. Find the minimum of

$$\frac{1}{a_1} + \frac{1}{2a_2^2} + \cdots + \frac{1}{na_n^n}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S560. Let a, b, c be nonnegative real numbers such that $a + b + c = 2$. Prove that

$$\frac{a}{b^2+bc+c^2} + \frac{b}{c^2+ca+a^2} + \frac{c}{a^2+ab+b^2} + 8 \geq \frac{10}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, România

S561. Let p be a prime. Solve in positive integers the equation

$$\left(x^2-yz\right)^3 + \left(y^2-zx\right)^3 + \left(z^2-xy\right)^3 - 3\left(x^2-yz\right)\left(y^2-zx\right)\left(z^2-xy\right) = p^2.$$

Proposed by Alessandro Ventullo, Milan, Italy

S562. Let a, b, c, d be nonnegative real numbers. Prove that

$$(a+b+c+d)^3 + 9(abc+abd+acd+bcd) \geq 4(a+b+c+d)(ab+ac+ad+bc+bd+cd).$$

Proposed by An Zhenping, Xianyang Normal University, China

S563. Let a, b, c be distinct positive real numbers. Prove that at least one of the numbers

$$\left(a+\frac{1}{a}\right)^2(1-b^4); \quad \left(b+\frac{1}{b}\right)^2(1-c^4); \quad \left(c+\frac{1}{c}\right)^2(1-a^4)$$

is not equal to 4.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S564. Let x, y, z be nonegative real numbers. Prove that

$$\frac{x^3+y^3+z^3+3xyz}{\sum_{cyc} xy(x+y)} + \frac{5}{4} \geq (xy+yz+zx)\left[\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right].$$

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U559. Evaluate

$$\int \sqrt{1+\frac{1}{x}}\,dx$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U560. Let $1, 1, 2, 3, 5, 8, ...$ be the Fibonacci sequence. Evaluate

$$\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{\cot(1)}{\cot(\frac{1}{2})}+\frac{\cot(\frac{1}{2})}{\cot(\frac{1}{3})}+\frac{\cot(\frac{1}{3})}{\cot(\frac{1}{5})}++\frac{\cot(\frac{1}{5})}{\cot(\frac{1}{8})}+\dots\right)$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

U561. The Fibonacci numbers F_n are defined as follows:

$$F_{n+2}=F_{n+1}+F_n,\ F_0=0,\ F_1=1.$$

Prove that

$$2F_nF_{n+1}^5-2F_n^5F_{n+1}=F_n^6+F_{n+1}^6-F_{n+1}^2-F_n^2.$$

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U562. Let $ABCD$ be a square. Variable points P and Q are taken on sides AB and BC, respectively, such that $\angle PDQ = 45^\circ$. Find the locus of the orthocenter of triangle PDQ.

Proposed by Mircea Becheanu, Canada

U563. Find all polynomials $P(X)$ with real coefficients such that for all real numbers x,

$$\left(P(x^2)+x\right)\left(P(x^3)-x^2\right)=P(x^5)+x.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U564. Evaluate $\lim_{n\to+\infty}(-1)^{n+1}\cdot\sin\left(\sum_{k=1}^{4n}\arctan\frac{k^4+3k^2-1}{k^4-k^2+7}\right)$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

Olympiad Problems

O559. Let x, y, z be real numbers such that none of them lies in the interval $(-1,1)$ and

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+x+y+z=0.$$

Find the minimum of $\dfrac{z}{x+y}$.

Proposed by Marius Stănean, Zalău, România

O560. Prove that there are infinitely many triples (a,b,c) of integers for which $ab+bc+ca=1$ and that for each such triple $(a^2+1)(b^2+1)(c^2+1)$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O561. Let ABC be a triangle. Prove that

$$\frac{a^2}{1+\cos^2 B+\cos^2 C}+\frac{b^2}{1+\cos^2 C+\cos^2 A}+\frac{c^2}{1+\cos^2 A+\cos^2 B}\le\frac{2}{3}(a^2+b^2+c^2).$$

Proposed by An Zhenping, Xianyang Normal University, China

O562. Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$x^2f(y)+f(y^2f(x))=xyf(x+y),$$

for all real numbers x,y.

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

O563. Prove that in any triangle ABC,

$$\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}+\frac{6(ab+bc+ca)}{(a+b+c)^2}\ge 5.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O564. Let $n\ge 3$ be an integer. For every sequence $-1=x_1<x_2<\cdots<x_n=1$ of real numbers and every $k=1,2,\dots,n$ we define

$$D_k(x_2,\dots,x_{n-1})=|(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)|$$

and denote

$$r_n=\max_{x_2,\dots,x_{n-1}}\min_k D_k(x_2,\dots,x_{n-1}).$$

Assume that r_n is achieved at the sequence $-1=a_1<a_2<\cdots<a_n=1$. Prove that

$$D_2(a_2,\dots,a_{n-1})=\cdots=D_{n-1}(a_2,\dots,a_{n-1}).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran