Junior Problems

J559. Let

\[ a_n = 1 - \frac{2n^2}{1 + \sqrt{1 + 4n^4}}, \quad n = 1, 2, 3, \ldots \]

Prove that \( \sqrt{a_1} + 2\sqrt{a_2} + \cdots + 20\sqrt{a_{20}} \) is an integer.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

J560. Let \( a, b, c \) be positive real numbers. Prove that

\[ \frac{2}{a^2} + \frac{5}{b^2} + \frac{45}{c^2} > \frac{16}{(a+b)^2} + \frac{24}{(b+c)^2} + \frac{48}{(c+a)^2}. \]

*Proposed by Kartik Vedula, James S. Rickards High School, Tallahassee, USA*

J561. Solve in nonzero real numbers the system of equations:

\[ x - \frac{1}{x} + \frac{2}{y} = y - \frac{1}{y} + \frac{2}{z} = z - \frac{1}{z} + \frac{2}{x} = 0. \]

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

J562. Let \( ABC \) be a triangle and let \( D, E, F \) be points on sides \( BC, CA, AB \), respectively, such that \( AD, BE, CF \) are concurrent in \( X \). Assume that the ratios \( \frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB} \) are in the interval \( \left[ \frac{1}{5}, 5 \right] \) and that \( \frac{BD}{DC} + \frac{CE}{EA} + \frac{AF}{FB} = \frac{31}{5} \). Evaluate

\[ \frac{AX}{XD} + \frac{BX}{XE} + \frac{CX}{XF}. \]

*Proposed by Mohammad Imran, India*

J563. Let \( a, b, c \geq 0 \) be real numbers such that \( ab + bc + ca = a + b + c > 0 \). Prove that

\[ 1 \leq \frac{1}{1 + 2a} + \frac{1}{1 + 2b} + \frac{1}{1 + 2c} \leq \frac{7}{5}. \]

*Proposed by An Zhenping, Xianyang Normal University, China*

J564. Find all complex numbers \( z \) such that for each real number \( a \) and each positive integer \( n \)

\[ (\cos a + z \sin a)^n = \cos na + z \sin na \]

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*
Senior Problems

S559. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n \leq n$. Find the minimum of

$$\frac{1}{a_1^2} + \frac{1}{2a_2^2} + \cdots + \frac{1}{na_n^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S560. Let $a, b, c$ be nonnegative real numbers such that $a + b + c = 2$. Prove that

$$\frac{a}{b^2 + bc + c^2} + \frac{b}{c^2 + ca + a^2} + \frac{c}{a^2 + ab + b^2} + 8 \geq \frac{10}{ab + bc + ca}.$$

Proposed by Marius Stănean, Zalău, România

S561. Let $p$ be a prime. Solve in positive integers the equation

$$(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx)(z^2 - xy) = p^2.$$

Proposed by Alessandro Ventullo, Milan, Italy

S562. Let $a, b, c, d$ be nonnegative real numbers. Prove that

$$(a + b + c + d)^3 + 9(abc + abd + acd + bcd) \geq 4(a + b + c + d)(ab + ac + ad + bc + bd + cd).$$

Proposed by An Zhenping, Xianyang Normal University, China

S563. Let $a, b, c$ be distinct positive real numbers. Prove that at least one of the numbers

$$\left(a + \frac{1}{a}\right)^2 (1 - b^4); \quad \left(b + \frac{1}{b}\right)^2 (1 - c^4); \quad \left(c + \frac{1}{c}\right)^2 (1 - a^4)$$

is not equal to 4.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S564. Let $x, y, z$ be nonnegative real numbers. Prove that

$$\frac{x^3 + y^3 + z^3 + 3xyz}{\sum_{cyc} xy(x + y)} + \frac{5}{4} \geq (xy + yz + zx)\left[\frac{1}{(x + y)^2} + \frac{1}{(y + z)^2} + \frac{1}{(z + x)^2}\right].$$

Proposed by Marius Stănean, Zalău, România
U559. Evaluate
\[ \int \sqrt{1 + \frac{1}{x}} \, dx \]

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

U560. Let 1, 1, 2, 3, 5, 8, ... be the Fibonacci sequence. Evaluate

\[ \lim_{n \to \infty} \frac{1}{n} \left( \frac{1}{\cot(\frac{1}{2})} + \frac{1}{\cot(\frac{1}{3})} + \frac{1}{\cot(\frac{1}{5})} + \frac{1}{\cot(\frac{1}{8})} + \ldots \right) \]

*Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India*

U561. The Fibonacci numbers \( F_n \) are defined as follows:

\[ F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1. \]

Prove that

\[ 2F_nF_{n+1}^5 - 2F_n^5F_{n+1} = F_n^6 + F_{n+1}^6 - F_{n+1}^2 - F_n^2. \]

*Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

U562. Let \( ABCD \) be a square. Variable points \( P \) and \( Q \) are taken on sides \( AB \) and \( BC \), respectively, such that \( \angle PDQ = 45^\circ \). Find the locus of the orthocenter of triangle \( PDQ \).

*Proposed by Mircea Becheanu, Canada*

U563. Find all polynomials \( P(X) \) with real coefficients such that for all real numbers \( x \),

\[ (P(x^2) + x) (P(x^3) - x^2) = P(x^5) + x. \]

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

U564. Evaluate \( \lim_{n \to +\infty} (-1)^{n+1} \cdot \sin \left( \sum_{k=1}^{4n} \arctan \frac{k^4 + 3k^2 - 1}{k^4 - k^2 + 7} \right) \)

*Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy*
O559. Let \( x, y, z \) be real numbers such that none of them lies in the interval \((-1, 1)\) and
\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + x + y + z = 0.
\]
Find the minimum of \( \frac{z}{x+y} \).

Proposed by Marius Stănean, Zalău, România

O560. Prove that there are infinitely many triples \((a, b, c)\) of integers for which 
\( ab + bc + ca = 1 \) and that for each such triple \((a^2 + 1)(b^2 + 1)(c^2 + 1)\) is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O561. Let \( ABC \) be a triangle. Prove that
\[
\frac{a^2}{1 + \cos^2 B + \cos^2 C} + \frac{b^2}{1 + \cos^2 C + \cos^2 A} + \frac{c^2}{1 + \cos^2 A + \cos^2 B} \leq \frac{2}{3}(a^2 + b^2 + c^2).
\]

Proposed by An Zhenping, Xianyang Normal University, China

O562. Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[
x^2f(y) + f(y^2f(x)) = xyf(x+y),
\]
for all real numbers \( x, y \).

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

O563. Prove that in any triangle \( ABC \),
\[
\sqrt{\frac{ma}{ha}} + \sqrt{\frac{mb}{hb}} + \sqrt{\frac{mc}{hc}} + \frac{6(ab + bc + ca)}{(a + b + c)^2} \geq 5.
\]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O564. Let \( n \geq 3 \) be an integer. For every sequence \(-1 = x_1 < x_2 < \cdots < x_n = 1\) of real numbers and every \( k = 1, 2, \ldots, n \) we define
\[
D_k(x_2, \ldots, x_{n-1}) = |(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)|
\]
and denote
\[
r_n = \max_{x_2, \ldots, x_{n-1}} \min_k D_k(x_2, \ldots, x_{n-1}).
\]
Assume that \( r_n \) is achieved at the sequence \(-1 = a_1 < a_2 < \cdots < a_n = 1\). Prove that
\[
D_2(a_2, \ldots, a_{n-1}) = \cdots = D_{n-1}(a_2, \ldots, a_{n-1}).
\]

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran