Junior problems

J559. Let

$$a_n = 1 - \frac{2n^2}{1+\sqrt{1+4n^4}}, \quad n = 1, 2, 3, \ldots.$$

Prove that $\sqrt{a_1} + 2\sqrt{a_2} + \cdots + 20\sqrt{a_{20}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

By rationalizing the denominator and Germain's factoring, we see that

$$\begin{aligned} n\sqrt{a_n} &= \sqrt{n^2 + \frac{1}{2} - \sqrt{\left(n^2 + \frac{1}{2} + n\right)\left(n^2 + \frac{1}{2} - n\right)}} \\ &= \sqrt{\frac{1}{2}\left(n^2 + \frac{1}{2} + n\right)} - \sqrt{\frac{1}{2}\left(n^2 + \frac{1}{2} - n\right)} \end{aligned}$$

Therefore, by telescoping,

$$\sqrt{a_1} + 2\sqrt{a_2} + \cdots + 20\sqrt{a_{20}} = \sqrt{\frac{1}{2}\left(20^2 + \frac{1}{2} + 20\right)} - \sqrt{\frac{1}{2}\left(1^2 + \frac{1}{2} - 1\right)} = \frac{29}{2} - \frac{1}{2} = 14.$$

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Alina Craciun, Theoretical High School Miron Costin, Pascani, Romania; G. C. Greubel, Newport News, VA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Mohammad Imran, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Kanav Talwar, Delhi Public School, Faridabad, India; Arkady Alt, San Jose, CA, USA.

J560. Let a, b, c be positive real numbers. Prove that

$$\frac{2}{a^2} + \frac{5}{b^2} + \frac{45}{c^2} > \frac{16}{(a+b)^2} + \frac{24}{(b+c)^2} + \frac{48}{(c+a)^2}.$$

Proposed by Kartik Vedula, James S. Rickards High School, Tallahassee, USA

Solution by the author

We claim that the expression $\frac{4}{ab} + \frac{6}{bc} + \frac{12}{ca}$ is between both expressions, but equality cannot occurs at the same time:

1. Note that $(a+b)^2 \geq 4ab$ and similar, so

$$\frac{16}{(a+b)^2} + \frac{24}{(b+c)^2} + \frac{48}{(c+a)^2} \leq \frac{16}{4ab} + \frac{24}{4bc} + \frac{48}{4ac} = \frac{4}{ab} + \frac{6}{bc} + \frac{12}{ca}$$

2. Note that $\left(\frac{1}{a} - \frac{2}{b}\right)^2, \left(\frac{1}{b} - \frac{3}{c}\right)^2, \left(\frac{1}{a} - \frac{6}{c}\right)^2 \geq 0$, so adding these together gives

$$\frac{2}{a^2} + \frac{5}{b^2} + \frac{45}{c^2} \geq \frac{4}{ab} + \frac{6}{bc} + \frac{12}{ca}$$

This means that we have shown $\frac{2}{a^2} + \frac{5}{b^2} + \frac{45}{c^2} \geq \frac{16}{(a+b)^2} + \frac{24}{(b+c)^2} + \frac{48}{(c+a)^2}$. However, equality for the first part occurs when $a = b = c$, but equality for the second part occurs when $\frac{1}{a} = \frac{2}{b} = \frac{6}{c}$. These cannot coincide, so we are done.

Also solved by Polyahedra, Polk State College, USA; Le Hoang Bao, Tien Giang, Vietnam; Daniel Văcaru, Pitești, Romania; Jiang Lianjun,Gui Lin,China; Arkady Alt, San Jose, CA, USA.

J561. Solve in nonzero real numbers the system

$$x-\frac{1}{x}+\frac{2}{y}=y-\frac{1}{y}+\frac{2}{z}=z-\frac{1}{z}+\frac{2}{x}=0.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Solving the equations

$$x-\frac{1}{x}+\frac{2}{y}=0,\quad y-\frac{1}{y}+\frac{2}{z}=0,\quad z-\frac{1}{z}+\frac{2}{x}=0$$

for y, z, and x, respectively, yields

$$y=\frac{2x}{1-x^2},\quad z=\frac{2y}{1-y^2},\quad \text{and}\quad x=\frac{2z}{1-z^2}.$$

Now, let $x=\tan\theta$. Then

$$y=\frac{2\tan\theta}{1-\tan^2\theta}=\tan 2\theta,\quad z=\frac{2\tan 2\theta}{1-\tan^2 2\theta}=\tan 4\theta,$$

and

$$x=\frac{2\tan 4\theta}{1-\tan^2 4\theta}=\tan 8\theta.$$

Thus,

$$\tan\theta=\tan 8\theta=\frac{\tan\theta+\tan 7\theta}{1-\tan\theta\tan 7\theta},$$

which reduces to

$$(1+\tan^2\theta)\tan 7\theta=0.$$

Because $x=\tan\theta$ must be nonzero, there are six solutions for θ:

$$\frac{\pi}{7},\frac{2\pi}{7},\frac{3\pi}{7},\frac{4\pi}{7},\frac{5\pi}{7},\frac{6\pi}{7}.$$

This yields six solutions to the original system:

$$\begin{aligned}(x,y,z) &= \left(\tan\frac{\pi}{7},\tan\frac{2\pi}{7},\tan\frac{4\pi}{7}\right)\\ &= \left(\tan\frac{2\pi}{7},\tan\frac{4\pi}{7},\tan\frac{\pi}{7}\right)\\ &= \left(\tan\frac{3\pi}{7},\tan\frac{6\pi}{7},\tan\frac{5\pi}{7}\right)\\ &= \left(\tan\frac{4\pi}{7},\tan\frac{\pi}{7},\tan\frac{2\pi}{7}\right)\\ &= \left(\tan\frac{5\pi}{7},\tan\frac{3\pi}{7},\tan\frac{6\pi}{7}\right)\\ &= \left(\tan\frac{6\pi}{7},\tan\frac{5\pi}{7},\tan\frac{3\pi}{7}\right)\end{aligned}$$

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Polyahedra, Polk State College, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J562. Let ABC be a triangle and let D, E, F be points on sides BC, CA, AB, respectively, such that AD, BE, CF are concurrent in X. Assume that the ratios $\frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB}$ are in the interval $\left[\frac{1}{5}, 5\right]$ and that $\frac{BD}{DC} + \frac{CE}{EA} + \frac{AF}{FB} = \frac{31}{5}$. Evaluate

$$\frac{AX}{XD} + \frac{BX}{XE} + \frac{CX}{XF}.$$

Proposed by Mohammad Imran, India

Solution by Polyahedra, Polk State College, USA

Let $x = BD/DC$, $y = CE/EA$, and $z = AF/FB$. By Ceva's theorem, $xyz = 1$. Notice that

$$\frac{AX}{XD} = \frac{[ABXC]}{[BCX]} = \frac{[ABX] + [CAX]}{[BCX]} = \frac{AE}{EC} + \frac{AF}{FB} = \frac{1}{y} + z.$$

Therefore,

$$\frac{AX}{XD} + \frac{BX}{XE} + \frac{CX}{XF} = \frac{1}{y} + z + \frac{1}{z} + x + \frac{1}{x} + y = zx + xy + yz + \frac{31}{5}.$$

Let $a = xy + yz + zx$ and

$$f(t) = (t - x)(t - y)(t - z) = t^3 - \frac{31}{5}t + at - 1 = \left(t - \frac{1}{5}\right)(t - 1)(t - 5) + \left(a - \frac{31}{5}\right)t.$$

If $a > 31/5$, then $f(0) = -1$ and $f(1/5) > 0$, so f has a zero in $(0, 1/5)$, a contradiction. If $a < 31/5$, then $f(5) < 0$, so f has a zero greater than 5, again a contradiction. Therefore, $a = 31/5$, attained by $\{x, y, z\} = \{1/5, 1, 5\}$. In conclusion, the value of the expression is $62/5$.

Also solved by Alina Craciun, Theoretical High School Miron Costin, Pascani, Romania; Daniel Văcaru, Pitești, Romania.

J563. Let $a, b, c \geq 0$ be real numbers such that $ab + bc + ca = a + b + c > 0$. Prove that

$$1 \leq \frac{1}{1+2a} + \frac{1}{1+2b} + \frac{1}{1+2c} \leq \frac{7}{5}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Polyahedra, Polk State College, USA

We may assume $a \geq b \geq c \geq 0$. If $c = 0$, then $0 \neq ab = a + b \geq 2\sqrt{ab}$, so $ab \geq 4$. Therefore,

$$0 < \frac{1}{1+2a} + \frac{1}{1+2b} = \frac{2(1+ab)}{1+6ab} \leq \frac{2}{5}.$$

Next, suppose $c > 0$. Let $x = 1/a$, $y = 1/b$, and $z = 1/c$. Then

$$x + y + z = \frac{bc + ca + ab}{abc} = \frac{c + a + b}{abc} = xy + yz + zx,$$

thus

$$a = \frac{xy + yz + zx}{x(x+y+z)}, \quad b = \frac{xy + yz + zx}{y(x+y+z)}, \quad c = \frac{xy + yz + zx}{z(x+y+z)}.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{cyc} \frac{1}{1+2a} &= \sum_{cyc} \frac{x(x+y+z)}{x(x+y+z) + 2(xy+yz+zx)} = (x+y+z) \sum_{cyc} \frac{x^2}{x^3 + 3x^2y + 3x^2z + 2xyz} \\ &\geq \frac{(x+y+z)^3}{\sum_{cyc}(x^3 + 3x^2y + 3x^2z + 2xyz)} = 1.\end{aligned}$$

On the other hand, since

$$\frac{1}{1+2a} + \frac{1}{1+2b} + \frac{1}{1+2c} = \frac{3 + 8(a+b+c)}{1 + 6(a+b+c) + 8abc},$$

the right inequality is equivalent to $a + b + c + 28abc \geq 4$, that is,

$$(x+y+z)^2(xy+yz+zx)^2 + 28(xy+yz+zx)^3 \geq 4xyz(x+y+z)^3.$$

Notice that $(xy + yz + zx)^2 \geq 3xyz(x + y + z)$ and

$$\begin{aligned}&(x+y+z)(xy+yz+zx)^2 + 84xyz(xy+yz+zx) - 4xyz(x+y+z)^2 \\ \geq &(x+y+z)(xy+yz+zx)^2 - 4xyz\left(x^2+y^2+z^2\right) \geq x^3(y-z)^2 + y^3(z-x)^2 + z^3(x-y)^2 \geq 0.\end{aligned}$$

Also solved by Christopher Newport University, Newport News, VA, USA; Daniel Văcaru, Pitești, Romania; Jiang Lianjun, Southwest University, Chong Qing, China; Arkady Alt, San Jose, CA, USA.

J564. Find all complex numbers z such that for each real number a and each positive integer n

$$(\cos a + z \sin a)^n = \cos na + z \sin na.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Letting $a = \pi/2$ and $n = 2$, we get $z^2 = -1$, so $z \in \{\pm i\}$.

By DeMoivre's formula, for all real a and natural n,

$$(\cos a + i \sin a)^n = \cos na + i \sin na$$

and

$$(\cos a - i \sin a)^n = [\cos(-a) + i \sin(-a)]^n = \cos na - i \sin na.$$

Therefore, $z = \pm i$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania.

Senior problems

S559. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n \leq n$. Find the minimum value of

$$\frac{1}{a_1} + \frac{1}{2a_2^2} + \cdots + \frac{1}{na_n^n}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

Using the AM-GM inequality we obtain

$$\frac{1}{a_1} + a_1 \geq 2,$$
$$\frac{1}{2a_2^2} + \frac{a_2}{2} + \frac{a_2}{2} \geq \frac{3}{2},$$
$$\cdots \quad \cdots \quad \cdots,$$
$$\frac{1}{na_n^n} + \underbrace{\frac{a_n}{n} + \cdots + \frac{a_n}{n}}_{n} \geq n + 1.$$

Summing up these inequalities we obtain

$$\frac{1}{a_1} + \frac{1}{2a_2^2} + \cdots + \frac{1}{na_n^n} + a_1 + a_2 + \cdots + a_n \geq \sum_{k=1}^{n} \frac{k+1}{k}.$$

From here and using the given condition we get

$$\begin{aligned}\frac{1}{a_1} + \frac{1}{2a_2^2} + \cdots + \frac{1}{na_n^n} &\geq \sum_{k=1}^{n} (1 + \frac{1}{k}) - n \\ &= \sum_{k=1}^{n} \frac{1}{k}.\end{aligned}$$

The equality occurs if and only if $a_1 = a_2 = \cdots = a_n$. Hence

$$\min\left(\frac{1}{a_1} + \frac{1}{2a_2^2} + \cdots + \frac{1}{na_n^n}\right) = \sum_{k=1}^{n} \frac{1}{k}.$$

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S560. Let a, b, c be nonnegative real numbers such that $a+b+c=2$. Prove that

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}+8 \geq \frac{10}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

After homogenizing, the inequality becomes as follows

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}+\frac{16}{a+b+c} \geq \frac{5(a+b+c)}{ab+bc+ca},$$

or multiplying by $a+b+c$

$$\sum_{cyc}\frac{a(a+b+c)}{b^2+bc+c^2}+16 \geq \frac{5(a+b+c)^2}{ab+bc+ca},$$

or

$$\sum_{cyc}\frac{a^2+b^2+c^2+ab+bc+ca-b^2-bc-c^2}{b^2+bc+c^2}+16 \geq \frac{5(a^2+b^2+c^2+ab+bc+ca)}{ab+bc+ca}+5,$$

or $f(a,b,c) \geq 5$, where

$$f(a,b,c)=\sum_{cyc}\frac{ab+bc+ca}{a^2+ab+b^2}+\frac{8(ab+bc+ca)}{a^2+b^2+c^2+ab+bc+ca}.$$

Without loss of generality, assume that $c=\min\{a,b,c\}$. Notice that we have equality for $a=b$, $c=0$ which leads us to prove a mixing of form $f(a,b,c) \geq f(a+c,b+c,0)$. Indeed, we have

$$\frac{ab+bc+ca}{b^2+bc+c^2}-\frac{a+c}{b+c}=\frac{c(ab-c^2)}{(b^2+bc+c^2)(b+c)} \geq 0,$$

$$\frac{ab+bc+ca}{c^2+ca+a^2}-\frac{b+c}{a+c}=\frac{c(ab-c^2)}{(c^2+ca+a^2)(a+c)} \geq 0,$$

$$\begin{aligned}\frac{ab+bc+ca}{a^2+ab+b^2}&-\frac{(a+c)(b+c)}{(a+c)^2+(a+c)(b+c)+(b+c)^2}\\ &\geq \frac{ab+bc+ca}{a^2+ab+b^2}-\frac{(a+c)(b+c)}{a^2+ab+b^2+c(a+b)}\\ &=\frac{abc(a+b+c)}{(a^2+ab+b^2)(a^2+ab+b^2+c(a+b))} \geq 0,\end{aligned}$$

$$\begin{aligned}&\frac{ab+bc+ca}{a^2+b^2+c^2+ab+bc+ca}-\frac{(a+c)(b+c)}{(a+c)^2+(a+c)(b+c)+(b+c)^2}\\ \geq &\frac{ab+bc+ca}{a^2+b^2+c^2+ab+bc+ca}-\frac{ab+bc+ca+c^2}{a^2+b^2+c^2+ab+bc+ca+c(a+b)}\\ = &\frac{c(ab-c^2)(a+b+c)}{(a^2+b^2+c^2+ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca+c(a+b))} \geq 0.\end{aligned}$$

It remains to show that $f(a+c, b+c, 0) \geq 5$, that is

$$\frac{a+c}{b+c} + \frac{b+c}{a+c} + \frac{9(a+c)(b+c)}{(a+c)^2 + (a+c)(b+c) + (b+c)^2} \geq 5,$$

or if we denote $t = \frac{a+c}{b+c} + \frac{b+c}{a+c} + 1 \geq 3$,

$$\frac{9}{t} + t - 1 \geq 5 \iff \frac{(t-3)^2}{t} \geq 0,$$

clearly true.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S561. Let p be a prime number. Solve in positive integers the equation

$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)=p^2.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

Let $P(x,y,z)=(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)$. Using the identity

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

with $a=x^2-yz$, $b=y^2-zx$ and $c=z^2-xy$, we have

$$\begin{aligned} P(x,y,z) &= (x^2+y^2+z^2-xy-yz-zx)(x+y+z)^2(x^2+y^2+z^2-xy-yz-zx) \\ &= [(x+y+z)(x^2+y^2+z^2-xy-yz-zx)]^2 \\ &= (x^3+y^3+z^3-3xyz)^2. \end{aligned}$$

So, the given equation becomes

$$x^3+y^3+z^3-3xyz=p. \qquad (1)$$

(i) If $p=2$, then the equation becomes $x^3+y^3+z^3-3xyz=2$, i.e.

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=2.$$

Since x,y,z are positive integers, then $x+y+z\geq 3$, contradiction.

(ii) If $p=3$, then the equation becomes $x^3+y^3+z^3-3xyz=3$, i.e.

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=3.$$

Since x,y,z are positive integers, then $x+y+z\geq 3$, which gives $x+y+z=3$, i.e. $x=y=z=1$. But then $x^2+y^2+z^2-xy-yz-zx=0$, contradiction.

(iii) If $p>3$, equation (1) has been solved in *T. Andreescu, D. Andrica, I. Cucurezeanu - An Introduction to Diophantine Equations*, pag. 9. We get the solutions $(x,y,z)\in\left(\frac{p+1}{3},\frac{p+1}{3},\frac{p-2}{3}\right)$ and the corresponding permutations or $(x,y,z)=\left(\frac{p+2}{3},\frac{p-1}{3},\frac{p-1}{3}\right)$ and the corresponding permutations.

Note: The problem can be formulated for the particular case $p=2017$.

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Kanav Talwar, Delhi Public School, Faridabad, India.

S562. Let a, b, c, d be nonnegative real numbers. Prove that

$$(a+b+c+d)^3 + 9(abc+abd+acd+bcd) \geq 4(a+b+c+d)(ab+ac+ad+bc+bd+cd).$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Daniel Văcaru, Pitești, Romania

By calculus, we obtain

$$(a+b+c+d)^3 + 9(abc+abd+acd+bcd) \geq 4(a+b+c+d)(ab+ac+ad+bc+bd+cd) \Leftrightarrow$$

$$a^3+b^3+c^3+d^3+3(abc+abd+acd+bcd) \geq a^2b+a^2c+a^2d+ab^2+b^2c+b^2d+ac^2+bc^2+c^2d+ad^2+bd^2+cd^2$$

Again, by calculus, this is equivalent to $4\left(a^3+b^3+c^3+d^3\right)+15(abc+abd+acd+bcd) \geq (a+b+c+d)^3$ $(*)$.

Let $E(a,b,c,d) = 4\left(a^3+b^3+c^3+d^3\right)+15(abc+abd+acd+bcd)-(a+b+c+d)^3$. Without loss of generality, assume $a \leq b \leq c \leq d$. We prove that

$$E(a,b,c,d) \geq E(0,a+b,c,d) \geq 0$$

We have $E(a,b,c,d) - E(0,a+b,c,d) = 4\left[a^3+b^3-(a+b)^3\right]+15ab(c+d) = 3ab\left[5(c+d)-4(a+b) \geq 0\right]$

Set $x \overset{\text{not}}{=} a+b$; we need to show that $E(0,x,c,d) \geq 0$,and $E(0,x,c,d) = 4\left(x^3+c^3+d^3\right)+15xcd-(x+c+d)^3$.

But that is equivalent to Schur's inequality

$$x^3+c^3+d^3+3xcd \geq xc(x+c)+cd(c+d)+dx(d+x).$$

We have equality for $a=0$ and $b=c=d$ (or any cyclic permutation), and also for $a=b=0$ and $c=d$ (or any permutation).

S563. Let a, b, c be distinct real numbers. Prove that at least one of the numbers

$$\left(a+\frac{1}{a}\right)^2\left(1-b^4\right), \quad \left(b+\frac{1}{b}\right)^2\left(1-c^4\right), \quad \left(c+\frac{1}{c}\right)^2\left(1-a^4\right)$$

is not equal to 4.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Assume for the sake of contradiction that all three numbers are equal to 4. Then a^2, b^2, c^2 are less than 1. The first equality can be rewritten as

$$a^2+\frac{1}{a^2}=\frac{4}{1-b^4}-2.$$

Let $a^2=\tan u$, $b^2=\tan v$, $c^2=\tan w$, where u, v, $w\in\left(0,\frac{\pi}{4}\right)$. We have

$$\frac{\tan^2 u+1}{\tan u}=2\frac{\tan^2 v+1}{1-\tan^2 v},$$

implying

$$\frac{2\tan u}{1+\tan^2 u}=\frac{1-\tan^2 v}{1+\tan^2 v}.$$

It follows that $\sin 2u=\cos 2v$ and, since $2u, 2v\in\left(0,\frac{\pi}{2}\right)$, we have $2u+2v=\frac{\pi}{2}$. Similarly, from the second equality, $2v+2w=\frac{\pi}{2}$, implying $u=w$, a contradiction. Hence the conclusion.

Second solution by the author

Assume by contradiction that all three numbers are equal to 4. The first equality can be rewritten as

$$\frac{2a^2}{1+a^4}=\frac{1-b^4}{1+b^4}.$$

Then,

$$1-\left(\frac{2a^2}{1+a^4}\right)^2=1-\left(\frac{1-b^4}{1+b^4}\right)^2,$$

i.e.

$$\frac{1-a^4}{1+a^4}=\frac{2b^2}{1+b^4}.$$

However, since

$$\frac{1-a^4}{1+a^4}=\frac{2c^2}{1+c^4},$$

we conclude that $b=c$, contradiction.

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Kanav Talwar, Delhi Public School, Faridabad, India.

S564. Let x, y, z be nonegative real numbers. Prove that

$$\frac{x^3+y^3+z^3+3xyz}{\sum_{cyc} xy(x+y)}+\frac{5}{4} \geq (xy+yz+zx)\left[\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right].$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

If one of x, y, z are equal with 0, say z, then the inequality becomes

$$\frac{(x-y)^2}{4(x+y)^2} \geq 0,$$

clearly true.

If x, y, z are positive numbers, then $a = y+z$, $b = z+x$, $c = x+y$ are side lengths of a triangle with the circumradius R, the inradius r, and the semiperimeter s. It follows that $x = s-a$, $y = s-b$, $z = s-c$, and since $ab+bc+ca = s^2+r^2+4Rr$ the inequality becomes

$$\frac{s^3-3r(4R-r)s}{s(r^2+4Rr)-3sr^2}+\frac{5}{4} \geq \frac{(r^2+4Rr)\left[(ab+bc+ca)^2-4abcs\right]}{16R^2r^2s^2},$$

or

$$\frac{s^2+3r^2-12Rr}{4Rr-2r^2}+\frac{5}{4} \geq \frac{(4R+r)\left[s^4-2r(4R-r)s^2+r^2(4R+r)^2\right]}{16R^2rs^2},$$

or

$$\frac{R^2}{2Rr-r^2}\left(\frac{s^2}{R^2}+\frac{3r^2}{R^2}-\frac{12r}{R}\right)+\frac{5}{2} \geq \frac{4R+r}{8r}\left[\frac{s^2}{R^2}+\left(\frac{r^2}{R^2}+\frac{4r}{R}\right)^2\frac{R^2}{s^2}-\frac{2r(4R-r)}{R^2}\right].$$

Consider the functions f, defined as

$$f\left(\frac{s^2}{R^2}\right)=\frac{R^2}{2Rr-r^2}\left(\frac{s^2}{R^2}\right)-\frac{4R+r}{8r}\left[\frac{s^2}{R^2}+\left(\frac{r^2}{R^2}+\frac{4r}{R}\right)^2\frac{R^2}{s^2}\right].$$

Since

$$f'\left(\frac{s^2}{R^2}\right)=\frac{R^2}{2Rr-r^2}-\frac{4R+r}{8r}+\left(\frac{4R+r}{8r}\right)\left(\frac{r^2}{R^2}+\frac{4r}{R}\right)^2\frac{R^4}{s^4}$$
$$=\frac{2R+r}{8(2R-r)}+\left(\frac{4R+r}{8r}\right)\left(\frac{r^2}{R^2}+\frac{4r}{R}\right)^2\frac{R^4}{s^4} \geq 0,$$

we deduce that f is a increasing function.

If we denote $t^2 = 1-\frac{2r}{R} \in [0,1)$, then by Blundon's Inequality

$$\frac{s^2}{R^2} \geq 2+5(1-t^2)-\frac{(1-t^2)^2}{4}-2t^3=\frac{(1-t)(t+3)^3}{4}.$$

Hence, it suffices to prove that

$$\frac{4}{(1-t^2)(3+t^2)}\left[\frac{(1-t)(t+3)^3}{4}+\frac{3(1-t^2)^2}{4}-6(1-t^2)\right]+\frac{5}{2}$$
$$\geq \frac{9-t^2}{8(1-t^2)}\left[\frac{(1-t)(t+3)^3}{4}+\frac{(1-t^2)^2(9-t^2)^2}{4(1-t)(t+3)^3}-\frac{(1-t^2)(7+t^2)}{2}\right],$$

or after some computations,

$$\frac{t^2(t-1)^2}{t^2+3} \geq 0,$$

clearly true. The equality holds when $t = 0$, so when the triangle is equilateral which means $x = y = z$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Undergraduate problems

U559. Evaluate

$$\int \sqrt{1+\frac{1}{x}}\mathrm{d}x$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

Letting

$$\sqrt{1+\frac{1}{x}} = t \Rightarrow x = \frac{1}{t^2-1} \Rightarrow \mathrm{d}x = \frac{-2t}{(t^2-1)^2}\mathrm{d}t.$$

Hence

$$\begin{aligned}
\int \sqrt{1+\frac{1}{x}}\mathrm{d}x &= \int \frac{-2t^2}{(t^2-1)^2}\mathrm{d}t \\
&= -2\int \left(\frac{1}{t^2-1}+\frac{1}{(t^2-1)^2}\right)\mathrm{d}t \\
&= -2\int \frac{1}{t^2-1}\mathrm{d}t - 2\int \left[\frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\right]^2 \mathrm{d}t \\
&= -2\int \frac{\mathrm{d}t}{t^2-1} - \frac{1}{2}\int \left(\frac{1}{(t-1)^2}+\frac{1}{(t+1)^2}-\frac{2}{t^2-1}\right)\mathrm{d}t \\
&= -\int \frac{\mathrm{d}t}{t^2-1} - \frac{1}{2}\int \left(\frac{1}{(t-1)^2}+\frac{1}{(t+1)^2}\right)\mathrm{d}t \\
&= \frac{1}{2}\int \left(\frac{1}{t+1}-\frac{1}{t-1}\right) - \frac{1}{2}\int \left(\frac{1}{(t-1)^2}\mathrm{d}t+\frac{1}{(t+1)^2}\right)\mathrm{d}t \\
&= \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + \frac{1}{2}\left(\frac{1}{t+1}+\frac{1}{t-1}\right) + C \\
&= \frac{1}{2}\ln\frac{(t+1)^2}{|t^2-1|} + \frac{t}{t^2-1} + C \\
&= \frac{1}{2}\ln\left[|x|\left(\sqrt{1+\frac{1}{x}}+1\right)^2\right] + x\sqrt{1+\frac{1}{x}} + C.
\end{aligned}$$

Also solved by Toyesh Prakash Sharma, St. C. F. Andrews School, Agra, India; Le Hoang Bao, Tien Giang, Vietnam; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Alina Craciun, Theoretical High School Miron Costin, Pascani, Romania; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Arkady Alt, San Jose, CA, USA.

U560. Let $1, 1, 2, 3, 5, 8, \cdots$ be the Fibonacci sequence. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \frac{\cot\left(\frac{1}{F_{j+1}}\right)}{\cot\left(\frac{1}{F_{j+2}}\right)}.$$

Proposed by Toyesh Prakash Sharma, St. C. F. Andrews School, Agra, India

Solution by G. C. Greubel, Newport News, VA, USA

The sum

$$S_n = \sum_{j=0}^{n} \frac{\cot\left(\frac{1}{F_{j+1}}\right)}{\cot\left(\frac{1}{F_{j+2}}\right)}$$

can also be seen in the form

$$S_n = \sum_{j=0}^{n} \cot\left(\frac{1}{F_{j+1}}\right) \cot\left(\frac{1}{F_{j+2}}\right).$$

Using

$$\cot\left(\frac{1}{x}\right) = x - \frac{1}{3\,x} + \cdots$$
$$\tan\left(\frac{1}{x}\right) = \frac{1}{x} + \frac{1}{3\,x^2} + \cdots$$

then

$$\begin{aligned} S_n &= \sum_{j=0}^{n} \left(\frac{F_{j+1}}{F_{j+2}} + \frac{F_{j+1}}{3\,F_{j+2}^3} - \frac{1}{3\,F_{j+1}F_{j+2}} + \cdots \right) \\ &\approx \sum_{j=0}^{n} \frac{F_{j+1}}{F_{j+2}} + \mathcal{O}(n^0) \\ &\approx \sum_{j=0}^{n} \frac{1}{\alpha} + \mathcal{O}(n^0) \\ &\approx \frac{n+1}{\alpha} + \mathcal{O}(n^0), \end{aligned}$$

where $F_n \approx \alpha^n$, $2\alpha = 1 + \sqrt{5}$, was used. Now,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \frac{\cot\left(\frac{1}{F_{j+1}}\right)}{\cot\left(\frac{1}{F_{j+2}}\right)} &= \lim_{n\to\infty} \frac{1}{n} \left(\frac{n+1}{\alpha} + \mathcal{O}(n^0) \right) \\ &= \frac{1}{\alpha} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \\ &= \frac{1}{\alpha}. \end{aligned}$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Toyesh Prakash Sharma, St. C. F. Andrews School, Agra, India; Alina Craciun, Theoretical High School Miron Costin, Pascani, Romania; Arkady Alt, San Jose, CA, USA.

U561. The Fibonacci numbers F_n an the Lucas nubers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Prove that

$$2F_nF_{n+1}^5 - 2F_n^5F_{n+1} = F_n^6 + F_{n+1}^6 - F_{n+1}^2 - F_n^2.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by the author

The proposed identity may be written as

$$2F_nF_{n+1}\left(F_{n+1}^4 - F_n^4\right) = F_n^2\left(F_n^4 - 1\right) + F_{n+1}^2\left(F_{n+1}^4 - 1\right).$$

Now, since $F_n^4 = F_{n-2}F_{n-1}F_{n+1}F_{n+2} - 1$, we have

$$\begin{aligned} 2F_nF_{n+1}\left(F_{n+1}^4 - F_n^4\right) &= F_n^2\left(F_n^4 - 1\right) + F_{n+1}^2\left(F_{n+1}^4 - 1\right) \\ 2F_{n-1}F_nF_{n+1}F_{n+2}\left(F_nF_{n+3} - F_{n-2}F_{n+1}\right) &= F_n^2\left(F_{n-2}F_{n-1}F_{n+1}F_{n+2}\right) + F_{n+1}^2\left(F_{n-1}F_nF_{n+2}F_{n+3}\right) \\ 2F_{n-1}F_nF_{n+1}F_{n+2}\left(F_nF_{n+3} - F_{n-2}F_{n+1}\right) &= F_{n-1}F_nF_{n+1}F_{n+2}\left(F_{n-2}F_n + F_{n+1}F_{n+3}\right). \end{aligned}$$

So the problem is reduced to prove that

$$2\left(F_nF_{n+3} - F_{n-2}F_{n+1}\right) = F_{n-2}F_n + F_{n+1}F_{n+3}.$$

Since $5F_nF_m = L_{n+m} - (-1)^mL_{n-m}$, last identity is equivalent to

$$\begin{aligned} 2\left(L_{2n+3} - (-1)^nL_3 - L_{2n-1} + (-1)^nL_3\right) &= L_{2n-2} - (-1)^nL_2 + L_{2n+4} - (-1)^{n+1}L_2 \\ 2\left(L_{2n+3} - L_{2n-1}\right) &= L_{2n+4} + L_{2n-2} \end{aligned}$$

which is true since both terms are equal to $10F_{2n+1}$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Le Hoang Bao, Tien Giang, Vietnam; G. C. Greubel, Newport News, VA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U562. Let $ABCD$ be a square. The variable points P and Q are taken on the side AB and BC, respectively such that $\angle PDQ = 45°$. Find the locus of the orthocenter of the triangle PDQ.

Proposed by Mircea Becheanu, Canada

Solution by the author

We shall use coordinates. Take $D \equiv O$ the origin of the coordinate axes, $A(1,0) \in (OX)$ and $C(0,1) \in (OY)$. Then, the points P and Q have coordinates $P(1,s)$ and $Q(t,1)$ where $0 < s, t < 1$ are variables related by

$$1 = \frac{s+t}{1-st} \tag{2}$$

Let H be the orthocenter of the triangle OPQ. In order to find faster the coordinates of H we remind the problem J551: if AC intersects OP and OQ in E and F, respectively then $QE \perp OP$ and $PF \perp OQ$. Hence, we can find H to be the intersection point of QE and PF. The coordinates of the point E are obtained by solving the system of linear equations:

$$x + y = 1 \text{ and } y = sx.$$

We obtain the solution:

$$x_E = \frac{1}{1+s}; \ y_E = \frac{s}{1+s}. \tag{3}$$

From the system of equations

$$x + y = 1 \text{ and } y = \frac{1}{t}x$$

we obtain the coordinates

$$x_F = \frac{t}{1+t}; \ y_F = \frac{1}{1+t} \tag{4}$$

Using formulae (2) and (3), the equation of the line QE is $x + sy = s + t$ and the equation of the line PF is $x + sy = s + t$. Solving the system given by these equations one obtains:

$$x_H = 1 - s; \ y_H = 1 - t. \tag{5}$$

Using the relation (1) we can eliminate the parameters s and t and we obtain that x_H and y_H satisfy the equation

$$xy - 2x - 2y + 2 = 0.$$

We can write it in the form

$$(x-2)(y-2) = 2,$$

which shows, after a translation given by the vector $v = (2,2)$, that the locus is an equilateral hyperbola. It passes through the points A and C.

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Sarah B. Seales, Prescott, AZ, USA.

U563. Find all polynomials $P(x)$ with real coefficients such that, for all real numbers x,

$$\left(P(x^2)+x\right)\left(P(x^3)-x^2\right)=P(x^5)+x.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by G. C. Greubel, Newport News, VA, USA

Let $P(x)$ be a polynomial with the expansion $P(x)=a_0+a_1\,x+a_2\,x^2+\cdots$ for which

$$\begin{aligned} P(x^2)+x &= a_0+x+a_1\,x^2+a_2\,x^4+a_3\,x^6+a_4\,x^8+a_5\,x^{10}+a_6\,x^{12}+a_7\,x^{14}+\cdots \\ P(x^3)-x^2 &= a_0-x+a_1\,x^3+a_2\,x^6+a_3\,x^9+a_4\,x^{12}+a_5\,x^{15}+\cdots \\ P(x^5)+x &= a_0+x+a_1\,x^5+a_2\,x^{10}+a_3\,x^{15}+\cdots \end{aligned}$$

and leads to

$$\begin{aligned} & a_0+x+a_1\,x^5+a_2\,x^{10}+a_3\,x^{15}+\cdots = \\ & \quad a_0^2+a_0\,x+a_0(a_1-1)\,x^2+(a_0a_1-1)\,x^3+a_0a_2\,x^4+a_1^2\,x^5+\cdots. \end{aligned}$$

Comparing the coefficients of the powers of x leads to the values of a_n being $a_n\in\{1,1,0,0,0,\cdots\}_{n\geq 0}$. This yields $P(x)=1+x$.

Also solved by Le Hoang Bao, Tien Giang, Vietnam.

U564. Evaluate $\lim_{n\to+\infty}(-1)^n \cdot \sin\left(\sum_{k=1}^{4n} \arctan \frac{-3k^2+1-k^4}{-k^2+7+k^4}\right)$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

Solution by the author

$$\frac{-3k^2+1-k^4}{-k^2+7+k^4} = \frac{-1+q_k}{1+q_k} \iff q_k = \frac{4-2k^2}{k^4+k^2+3}$$

thus

$$\arctan \frac{-3k^2+1-k^4}{-k^2+7+k^4} = \arctan \frac{-1+q_k}{1+q_k} = \arctan(-1) + \arctan q_k =$$
$$= \frac{-\pi}{4} + \arctan \frac{4-2k^2}{k^4+k^2+3}$$

Moreover

$$\arctan \frac{k+1}{1+(k+1)^2} - \arctan \frac{k-1}{1+(k-1)^2} = \arctan \frac{\frac{k+1}{1+(k+1)^2} - \frac{k-1}{1+(k-1)^2}}{1+\frac{k+1}{1+(k+1)^2}\frac{k-1}{1+(k-1)^2}} =$$
$$= \arctan \frac{4-2k^2}{k^4+k^2+3}$$

and

$$\arctan \sum_{k=1}^{n} \frac{-3k^2+1-k^4}{-k^2+7+k^4} = \sum_{k=1}^{n}\left[\frac{-\pi}{4} + \arctan \frac{k+1}{1+(k+1)^2} - \arctan \frac{k-1}{1+(k-1)^2}\right] =$$
$$= \frac{-n\pi}{4} + \sum_{k=1}^{n}\left[\arctan \frac{k+1}{1+(k+1)^2} - \arctan \frac{k}{1+k^2} + \right.$$
$$+ \left. \arctan \frac{k}{1+k^2} - \arctan \frac{k-1}{1+(k-1)^2}\right] =$$
$$= \frac{-n\pi}{4} + \arctan \frac{n+1}{1+(n+1)^2} - \arctan \frac{1}{2} + \arctan \frac{n}{1+n^2} =$$
$$= \frac{-n\pi}{4} + \arctan \frac{\frac{n+1}{1+(n+1)^2} + \frac{n}{1+n^2}}{1-\frac{n+1}{1+(n+1)^2}\frac{n}{1+n^2}} - \arctan \frac{1}{2} =$$
$$= \frac{-n\pi}{4} + \arctan \frac{3n+2n^3+1+3n^2}{2+2n^2+n^4+n+2n^3} - \arctan \frac{1}{2} = \arctan \frac{\frac{3n+2n^3+1+3n^2}{2+2n^2+n^4+n+2n^3} - \frac{1}{2}}{1+\frac{3n+2n^3+1+3n^2}{2+2n^2+n^4+n+2n^3}\frac{1}{2}} =$$
$$= \frac{-n\pi}{4} + \arctan \frac{-n^4+2n^3+4n^2-5n}{2n^4+6n^3+7n^2+5n+5}$$

$$\sum_{k=1}^{4n} \arctan \frac{-3k^2+1-k^4}{-k^2+7+k^4} = -n\pi + \arctan \frac{-(4n)^4+2(4n)^3+4(4n)^2-20n}{2(4n)^4+6(4n)^3+7(4n)^2+20n+5}$$

and finally

$$\begin{aligned}
&\lim_{n\to+\infty} (-1)^n \sin\left[\sum_{k=1}^{4n} \arctan \frac{-3k^2+1-k^4}{-k^2+7+k^4}\right] = \\
&= \lim_{n\to+\infty} (-1)^n (-1)^n \sin\left[\arctan \frac{-(4n)^4+2(4n)^3+4(4n)^2-20n}{2(4n)^4+6(4n)^3+7(4n)^2+20n+5}\right] = \\
&= \sin\left[\arctan \lim_{n\to+\infty} \frac{-(4n)^4+2(4n)^3+4(4n)^2-20n}{2(4n)^4+6(4n)^3+7(4n)^2+20n+5}\right] = \\
&= \sin\arctan \frac{-1}{2} = \frac{-1}{2}\frac{1}{\sqrt{1+\frac{1}{4}}} = \frac{-1}{\sqrt{5}}
\end{aligned}$$

O559. Let x, y, z be real numbers such that none of them lies in the open interval $(-1, 1)$ and

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + x + y + z = 0.$$

Find the minimum value of $\frac{z}{x+y}$.

Proposed by Marius Stănean, Zalău, România

Solution by the author

Notice that if $\{a, b, c\} = \{x, y, z\}$, then

$$(a+b)\left(1+\frac{1}{ab}\right) = -\frac{c^2+1}{c} \Longrightarrow \frac{c}{a+b} < 0.$$

Suppose that $a \geq b \geq c$, then obviously $a \geq 1$, $c \leq -1$ and

$$\frac{a+b+c}{a+c} = -\frac{ab+bc+ca}{abc(a+c)} = -\frac{1}{ac} - \frac{1}{b(a+c)} > 0 \Longrightarrow \frac{b}{a+c} > -1.$$

We have 2 more cases to evaluate:

Case 1: $b \geq 1$, then as above $\frac{a}{b+c} > -1$.

Let us denote $A = a + b + \frac{1}{a} + \frac{1}{b} \geq 4$, so $c^2 + Ac + 1 = 0$. Solving the quadratic equation it follows that $c = \frac{-A - \sqrt{A^2-4}}{2} \leq -1$, and hence

$$\begin{aligned}\frac{2c}{a+b} = \frac{-A-\sqrt{A^2-4}}{a+b} &= -1 - \frac{1}{ab} - \sqrt{\left(1+\frac{1}{ab}\right)^2 - \frac{4}{(a+b)^2}} \\ &\geq -1 - \frac{1}{ab} - \sqrt{\left(1+\frac{1}{ab}\right)^2 - \frac{1}{(ab)^2}} \\ &= -1 - \frac{1}{ab} - \sqrt{1+\frac{2}{ab}} \geq -2-\sqrt{3},\end{aligned}$$

where above we used that

$$\frac{4}{(a+b)^2} \geq \frac{1}{(ab)^2} \Longleftrightarrow (ab)^2 - 1 + a^2(b^2-1) + b^2(a^2-1) + (ab-1)^2 \geq 0,$$

so

$$\frac{c}{a+b} \geq -1 - \frac{\sqrt{3}}{2}.$$

The equality holds when $a = b = 1$, $c = -2 - \sqrt{3}$.

Case 2: $b \le -1$, then as above $\dfrac{c}{a+b} > -1$.

Let us denote $B = -b - c - \dfrac{1}{b} - \dfrac{1}{c} \ge 4$, so $a^2 - Ba + 1 = 0$. Solving the quadratic equation it follows that $a = \dfrac{B + \sqrt{B^2 - 4}}{2} \ge 1$, and hence

$$\frac{2a}{b+c} = \frac{B + \sqrt{B^2-4}}{b+c} = -1 - \frac{1}{bc} - \sqrt{\left(1 + \frac{1}{bc}\right)^2 - \frac{4}{(b+c)^2}}$$
$$\ge -2 - \sqrt{3},$$

as in Case 1, so

$$\frac{a}{b+c} \ge -1 - \frac{\sqrt{3}}{2}.$$

The equality holds when $a = 2 + \sqrt{3}$, $b = c = -1$.

We conclude that the required minimum is $-1 - \frac{\sqrt{3}}{2}$.

O560. Prove that there are infinitely many triples (a, b, c) of integers for which $ab + bc + ca = 1$ an that for each such triple $\left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right)$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Văcaru, Pitești, Romania

We have $a^2 + 1 = (a+b)(a+c)$ (1)We obtain, for example $\begin{cases} a+b & = 1 \\ a+c & = a^2+1 \end{cases} \Rightarrow \begin{cases} b & = 1-a \\ c & = a^2-a+1 \end{cases}$.It is clear that $\left(a, 1-a, a^2-a+1\right), a \in \mathbb{Z}$is (an) triple which fulfilled $ab + bc + ca = 1$. Observe relationships $b^2 + 1 = (b+a)(b+c)$ (2) and $c^2 + 1 = (c+a)(c+b)$.We obtain

$$\left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right) = \left[(a+b)(b+c)(c+a)\right]^2,$$

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Kanav Talwar, Delhi Public School, Faridabad, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Mohammad Imran, India; Prithwijit De, HBCSE, Mumbai, India; Arkady Alt, San Jose, CA, USA; Taes Padhihary, Disha Delphi Public School, Rajasthan, India .

O561. Let ABC be a triangle. Prove that

$$\frac{a^2}{1+\cos^2 B+\cos^2 C}+\frac{b^2}{1+\cos^2 C+\cos^2 A}+\frac{c^2}{1+\cos^2 A+\cos^2 B}\le\frac{2}{3}\left(a^2+b^2+c^2\right)$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Taes Padhihary, Disha Delphi Public School, Rajasthan, India

Observe that

$$\begin{aligned}\frac{a^2}{1+\cos^2 B+\cos^2 C}&=\frac{a^2}{1+(\frac{c^2+a^2-b^2}{2ca})^2+(\frac{a^2+b^2-c^2}{2ab})^2}\\&\le\frac{a^2}{1+\frac{(c^2+a^2-b^2+a^2+b^2-c^2)^2}{4c^2a^2+4a^2b^2}}\\&=\frac{a^2}{1+\frac{a^2}{b^2+c^2}}\\&=\frac{(ab)^2+(ac)^2}{a^2+b^2+c^2}.\end{aligned}$$

Summing up the three relations, we obtain

$$\begin{aligned}LHS\le\frac{2((ab)^2+(bc)^2+(ca)^2)}{a^2+b^2+c^2}&=\frac{2}{a^2+b^2+c^2}\left((ab)^2+(bc)^2+(ca)^2\right)\\&\le\frac{2}{a^2+b^2+c^2}\cdot\frac{(a^2+b^2+c^2)^2}{3}\\&\le\frac{2}{3}(a^2+b^2+c^2).\end{aligned}$$

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Arkady Alt, San Jose, CA, USA.

O562. Find all functions $f:\mathbb{R}\leftarrow\mathbb{R}$ such that

$$x^2f(y)+f(y^2f(x))=xyf(x+y),$$

for all real numbers x,y.

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

Solution by the author

It's clear that $f(x)=x,\forall x\in\mathbb{R}$ and $f(x)=0,\forall x\in\mathbb{R}$ satisfy the given condition.
Let $P(x,y): x^2f(y)+f(y^2f(x))=xyf(x+y)$ $P(0,0): f(0)=0$. Suppose that there exist $t\in\mathbb{R}^*$ such that $f(t)=0$. $P(t,t): f(2t)=0$.

$$P(t,x): t^2f(x)=txf(x+t)\overset{t\neq0}{\Longleftrightarrow} tf(x)=xf(x+t) \tag{1}$$

Since $f(2t)=0$ we have similarly

$$2tf(x)=xf(x+2t) \tag{2}$$

Comparing (1) and (2) we have that $2f(x+t)=f(x+2t),\forall x\in\mathbb{R}^*$. Also $2f(0+t)=0=f(2t)=f(2t)=f(0+2t)$ so $2f(x)=f(x+t),\forall x\in\mathbb{R}$. Using (1) we have that $tf(x)=2xf(x)$ which means that $f(x)=0$ or x $t/2$. If $f(t/2)=0$ then $f(x)=0,\forall x\in\mathbb{R}$. If $f(t/2)\neq0$ then putting $x=-t/2$ in (1) we get $tf(-t/2)=\dfrac{t}{2}f(t/2)\neq0$, a contradiction since $-t/2\neq t/2\Rightarrow f(-t/2)=0$. Suppose now that there is no such t, i.e. $f(u)=0\Leftrightarrow u=0$. $P(x,-y): x^2f(-y)+f(y^2f(x))=-xyf(x-y)$. For the following suppose that $x,y\neq0$. Let $Q(x,y)\equiv P(x,y)-P(x,-y)$ i.e. $Q(x,y): x(f(y)-f(-y))=yf(x+y)+yf(x-y)$.

$$Q(x,x): x(f(x)-f(-x))=xf(2x)+xf(0)\Leftrightarrow f(x)-f(-x)=f(2x) \tag{3}$$

$$Q(-x,x): -x(f(x)-f(-x))=xf(0)+xf(-2x)\Leftrightarrow f(x)-f(-x)=-f(-2x) \tag{4}$$

Comparing (3), (4) we get $f(-2x)=-f(2x)$ i.e. f is odd. Now $Q(x,y)$ becomes $R(x,y): 2\dfrac{x}{y}f(y)=f(x+y)+f(x-y)$. $R(y,x)+R(x,y)\Rightarrow x^2f(y)+y^2f(x)=xyf(x+y)$. However, $P(x,y): x^2f(y)+f(y^2f(x))=xyf(x+y)$ so $f(y^2f(x))=y^2f(x)$. Put $x=1: f(y^2f(1))=y^2f(1)$. Also $f(-y^2f(1))=-f(y^2f(1))=-y^2f(1)$.
Since $f(1)\neq0$ we have that $f(x)=x,\forall x\in\mathbb{R}$.
Therefore $f(x)=x,\forall x\in\mathbb{R}$ and $f(x)=0,\forall x\in\mathbb{R}$ are the only solutions.

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Kanav Talwar, Delhi Public School, Faridabad, India.

O563. Prove that in any triangle ABC,

$$\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} + \frac{6(ab+bc+ca)}{(a+b+c)^2} \geq 5.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

$$m_a \geq \frac{b^2+c^2}{4R}.$$

This result can be rewritten as

$$m_a \geq \frac{abc}{4R} \cdot \frac{b^2+c^2}{abc} = \frac{S}{a} \cdot \frac{b^2+c^2}{bc} = h_a \cdot \frac{b^2+c^2}{2bc}.$$

It follows that

$$\frac{m_a}{h_a} \geq \frac{b^2+c^2}{2bc}.$$

Therefore

$$\begin{aligned}\sqrt{\frac{m_a}{h_a}} &\geq \sqrt{\frac{b^2+c^2}{2bc}} \\ &= \frac{b^2+c^2}{\sqrt{(2bc)(b^2+c^2)}} \\ &\geq \frac{2(b^2+c^2)}{(b+c)^2}.\end{aligned}$$

It is enough to show that

$$\sum_{\text{cyc}} \frac{2(b^2+c^2)}{(b+c)^2} + \frac{6(ab+bc+ca)}{(a+b+c)^2} \geq 5.$$

This is equivalent to

$$\sum_{\text{cyc}} \left(\frac{2(b^2+c^2)}{(b+c)^2} - 1 \right) + \frac{6(ab+bc+ca)}{(a+b+c)^2} - 2 \geq 0,$$

or

$$\sum_{\text{cyc}} \frac{(b-c)^2}{(b+c)^2} + \frac{2(ab+bc+ca-a^2-b^2-c^2)}{(a+b+c)^2} \geq 0,$$

or

$$\sum_{\text{cyc}} \frac{(b-c)^2}{(b+c)^2} - \frac{(b-c)^2+(c-a)^2+(a-b)^2}{(a+b+c)^2} \geq 0,$$

or

$$\sum_{\text{cyc}} \left(\frac{1}{(b+c)^2} - \frac{1}{(a+b+c)^2} \right) (b-c)^2 \geq 0$$

which is true because

$$\frac{1}{(b+c)^2} - \frac{1}{(a+b+c)^2} = \frac{a(a+2b+2c)}{(b+c)^2(a+b+c)^2} \geq 0.$$

The proof is completed. The equality occurs is and only if the triangle ABC is equilateral.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania.

O564. Let $n \geq 3$ be an integer. For every sequence $-1 = x_1 < x_2 < \cdots < x_n = 1$ of real numbers and every $k = 1, 2, \ldots, n$ we define

$$D_k(x_2, \ldots, x_{n-1}) = |(x_k - x_1)| \ldots (x_k - x_{k-1}(x_k - x_{k+1} \ldots (x_k - x_n)))$$

and denote

$$r_n = \max{}_{x_2,\ldots,x_{n-1}} \min{}_k D_k(x_2, \ldots, x_{n-1}).$$

Assume that r_n is achieved at the sequence $-1 = a_1 < a_2 < \cdots < a_n = 1$. Prove that

$$D_2(a_2, \ldots, a_{n-1}) = \cdots = D_{n-1}(a_2, \ldots, a_{n-1}).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

At the outset, we shall prove that for each $k = 1, \ldots, n-1$, we have

$$\min(D_k(a_2, \ldots, a_{n-1}), D_{k+1}(a_2, \ldots, a_{n-1})) = r_n.$$

Assume by contradiction that $\min(D_k(a_2, \ldots, a_{n-1}), D_{k+1}(a_2, \ldots, a_{n-1})) > r_n$. Let's define

$$b_i = \begin{cases} a_k + \varepsilon, & i = k > 1 \\ a_{k+1} - \varepsilon & i = k+1 < n \\ a_i & \text{otherwise.} \end{cases}$$

For some $\varepsilon > 0$ that satisfies $\varepsilon < \dfrac{a_{k+1} - a_k}{2}$.

Then, the fact that $\min(D_k(b_2, \ldots, b_{n-1}), D_{k+1}(b_2, \ldots, b_{n-1})) > r_n$ remains unchanged. Now, for $i \neq k, k+1, 2 \leq i \leq n-1$, one can find that $D_k(b_2, \ldots, b_{n-1}) - D_i(a_2, \ldots, a_{n-1}) =$

$$= \left| \prod_{\substack{j=1 \\ j \neq i,k,k+1}}^{n} (a_i - a_j) \right| |(a_i - a_k - \varepsilon)(a_i - a_{k+1} + \varepsilon) - (a_i - a_k)(a_i - a_{k+1})|$$

$$= \left| \prod_{\substack{j=1 \\ j \neq i,k,k+1}}^{n} (a_i - a_j) \right| |\varepsilon(a_{k+1} - a_k - \varepsilon)| > 0.$$

This means $\min{}_{1 \leq i \leq n} D_i(b_2, \ldots, b_{n-1}) = \min{}_{2 \leq i \leq n-1} D_i(b_2, \ldots, b_{n-1}) > r_n$. This contradicts with the choice of r_n. Thus,

$$D_2(a_2, \ldots, a_{n-1}) = \cdots = D_{n-1}(a_2, \ldots, a_{n-1}) = r_n.$$