On the Relationship Between Inversive Geometry and the Cardinalities of Infinite Sets

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Abstract

I will define inversion in a circle, and prove that cardinalities of different infinite sets are equal using inversive geometry. I will show examples with the graphing tool Desmos. Finally, I will extend my theorems to n dimensions.

1 Background

The inverse of a point P with respect to a circle C, center O, and radius r is the point P' lying on the ray OP such that $OP * OP' = r^2$. A reference circle is the circle to which the inversion is taking place. Similarly, the inverse of a point P with respect to an n dimensional sphere C, center O, and radius r is the point P' lying on the ray OP such that $OP * OP' = r^2$.

The following theorems were proven in [1].

Theorem 1.1. *The inversion of a line through O is a line through O.*

Theorem 1.2. *The inversion of a line not through O is a circle through O.*

Theorem 1.3. *The inversion of a circle through O is a line not through O.*

Theorem 1.4. *The inversion of a circle not through O is a circle not through O.*

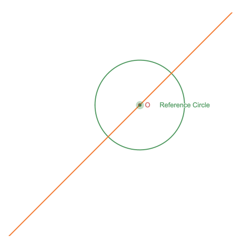

Figure 1: Inversion of a line through O

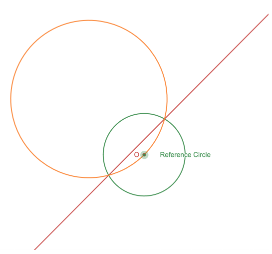

Figure 2: Inversion of a line not through O

We can extend this to any dimension:

Theorem 1.5. *The inversion of a n - dimensional plane through O is a n - dimensional plane through O.*

Theorem 1.6. *The inversion of a n - dimensional plane not through O is a (n+1) - dimensional sphere through O.*

Theorem 1.7. *The inversion of a (n+1) - dimensional sphere through O is a n - dimensional plane not through O.*

Theorem 1.8. *The inversion of a n - dimensional sphere not through O is a n - dimensional sphere not through O.*

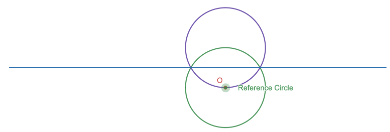

Figure 3: Inversion of a circle through O

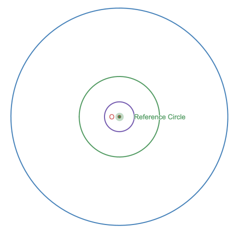

Figure 4: Inversion of a circle not through O

2 Inversion in two dimensions and Cardinalities of Infinite Sets

With these theorems, we will prove that the cardinalities of different infinite sets are equal. This follows from a very important property of inversion: **Inversion is biunique.** This means that if the point P' is the inverse of point P, then the point P is the inverse of point P'. This simple property of inversion allows us to prove many theorems related to cardinalities of sets. For example, because the inversion of a line not through O is a circle through O, we get: **The number of points on a line segment is the number of points on a line.** This is because we can "unravel" the circumference of the circle to get a line segment.

Theorem 2.1. *The number of points in a filled circle with any finite radius is equal to the number of points on a closed half-plane.*

Proof.

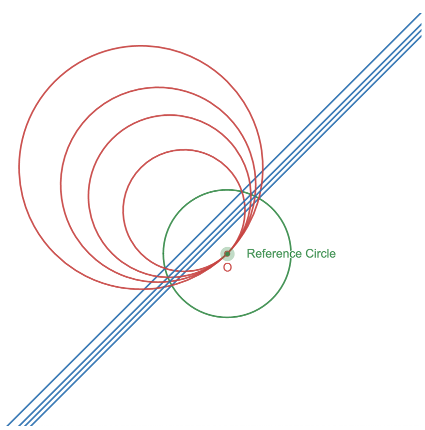

Figure 5: Inversion of closed half-plane not through O

Recall Theorem 1.2 (the inversion of a line not through O is a circle through O). Let's pick a line, L on either side of O. Let r be the radius of the reference circle, A be the point where the perpendicular from O meets L, and k be the distance from O to A. Then, consider the closed half-plane from L away from O. Note that this is just infinitely many parallel lines. Let's take the inversion of these infinitely many lines. Notice that by the definition of inversion, the image is a closed disk (with infinitely many tangent circles through O). The radius of this circle is finite, because by the definition of inversion, the radius of the circle will decrease as the distance between O and the line increases. Thus, the number of points on a closed half-plane is the number of points in a filled circle with any finite radius.

□

Theorem 2.2. *The number of points inside a filled circle (not including the boundary) equals the number of points outside the filled circle (not including the boundary).*

Proof.

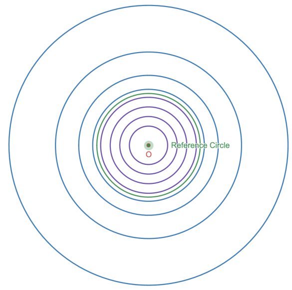

Figure 6: Inversion of infinitely many concentric circles not through O

Recall that the inversion of a circle not through O is a circle not through O. Now, consider an open disk with finite radius. Notice that this is just infinitely many concentric circles. Let's take the inversion of these infinitely many circles not through O. Note that as the radius of the pre-image circle tends to 0, the radius of the image tends to infinity (by the definition of inversion). Thus, when we take the inversion of these infinitely many circles, they get inverted to infinitely many circles outside the reference circle. Therefore, the number of points inside a filled circle (not including the boundary) equals the number of points outside the filled circle (not including the boundary). □

3 Inversion in n dimensions and Cardinalities of Infinite Sets

Using the previous theorems as a guide, we can generalize them to n dimensions.

Theorem 3.1. *The number of points in a filled n dimensional sphere with any finite radius is equal to the number of points in n dimensional closed half-space.*

Proof. Recall Theorem 1.6: The inversion of a n dimensional plane not through O is a $(n+1)$ dimensional sphere through O. Let's pick a n dimensional plane, L on either side of O. Let r be the radius of the reference $(n+1)$ dimensional sphere and let A be the point where the perpendicular from O meets L. Then, let k be the distance from O to A. Consider the closed $(n+1)$ dimensional half-space from L away from O. Note that this is just infinitely many n dimensional parallel planes. Let's take the inversion of these planes. Notice that by the definition of inversion, the image is a filled $(n+1)$ dimensional sphere (with infinitely many tangent $(n+1)$ dimensional spheres through O). Note that the radius of this $(n+1)$ dimensional sphere is finite, because by the definition of inversion, the radius of the $n+1$ dimensional sphere will decrease as the distance between the reference $n+1$ dimensional sphere and the n dimensional plane increases. Thus, the number of points on a closed n dimensional half-space is the number of points in a filled n dimensional sphere with a finite radius. □

Theorem 3.2. *The number of points in a filled n dimensional sphere (not including the boundary) equals the number of points outside the n dimensional sphere (not including the boundary).*

Proof. Recall that the inversion of a n dimensional sphere not through O is a n dimensional sphere not through O. Now, consider a open n dimensional disk with finite radius. Notice that this is just infinitely many n dimensional spheres. Let's take the inversion of these infinitely many n dimensional spheres not through O. Note that as the radius of the pre-image n dimensional sphere tends to 0, the radius of the image tends to infinity (by the definition of inversion). Thus, when we take the inversion of these infinitely many n dimensional spheres, they get inverted to infinitely many n dimensional spheres outside the reference n dimensional sphere. Therefore, the number of points in a filled n dimensional sphere (not including the boundary) equals the number of points outside the n dimensional sphere (not including the boundary). □

References

[1] What is Mathematics, Courant, Richard, Robbins, Herbert, Revised by Ian Stewart 1996.

[2] Geometry with an Introduction to Cosmic Topology, Hitchman, Michael 2018.