Junior Problems

J565. Let $f(m,n) = (mn+4)^2 + 4(m-n)^2$. Prove that $f(2021^2, 2023^2)$ is divisible by $(2022^2+1)^2$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J566. Let a, b, c, d be positive real numbers such that $abc + bcd + cda + dab = 1$. Prove that

$$\frac{1}{1+a^3} + \frac{1}{1+b^3} + \frac{1}{1+c^3} + \frac{1}{1+d^3} \le \frac{16}{5}$$

Proposed by An Zhenping, Xianyang Normal University, China

J567. Let x, y, z be real numbers, $z \neq 0$, such that

$$\left|\frac{y^2}{z} - 2xz\right| \le 2 \quad \text{and} \quad \left|y^2 z + \frac{2x}{z}\right| \le 2.$$

Find the maximum of $x^{2022} + y^2$.

Proposed by Mihaela Berindeanu, Bucharest ,România

J568. Let ABC be a scalene triangle and let M be the midpoint of BC. The circumcircle of ΔAMB meets AC at D, other than A. Similarly, the circumcircle of ΔAMC meets AB at E, other than A. Let N be the midpoint of DE. Prove that MN is parallel to the A-symmedian of ΔABC.

Proposed by Ana Boiangiu, Bucharest, România

J569. Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{\frac{2ab}{a^2+b^2}} + \sqrt[4]{\frac{2bc}{b^2+c^2}} + \sqrt[4]{\frac{2ca}{c^2+a^2}} + \frac{(a+b)(b+c)(c+a)}{8abc} \ge 4.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J570. Let ABC be an acute triangle. Prove that

$$\left(\frac{\sin A + \sin B}{\cos C}\right)^2 + \left(\frac{\sin B + \sin C}{\cos A}\right)^2 + \left(\frac{\sin C + \sin A}{\cos B}\right)^2 \ge 36.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S565. Let a, b, c, λ be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)} \geq 4.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S566. Let a, b, c, d be positive real numbers such that

$$abcd = 3 + 2(a+b+c+d) + (ab+ac+ad+bc+bd+cd)$$

Prove that

$$ab+ac+ad+bc+bd+cd \geq 3(a+b+c+d)+18$$

Proposed by An Zhenping, Xianyang Normal University, China

S567. Let x, y, z be positive real numbers such that $x+y+z=1$ and $(x-yz)(y-zx)(z-xy)>0$. Prove that

$$\frac{1}{x-yz}+\frac{1}{y-zx}+\frac{1}{z-xy} \geq \frac{2}{x+yz}+\frac{2}{y+zx}+\frac{2}{z+xy}$$

Proposed by Mircea Becheanu, Canada

S568. Let ABC be a triangle with $\angle ABC = 60^\circ$, O its circumcenter, and I its incenter. Let M be the intersection of AI with the circumcircle of $\triangle ABC$. Prove that if $OI = IM$, then $AB = \sqrt{2}AI$.

Proposed by Mihaela Berindeanu, Bucharest, România

S569. Find all perfect squares written in base 10 with one digit of 6, and n digits of 1, for some positive integer n.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

S570. Let a, b, c, d be positive numbers satisfying the equality

$$abc+abd+acd+bcd = ab+ac+ad+bc+bd+cd$$

and such that no two of them are less than 1 and the other two are greater than 1. Prove that

$$a+b+c+d-abcd \geq \frac{15}{16}.$$

Proposed by Marian Tetiva, România

Undergraduate Problems

U565. Let p be a prime and let $b_1, \ldots, b_{p-1}$ be integers such that they are congruent (in some order) to $1, \ldots, p-1$ modulo p. Also, let $a_1, \ldots, a_{p-1}$ be integers such that p divides $a_1 b_1 + \cdots + a_{p-1} b_{p-1}$. Prove that there is a permutation $i_1, \ldots, i_{p-1}$ of $1, \ldots, p-1$ such that the determinant of the circulant matrix

$$\begin{pmatrix} a_{i_1} & a_{i_2} & \ldots & a_{i_{p-2}} & a_{i_{p-1}} \\ a_{i_{p-1}} & a_{i_1} & a_{i_2} & & a_{i_{p-2}} \\ \vdots & a_{i_{p-1}} & a_{i_1} & \ddots & \vdots \\ a_{i_3} & & \ddots & \ddots & a_{i_2} \\ a_{i_2} & a_{i_3} & \ldots & a_{i_{p-1}} & a_{i_1} \end{pmatrix}$$

is also divisible by p.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

U566. Solve in real numbers the equation

$$125^x + 64^{\frac{1}{x}} + 81 \cdot 5^x \cdot 4^{\frac{1}{x}} = 27^3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U567. Let d be an even positive integer and let C be a complex number. Prove that there are no polynomials $Q(x)$ and $R(x)$ with complex coefficients and of degree at least two such that

$$(x - 1^2)(x - 3^2) \ldots (x - (d-1)^2) + C = Q(R(x)).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U568. Let $a > 2$ be a real number. Evaluate

$$\int_0^a \frac{\tan^{-1} x}{ax^2 - ax + a - 1}\, dx.$$

Proposed by Nicusor Zlota, Focşani, România

U569. Let d be a positive integer and let $P(X) = a_0 + a_1 X + ... + a_d X^d$ be a polynomial with positive coefficients. Prove that for any monic polynomial $f \in \mathbb{R}[X]$ taking positive values on $(0, \infty)$ there is a positive integer m such that all coefficients of $P(X)^m f(X)$ are nonnegative.

Proposed by Titu Andreescu, USA, and Navid Safaei, Iran

U570. Solve the following differential equation

$$\frac{dy}{dx} = \tan(x - y) + \cot(x - y).$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

Olympiad Problems

O565. Let a, b, c be the sidelengths of a triangle, $s = (a+b+c)/2$ its semiperimeter, and r its inradius. We denote

$$x = \sqrt{\frac{s-a}{s}},\ y = \sqrt{\frac{s-b}{s}},\ \text{and}\ z = \sqrt{\frac{s-c}{s}}.$$

Let $S = x+y+z$ and $Q = xy+xz+yz$. Prove that

$$\frac{r}{s} \le \frac{2S - \sqrt{4-Q}}{9} \le \frac{1}{3\sqrt{3}}.$$

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

O566. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{(a+b+1)^2}{a^3+b^3+1} + \frac{(b+c+1)^2}{b^3+c^3+1} + \frac{(c+a+1)^2}{c^3+a^3+1} \le 9.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O567. Let ABC be a scalene triangle with incenter I and circumcenter O. Let N be the center of the nine-point circle of $\triangle ABC$ and M be the midpoint of BC. Knowing that the midpoint of OI lies on side BC, prove that IM is parallel to AN.

Proposed by Todor Zaharinov, Sofia, Bulgaria

O568. Let x, y, z be positive real numbers such that $x^2+y^2+z^2+xyz = 4$. Prove that

$$\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 + 11xyz \ge 20.$$

Proposed by Marius Stănean, Zalău, România

O569. Let a, b, c be positive real numbers such that $a+b+c = ab+bc+ca$. Prove that

$$\frac{4abc}{(1+a)(1+b)(1+c)} + 4 \le \frac{3a}{1+a} + \frac{3b}{1+b} + \frac{3c}{1+c} \le 5.$$

Proposed by An Zhenping, Xianyang Normal University, China

O570. Find all perfect squares in base 10 with one digit of 4, and n digits of 9, for some positive integer n.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România