Junior Problems

**J565.** Let \( f(m, n) = (mn + 4)^2 + 4(m - n)^2 \). Prove that \( f(2021^2, 2023^2) \) is divisible by \((2022^2 + 1)^2\).

*Proposed by Adrian Andreeescu, University of Texas at Dallas, USA*

**J566.** Let \( a, b, c, d \) be positive real numbers such that \( abc + bcd + cda + dab = 1 \). Prove that
\[
\frac{1}{1 + a^2} + \frac{1}{1 + b^2} + \frac{1}{1 + c^2} + \frac{1}{1 + d^2} \leq \frac{16}{5}
\]

*Proposed by An Zhenping, Xianyang Normal University, China*

**J567.** Let \( x, y, z \) be real numbers, \( z \neq 0 \), such that
\[
\left| \frac{y^2}{z} - 2xz \right| \leq 2 \quad \text{and} \quad \left| y^2 z + 2xz \right| \leq 2.
\]
Find the maximum of \( x^{2022} + y^2 \).

*Proposed by Mihaela Berindeanu, Bucharest, România*

**J568.** Let \( ABC \) be a scalene triangle and let \( M \) be the midpoint of \( BC \). The circumcircle of \( \triangle AMB \) meets \( AC \) at \( D \), other than \( A \). Similarly, the circumcircle of \( \triangle AMC \) meets \( AB \) at \( E \), other than \( A \). Let \( N \) be the midpoint of \( DE \). Prove that \( MN \) is parallel to the \( A \)-symmedian of \( \triangle ABC \).

*Proposed by Ana Boiștia, Bucharest, România*

**J569.** Let \( a, b, c \) be positive real numbers. Prove that
\[
\sqrt[4]{\frac{2ab}{a^2 + b^2}} + \sqrt[4]{\frac{2bc}{b^2 + c^2}} + \sqrt[4]{\frac{2ca}{c^2 + a^2}} + \frac{(a + b)(b + c)(c + a)}{8abc} \geq 4.
\]

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**J570.** Let \( ABC \) be an acute triangle. Prove that
\[
\left( \frac{\sin A + \sin B}{\cos C} \right)^2 + \left( \frac{\sin B + \sin C}{\cos A} \right)^2 + \left( \frac{\sin C + \sin A}{\cos B} \right)^2 \geq 36.
\]

*Proposed by Marius Stănean, Zalău, România*
S565. Let $a, b, c, \lambda$ be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{abc(\lambda + 1)^3}{(a + \lambda b)(b + \lambda c)(c + \lambda a)} \geq 4.$$ 

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

S566. Let $a, b, c, d$ be positive real numbers such that

$$abcd = 3 + 2(a + b + c + d) + (ab + ac + ad + bc + bd + cd)$$

Prove that

$$ab + ac + ad + bc + bd + cd \geq 3(a + b + c + d) + 18$$

*Proposed by An Zhenping, Xianyang Normal University, China*

S567. Let $x, y, z$ be positive real numbers such that $x + y + z = 1$ and $(x - yz)(y - zx)(z - xy) > 0$. Prove that

$$\frac{1}{x - yz} + \frac{1}{y - zx} + \frac{1}{z - xy} \geq \frac{2}{x + yz} + \frac{2}{y + zx} + \frac{2}{z + xy}$$

*Proposed by Mircea Becheanu, Canada*

S568. Let $ABC$ be a triangle with $\angle ABC = 60^\circ$, $O$ its circumcenter, and $I$ its incenter. Let $M$ be the intersection of $AI$ with the circumcircle of $\triangle ABC$. Prove that if $OI = IM$, then $AB = \sqrt{2}AI$.

*Proposed by Mihaela Berindeanu, Bucharest, România*

S569. Find all perfect squares written in base 10 with one digit of 6, and $n$ digits of 1, for some positive integer $n$.

*Proposed by Titu Andreescu, USA, and Marian Tetiva, România*

S570. Let $a, b, c, d$ be positive numbers satisfying the equality

$$abc + abd + acd + bcd = ab + ac + ad + bc + bd + cd$$

and such that no two of them are less than 1 and the other two are greater than 1. Prove that

$$a + b + c + d - abcd \geq \frac{15}{16}.$$ 

*Proposed by Marian Tetiva, România*
Undergraduate Problems

U565. Let $p$ be a prime and let $b_1, \ldots, b_{p-1}$ be integers such that they are congruent (in some order) to $1, \ldots, p-1$ modulo $p$. Also, let $a_1, \ldots, a_{p-1}$ be integers such that $p$ divides $a_1 b_1 + \cdots + a_{p-1} b_{p-1}$. Prove that there is a permutation $i_1, \ldots, i_{p-1}$ of $1, \ldots, p-1$ such that the determinant of the circulant matrix

$$
\begin{pmatrix}
  a_{i_1} & a_{i_2} & \cdots & a_{i_{p-2}} & a_{i_{p-1}} \\
  a_{i_{p-1}} & a_{i_1} & a_{i_2} & \cdots & a_{i_{p-2}} \\
  \vdots & a_{i_{p-1}} & a_{i_1} & \ddots & \vdots \\
  a_{i_3} & \ddots & \ddots & \ddots & a_{i_2} \\
  a_{i_2} & a_{i_3} & \cdots & a_{i_{p-1}} & a_{i_1}
\end{pmatrix}
$$

is also divisible by $p$.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

U566. Solve in real numbers the equation

$$
125^x + 64^{1\over 2} + 81 \cdot 5^{x \cdot 4^{1\over 2}} = 27^3.
$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U567. Let $d$ be an even positive integer and let $C$ be a complex number. Prove that there are no polynomials $Q(x)$ and $R(x)$ with complex coefficients and of degree at least two such that

$$(x - 1^2)(x - 3^2) \cdots (x - (d - 1)^2) + C = Q(R(x)).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U568. Let $a > 2$ be a real number. Evaluate

$$
\int_0^a \frac{\tan^{-1} x}{ax^2 - ax + a - 1} \, dx.
$$

Proposed by Nicusor Zlota, Focșani, România

U569. Let $d$ be a positive integer and let $P(X) = a_0 + a_1 X + \ldots + a_d X^d$ be a polynomial with positive coefficients. Prove that for any monic polynomial $f \in \mathbb{R}[X]$ taking positive values on $(0, \infty)$ there is a positive integer $m$ such that all coefficients of $P(X)^m f(X)$ are nonnegative.

Proposed by Titu Andreescu, USA, and Navid Safaei, Iran

U570. Solve the following differential equation

$$
\frac{dy}{dx} = \tan(x - y) + \cot(x - y).
$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India
Olympiad Problems

O565. Let \(a, b, c\) be the sidelengths of a triangle, \(s = (a + b + c)/2\) its semiperimeter, and \(r\) its inradius. We denote
\[
x = \sqrt{s - \frac{a}{s}}, \quad y = \sqrt{s - \frac{b}{s}}, \quad \text{and} \quad z = \sqrt{s - \frac{c}{s}}.
\]
Let \(S = x + y + z\) and \(Q = xy + xz + yz\). Prove that
\[
\frac{r}{s} \leq \frac{2S - \sqrt{4 - Q}}{9} \leq \frac{1}{3\sqrt{3}}.
\]
Proposed by Titu Andreescu, USA, and Marian Tetiva, România

O566. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that
\[
\frac{(a + b + 1)^2}{a^3 + b^3 + 1} + \frac{(b + c + 1)^2}{b^3 + c^3 + 1} + \frac{(c + a + 1)^2}{c^3 + a^3 + 1} \leq 9.
\]
Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O567. Let \(ABC\) be a scalene triangle with incenter \(I\) and circumcenter \(O\). Let \(N\) be the center of the nine-point circle of \(\triangle ABC\) and \(M\) be the midpoint of \(BC\). Knowing that the midpoint of \(OI\) lies on side \(BC\), prove that \(IM\) is parallel to \(AN\).

Proposed by Todor Zaharinov, Sofia, Bulgaria

O568. Let \(x, y, z\) be positive real numbers such that \(x^2 + y^2 + z^2 + xyz = 4\). Prove that
\[
\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 + 11xyz \geq 20.
\]
Proposed by Marius Stânean, Zalău, România

O569. Let \(a, b, c\) be positive real numbers such that \(a + b + c = ab + bc + ca\). Prove that
\[
\frac{4abc}{(1 + a)(1 + b)(1 + c)} + 4 \leq \frac{3a}{1 + a} + \frac{3b}{1 + b} + \frac{3c}{1 + c} \leq 5.
\]
Proposed by An Zhenping, Xianyang Normal University, China

O570. Find all perfect squares in base 10 with one digit of 4, and \(n\) digits of 9, for some positive integer \(n\).

Proposed by Titu Andreescu, USA, and Marian Tetiva, România