Junior problems

J565. Let $f(m,n) = (mn+4)^2 + 4(m-n)^2$. Prove that $f(2021^2, 2023^2)$ is divisible by $(2022^2+1)^2$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
With $f(m,n) = (mn+4)^2 + 4(m-n)^2$,

$$\begin{aligned}
f(2021^2, 2023^2) &= (2021^2 \cdot 2023^2 + 4)^2 + 4(2021^2 - 2023^2)^2 \\
&= ((2022^2-1)^2+4)^2 + 64 \cdot 2022^2 \\
&= ((2022^2+1)^2 - 4 \cdot 2022^2 + 4)^2 + 64 \cdot 2022^2 \\
&= (2022^2+1)^4 + 8(1-2022^2)(2022^2+1)^2 + 16(1-2022^2)^2 + 64 \cdot 2022^2 \\
&= (2022^2+1)^4 + 8(1-2022^2)(2022^2+1)^2 + 16(1+2022^2)^2 \\
&= (2022^2+1)^2((2022^2+1)^2 - 8 \cdot 2022^2 + 24).
\end{aligned}$$

Thus, $f(2021^2, 2023^2)$ is divisible by $(2022^2+1)^2$.

Also solved by Polyahedra, Polk State College, USA; Joel Schlosberg, Bayside, NY, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; G. C. Greubel, Newport News, VA, USA; Jiang Lianjun, Quanzhou No.2 Middle School of Quan Zhou, GuiLin, China; Lames Neama, SUNY Brockport, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh City, Vietnam; Joe Simons, Utah Valley University Orem, UT, USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Kanav Talwar, Delhi Public School, Faridabad, India; Le Hoang Bao, Tien Giang, Vietnam; Prajnanaswaroopa S, Bangalore, India; Arkady Alt, San Jose, CA, USA.

J566. Let a, b, c, d be positive real numbers such that $abc + bcd + cda + dab = 1$. Prove that

$$\frac{1}{1+a^3} + \frac{1}{1+b^3} + \frac{1}{1+c^3} + \frac{1}{1+d^3} \le \frac{16}{5}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

The inequality is equivalent to

$$\frac{a^3}{1+a^3} + \frac{b^3}{1+b^3} + \frac{c^3}{1+c^3} + \frac{d^3}{1+d^3} \ge \frac{4}{5}.$$

By applying Cauchy inequality, we obtain

$$\sum \frac{a^3}{1+a^3} = \sum \frac{a^4}{a+a^4} \ge \frac{\left(\sum a^2\right)^2}{\sum a + \sum a^4}.$$

So, we have to prove that

$$\frac{\left(\sum a^2\right)^2}{\sum a + \sum a^4} \ge \frac{4}{5} \Leftrightarrow 5\left(\sum a^2\right)^2 \ge 4\sum a \sum abc + 4\sum a^4$$

which is true. In fact,

$$4\sum a \sum abc + 4\sum a^4 = 16\prod a + 4\sum a^2(bc+bd+cd) + 4\sum a^4 \le \left(\sum a^2\right)^2 + 4\sum a^2(b^2+c^2+d^2) + 4\sum a^4$$

$$= \left(\sum a^2\right)^2 + 4\left(\sum a^2\right)^2 = 5\left(\sum a^2\right)^2.$$

Also solved by Polyahedra, Polk State College, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Nicusor Zlota, Focşani, România.

J567. If $x, y, z \in \mathbb{R},\ z \neq 0$ such that $\left|\frac{y^2}{z} - 2xz\right| \leq 2,$ and $\left|y^2 z + 2\frac{x}{z}\right| \leq 2$, find the maxim value of $x^{2022} + y^2$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by the author

$\left(\left|\frac{y^2}{z} - 2xz\right|\right)^2 \leq (2)^2 \Rightarrow$

$$\frac{y^4}{z^2} + 4x^2z^2 - 4y^2x \leq 4 \tag{1}$$

$\left(\left|y^2z + 2\frac{x}{z}\right|\right)^2 \leq (2)^2 \Rightarrow$

$$y^4z^2 + \frac{4x^2}{z^2} + 4y^2z \leq 4 \tag{2}$$

From $(1) + (2) \Rightarrow$

$$y^4\left(z^2 + \frac{1}{z^2}\right) + 4x^2\left(z^2 + \frac{1}{z^2}\right) \leq 8 \Rightarrow \underbrace{\left(z^2 + \frac{1}{z^2}\right)}_{\geq 2}\underbrace{\left(y^4 + 4x^2\right)}_{\leq 4} \leq 8$$

$z \neq 0 \Rightarrow z^2 + \frac{1}{z^2} \geq 2$, so $y^4 + 4x^2 \leq 4$.

$$\begin{cases} y^4 + 4x^2 \leq 4 \\ y^4 \geq 0 \\ x^2 \geq 0 \end{cases} \Rightarrow \begin{cases} y^4 \leq 4 \\ 4x^2 \leq 4 \Rightarrow x^2 \leq 1 \Rightarrow x^{2021} \leq x^2 \end{cases}$$

$$x^{2021} + y^2 \leq x^2 + y^2 \leq x^2 + \frac{y^4 + 4}{4} = \frac{4x^2 + y^4 + 4}{4} \leq \frac{8}{4} = 2, \text{ with equality for } x = 0,\ y = \pm\sqrt{2},\ z = \pm 1$$

Conclusion:

The maximum value of $x^{2022} + y^2$ is 2.

Also solved by Polyahedra, Polk State College, USA; Mai Vo Phuc Thanh, Tien Giang High School for the Gifted, Vietnam; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.

J568. Let ABC be a scalene triangle and let M be the midpoint of BC. The circumcircle of $\triangle AMB$ meets AC at D, other than A. Similarly, the circumcircle of $\triangle AMC$ meets AB at E, other than A. Let N be the midpoint of DE. Prove that MN is parallel to the A-symmedian of $\triangle ABC$.

Proposed by Ana Boiangiu, Bucharest, România

First solution by the author

Witout loss of generality, assume that $AC > AB$.

Note that by power of a point, $BE \cdot BA = BM \cdot BC = \frac{BC^2}{2}$ and $CD \cdot CA = CM \cdot CB = \frac{BC^2}{2}$. Therefore, $BE \cdot BA = CD \cdot CA$, or equivalently $\frac{BA}{CA} = \frac{CD}{BE}$.

Let (AED) meet (ABC) a second time at K. It is well-known that K is the center of spiral similarity sending BE to CD. Therefore $\Delta KEB \sim \Delta KDC$ and thus $\frac{CD}{BE} = \frac{KC}{KB}$.

We can now conclude that $\frac{KB}{KC} = \frac{CA}{BA}$ and since $\angle BKC = \angle BAC$, we have $\Delta BKC \sim \Delta CAB$. This proves that $\angle KCB = \angle ABC$ or equivalently that $BAKC$ is an isosceles trapezoid.

Now since $\frac{NE}{ND} = \frac{MB}{MC} = \frac{1}{2}$, K is also the center of the spiral similarity sending MN to CD and therefore $\Delta KMN \sim \Delta KCD$. Consequently

$$\angle KMN = \angle KCD = \angle KCA = \angle KCB - \angle ACB = \angle ABC - \angle ACB$$

since $BAKC$ is an isosceles trapezoid. But by symmetry $\angle KMC = \angle AMB$ and thus

$$\angle NMC = \angle KMN + \angle KMC = \angle ABC - \angle ACB + \angle AMB.$$

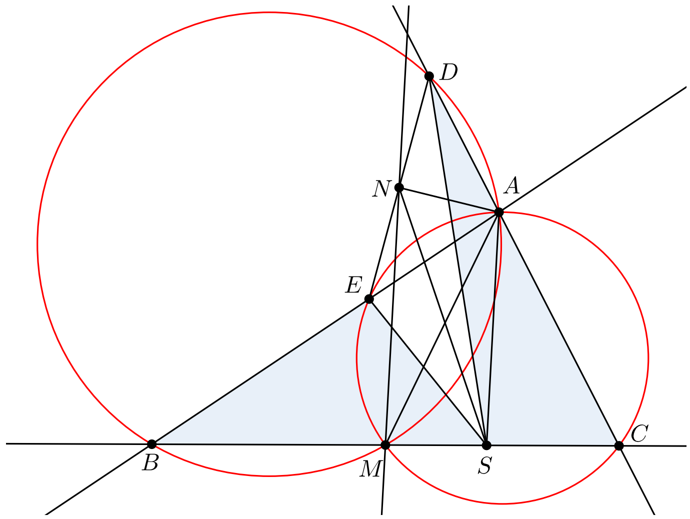
Let T be the foot of the A-symmedian of ΔABC. We wish to compute angle $\angle ATC$, that is, show that it is equal to angle $\angle NMC$. Noting that $\angle TAC = \angle BAM$, we have

$$\angle ATC = 180^\circ - \angle TAC - \angle TCA = 180^\circ - \angle BAM - \angle ACB$$

$$= 180^\circ - (180^\circ - \angle AMB - \angle ABM) - \angle ACB = \angle ABC - \angle ACB + \angle AMB.$$

Finally, $\angle NMC = \angle ATC$, which lets us conclude that $MN \parallel AT$, as desired.

Second solution by Polyahedra, Polk State College, USA



Suppose that the A-symmedian of $\triangle ABC$ meets BC at S. Let h_D and h_E be the distances from S to CD and BE, respectively. By the power of a point, $BE \cdot BA = BM \cdot BC = CM \cdot CB = CD \cdot CA$. By the law of sines,

$$\frac{h_D}{h_E} = \frac{\sin \angle SAC}{\sin \angle BAS} = \frac{\sin \angle BAM}{\sin \angle MAC} = \frac{CA}{BA} = \frac{BE}{CD},$$

thus $[SCD] = \frac{1}{2}CD \cdot h_D = \frac{1}{2}BE \cdot h_E = [SEB]$. Since N is the midpoint of DE,

$$[SCAN] = \frac{[SCD] + [SCAE]}{2} = \frac{[SEB] + [SCAE]}{2} = \frac{[ABC]}{2} = [MCA].$$

Therefore, $[SAN] = [SAM]$, from which the claim follows.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Mai Vo Phuc Thanh, Tien Giang High School for the Gifted, Vietnam; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.

J569. Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{\frac{2ab}{a^2+b^2}}+\sqrt[4]{\frac{2bc}{b^2+c^2}}+\sqrt[4]{\frac{2ca}{c^2+a^2}}+\frac{(a+b)(b+c)(c+a)}{8abc}\geq 4.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

The AM-GM inequality gives us

$$\sqrt[4]{\frac{2bc}{b^2+c^2}}=\frac{\sqrt{2bc}}{\sqrt[4]{(2bc)(b^2+c^2)}}=\sqrt{\frac{2bc}{\sqrt{(2bc)(b^2+c^2)}}}\geq\sqrt{\frac{4bc}{2bc+b^2+c^2}}=\frac{2\sqrt{bc}}{b+c}.$$

Similarly

$$\sqrt[4]{\frac{2ca}{c^2+a^2}}\geq\frac{2\sqrt{ca}}{c+a},$$

$$\sqrt[4]{\frac{2ab}{a^2+b^2}}\geq\frac{2\sqrt{ab}}{a+b}.$$

From here and by using the AM-GM inequality again we obtain

$$\begin{aligned}\text{LHS}&\geq\frac{2\sqrt{bc}}{b+c}+\frac{2\sqrt{ca}}{c+a}+\frac{2\sqrt{ab}}{a+b}+\frac{(a+b)(b+c)(c+a)}{8abc}\\&\geq 4\sqrt[4]{\frac{2\sqrt{bc}}{b+c}\cdot\frac{2\sqrt{ca}}{c+a}\cdot\frac{2\sqrt{ab}}{a+b}\cdot\frac{(a+b)(b+c)(c+a)}{8abc}}\\&=4.\end{aligned}$$

The proof is completed. The equality holds if and only if $a=b=c$.

Also solved by Arkady Alt, San Jose, CA, USA; Polyahedra, Polk State College, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicusor Zlota, Focşani, România; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.

J570. Let ABC be an acute triangle. Prove that

$$\left(\frac{\sin A+\sin B}{\cos C}\right)^2+\left(\frac{\sin B+\sin C}{\cos A}\right)^2+\left(\frac{\sin C+\sin A}{\cos B}\right)^2\geq 36.$$

Proposed by Marius Stănean, Zalău, România

First solution by the author

With the substitutions $A\longrightarrow\frac{\pi-A}{2}$, $B\longrightarrow\frac{\pi-B}{2}$, $C\longrightarrow\frac{\pi-C}{2}$, we need to show that in any triangle ABC, we have

$$\sum_{cyc}\left(\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\sin\frac{C}{2}}\right)^2\geq 36,$$

or

$$\sum_{cyc}\frac{pab}{(p-a)(p-b)}\left[\frac{p-a}{bc}+\frac{p-b}{ac}+\frac{2\sqrt{(p-a)(p-b)}}{c\sqrt{ab}}\right]\geq 36,$$

or

$$\sum_{cyc}\frac{pab}{(p-a)(p-b)}\left[a(p-a)+b(p-b)+2\sqrt{ab(p-a)(p-b)}\right]\geq 36abc.$$

Now, using Ravi's substitutions, i.e. $a=y+z$, $b=z+x$, $c=x+y$, $x,y,z\geq 0$, this becomes

$$\sum_{cyc}\frac{(x+y+z)(y+z)(z+x)}{xy}\left(xy+zx+xy+yz+2\sqrt{xy(y+z)(z+x)}\right)$$

$$\geq 36(x+y)(y+z)(z+x),$$

or

$$\frac{(x+y)(y+z)(z+x)(x^2+y^2+z^2)(x+y+z)}{xyz}+2(x+y+z)\sum_{cyc}(y+z)(z+x)$$

$$+2(x+y+z)\sum_{cyc}\sqrt{\frac{(y+z)^3(z+x)^3}{xy}}\geq 36(x+y)(y+z)(z+x).$$

But this is true because

$$\frac{(x+y)(y+z)(z+x)(x^2+y^2+z^2)(x+y+z)}{xyz}\geq 9(x+y)(y+z)(z+x),$$

$$\begin{aligned}2(x+y+z)\sum_{cyc}(y+z)(z+x) &=2(x+y)(y+z)(z+x)(x+y+z)\sum_{cyc}\frac{1}{x+y}\\ &\geq 2(x+y)(y+z)(z+x)(x+y+z)\cdot\frac{9}{2(x+y+z)}\\ &=9(x+y)(y+z)(z+x),\end{aligned}$$

and from the AM-GM Inequality, we have

$$\begin{aligned}2(x+y+z)\sum_{cyc}\sqrt{\frac{(y+z)^3(z+x)^3}{xy}} &\geq\frac{6(x+y)(y+z)(z+x)(x+y+z)}{\sqrt[3]{xyz}}\\ &\geq 18(x+y)(y+z)(z+x).\end{aligned}$$

Second solution by Polyahedra, Polk State College, USA

Let R and Δ be the circumradius and area of $\triangle ABC$. Then

$$\left(\frac{abc}{R}\right)^2 = 16\Delta^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4 = \sum_{cyclic} c^2(a^2 + b^2 - c^2).$$

Hence, by the Cauchy-Schwarz inequality,

$$\sum_{cyclic}\left(\frac{\sin A + \sin B}{\cos C}\right)^2 = \sum_{cyclic}\frac{a^2b^2(a+b)^2}{R^2(a^2+b^2-c^2)^2} = 16\Delta^2\sum_{cyclic}\frac{(a+b)^2}{c^2(a^2+b^2-c^2)^2}$$
$$\geq \left(\sum_{cyclic}\frac{a+b}{\sqrt{a^2+b^2-c^2}}\right)^2.$$

Applying Jensen's inequality to the convex function $1/\sqrt{x}$ we get

$$\frac{a}{\sqrt{a^2+b^2-c^2}} + \frac{a}{\sqrt{c^2+a^2-b^2}} \geq \frac{2a\sqrt{2}}{\sqrt{a^2+b^2-c^2+c^2+a^2-b^2}} = 2.$$

Summing this with the other two analogous inequalities completes the proof.

Also solved by Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicusor Zlota, Focşani, România; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.

Senior problems

S565. Let a, b, c, λ be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)} \geq 4.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

We have

$$\begin{aligned}
&\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)}\\
&=\frac{\lambda}{\lambda+1}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{\lambda+1}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)}\\
&=\frac{\lambda}{\lambda+1}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{\lambda+1}\left(\frac{a+\lambda b}{b}+\frac{b+\lambda c}{c}+\frac{c+\lambda a}{a}-3\lambda\right)+\frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)}\\
&=\frac{\lambda}{\lambda+1}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)+\frac{a+\lambda b}{(\lambda+1)b}+\frac{b+\lambda c}{(\lambda+1)c}+\frac{c+\lambda a}{(\lambda+1)a}+\frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)}\\
&\geq 4.
\end{aligned}$$

The last inequality is true by the AM-GM inequality as follows

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3,$$

and

$$\frac{a+\lambda b}{(\lambda+1)b}+\frac{b+\lambda c}{(\lambda+1)c}+\frac{c+\lambda a}{(\lambda+1)a}+\frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)} \geq 4.$$

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicusor Zlota, Focşani, România; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Arkady Alt, San Jose, CA, USA.

S566. Let a, b, c, d be positive real numbers such that

$$abcd = 3 + 2(a + b + c + d) + (ab + ac + ad + bc + bd + cd)$$

Prove that

$$ab + ac + ad + bc + bd + cd \geq 3(a + b + c + d) + 18$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Theo Koupelis, Broward College, Pembroke Pines, FL, USA

Let $f(a, b, c, d) = \sum ab - 3\sum a - 18$ and $g(a, b, c, d) = abcd - 3 - 2\sum a - \sum ab = 0$. Because f and g are polynomials, they have continuous first partial derivatives. Also, the gradient of g is never zero; indeed, if $\partial g/\partial a = bcd - 2 - b - c - d = 0$, then $abcd = 2a + ab + ac + ad$, which would contradict $g = 0$. Therefore, by the theorem of Lagrange multipliers, any extrema occur on the boundary or where $\nabla f = \lambda \nabla g$, for suitable scalars λ. We have

$$\begin{aligned} b + c + d - 3 &= \lambda(bcd - 2 - b - c - d), \\ a + c + d - 3 &= \lambda(acd - 2 - a - c - d), \\ a + b + d - 3 &= \lambda(abd - 2 - a - b - d), \\ a + b + c - 3 &= \lambda(abc - 2 - a - b - c). \end{aligned}$$

Subtracting the first equation above from each of the other three we get

$$\begin{aligned} a - b &= \lambda(a - b)(cd - 1), \\ a - c &= \lambda(a - c)(bd - 1), \\ a - d &= \lambda(a - d)(bc - 1). \end{aligned}$$

From the given condition it is obvious that the product of any two of the variables a, b, c, d cannot equal 1. Therefore, we get

$$\begin{aligned} a = b \quad &\text{or} \quad \lambda = \frac{1}{cd - 1}, \\ a = c \quad &\text{or} \quad \lambda = \frac{1}{bd - 1}, \\ a = d \quad &\text{or} \quad \lambda = \frac{1}{bc - 1}. \end{aligned}$$

From the above we see that setting two expressions for λ equal to each other leads to two variables being equal to each other. Thus, we need to examine the following cases.

(i) If $a = b = c = d$, the condition becomes $a^4 - 6a^2 - 8a - 3 = 0$ or $(a - 3)(a^3 + 3a^2 + 3a + 1) = 0$. Therefore $a = b = c = d = 3$, which leads to $f = 0$.

(ii) If $a \neq b = c = d$, the condition leads to $a = \frac{3}{b-2}$. Thus $b > 2$ and

$$f = 3ab + 3b^2 - 3a - 9b - 18 = \frac{9(b - 1)}{b - 2} + 3(b^2 - 3b - 6) = \frac{3(b - 3)^2(b + 1)}{b - 2}.$$

Therefore $f \geq 0$, with equality when $a = b = c = d = 3$.

(iii) If $a = b$ and $c = d$, the given condition leads to $(a+c-ac+3)(a+c+ac+1) = 0$, and thus $ac = a+c+3$. Using AM-GM, we get $ac \geq 3 + 2\sqrt{ac} \Longrightarrow (\sqrt{ac}+1)(\sqrt{ac}-3) \geq 0$, and thus $ac \geq 9$. Therefore,

$$f = (a+c)^2 - 6(a+c) + 2ac - 18 = (a+c)^2 - 4(a+c) - 12 = (a+c+2)(a+c-6).$$

But $a + c \geq 2\sqrt{ac} \geq 6$ and thus $f \geq 0$. Equality occurs when $a = c$ and thus $a = b = c = d = 3$.

We now check for extrema on the boundary. From the condition we get $a[bcd - 2 - (b + c + d)] > 0$, and therefore $bcd > 2+(b+c+d)$; using AM-GM we find that $bcd \geq 8$. Also, $b(cd-1) > 2+c+d > 0$, and thus $cd > 1$. Similarly we find that the product of any two of the variables is greater than 1. Therefore, at most one of the variables could tend to 0^+, which would lead to the other three variables tending to $+\infty$; in this case, clearly $f > 0$. On the other hand, because at most one of the variables could be less than 1, as one or more of the other three tend to $+\infty$, we clearly have $f > 0$. In summary, $f \geq 0$, with equality when $a = b = c = d = 3$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicusor Zlota, Focşani, România.

S567. Let x,y,z be positive real numbers such that $x+y+z=1$ and $(x-yz)(y-zx)(z-xy)>0$. Prove that

$$\frac{1}{x-yz}+\frac{1}{y-zx}+\frac{1}{z-xy}\geq\frac{2}{x+yz}+\frac{2}{y+zx}+\frac{2}{z+xy}$$

Proposed by Mircea Becheanu, Canada

Solution by the author

From the condition $(x-yz)(y-zx)(z-xy)>0$ it follows that all factors are positive or two factors are negative and the third is positive. Assuming that $x<yz$ and $y<zx$ we obtain, by multiplication $xy<xyz^2$, hence $z^2>1$, which is a contradiction. Therefore, all factors are positive.

From the following computation

$$x-yz=x(x+y+z)-yz=x^2+xy+xz-yz=\frac{1}{2}\left((x+y)^2+(z+x)^2-(y+z)^2\right)$$

we are led to denote $x+y=c$, $y+z=a$ and $z+x=b$, where a,b,c are the sides of an acute triangle ABC. Then we have

$$x-yz=(1/2)(b^2+c^2-a^2),$$
$$y-zx=(1/2)(c^2+a^2-b^2),$$
$$z-xy=(1/2)(a^2+b^2-c^2).$$

Similarly, one obtains $x+yz=bc$, $y+zx=ca$ and $z+xy=ab$. Now. the given inequality is equivalent to:

$$\sum_{cyc}\frac{2}{b^2+c^2-a^2}\geq\sum_{cyc}\frac{2}{ab}\Leftrightarrow\sum_{cyc}\frac{abc}{b^2+c^2-a^2}\geq\sum_{cyc}\frac{abc}{ab}\Leftrightarrow$$

$$\sum_{cyc}\frac{a}{\frac{b^2+c^2-a^2}{2bc}}\geq2\sum_{cyc}a\Leftrightarrow\sum_{cyc}\frac{a}{\cos A}\geq2\sum_{cyc}a\Leftrightarrow$$

$$\sum_{cyc}\frac{2R\sin A}{\cos A}\geq2\sum_{cyc}2R\sin A\Leftrightarrow\sum_{cyc}\tan A\geq2\sum_{cyc}\sin A.$$

The last inequality is obtained by Jensen inequalities in an acute triangle ABC:

$$\tan A+\tan B+\tan C\geq3\sqrt{3}$$

and

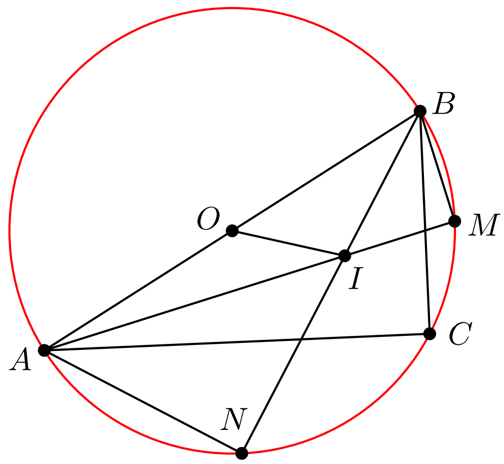
$$\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}.$$

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Arkady Alt, San Jose, CA, USA; Nicusor Zlota, Focşani, România; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.

S568. Let ABC be a triangle with $\measuredangle(B) = 60°$. If $\triangle ABC$ circumcircle is Γ, O is the circumcenter, I is the incenter, $AI \cap \Gamma = M$ and $OI = IM$, show that $AB = AI\sqrt{2}$.

Proposed by Mihaela Berindeanu, Bucharest, România

First solution by Li Zhou, Polk State College, USA



Let R and r be the circumradius and inradius of $\triangle ABC$. It is well known that $r/R = \cos A + \cos B + \cos C - 1$, $OI^2 = R(R-2r)$, and $IM = BM$. Therefore, if $B = 60°$ and $OI = IM$, then

$$2\cos A + 2\cos C - 1 = \frac{2r}{R} = 1 - \left(\frac{BM}{R}\right)^2 = 1 - 4\sin^2\frac{A}{2} = 2\cos A - 1,$$

thus $C = 90°$. Suppose that BI intersects the circumcircle of $\triangle ABC$ at N. Then $AN = IN$, so $AI = \sqrt{2}AN = AB/\sqrt{2}$.

Second solution by the author

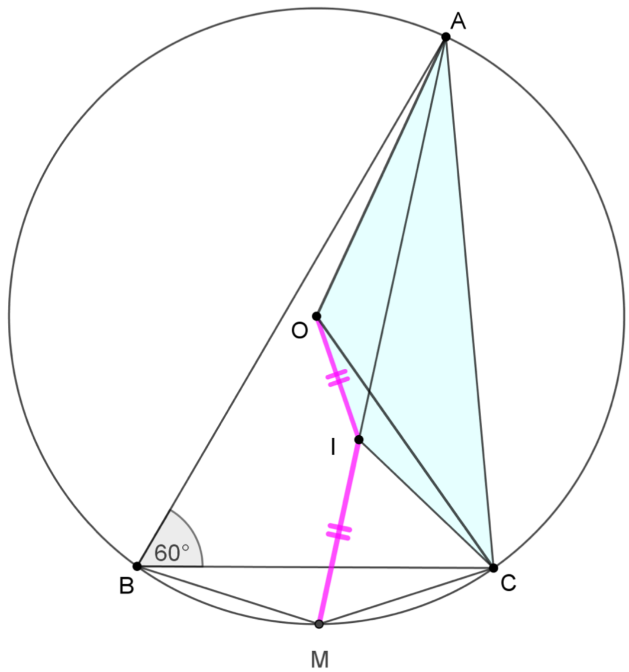


Figura 1:

- Show that $AOIC$ is an inscriptible quadrilateral
  $\angle ABC$ is an inscribed angle and $\measuredangle(ABC) = 60°$.
  $\angle AOC$ is a central angle, so $\measuredangle(AOC) = 2\measuredangle(ABC) = 120°$
  $AI,\ CI$ are bisectors, so
  $\measuredangle(AIC) = 180° - \dfrac{\measuredangle(A)}{2} - \dfrac{\measuredangle(C)}{2} = 180° - \dfrac{\measuredangle(A+C)}{2} = 180° - \dfrac{90° - \measuredangle(B)}{2} = \dfrac{\measuredangle(B)}{2} + 90° = 120°$
  $\measuredangle(AOC) = \measuredangle(AIC) \Rightarrow AOIC$ is an inscriptible quadrilateral.

- Show that $BM = MC = MI$

  $\measuredangle(BAM) = \measuredangle(MAC) \Rightarrow \overset{\frown}{BM} \equiv \overset{\frown}{MC} \Rightarrow BM = MC$

  $$\left.\begin{array}{l} \triangle ABI: \quad \angle BIM \text{ exterior angle } \Rightarrow \measuredangle(BIM) = \dfrac{\measuredangle(A)}{2} + \dfrac{\measuredangle(B)}{2} \\ \measuredangle(IBM) = \measuredangle(IBC) + \measuredangle(CBM) = \dfrac{\measuredangle(A)}{2} + \dfrac{\measuredangle(B)}{2} \end{array}\right\} \Rightarrow \measuredangle(BIM) = \measuredangle(IBM) \Rightarrow BM = IM$$

  $$\begin{cases} BM = IM \\ OI = IM(\text{hypothesis}) \end{cases} \Rightarrow OI = IM = MB = MC$$

- Show that MIC is an equilateral triangle

  $$\begin{cases} \measuredangle(MIC) = 180° - 120° = 60° \\ IM = MC \end{cases} \Rightarrow IC = IM = MC \Rightarrow \triangle MIC \text{ is an equilateral triangle}$$

- Show that $O \in AB$
  In the inscriptible quadrilateral $AOIC$:
  $$\begin{cases} OI = IC \Rightarrow \measuredangle(OAI) = \measuredangle(IAC) \\ AI = \text{ bisector } \Rightarrow \measuredangle(IAC) = \measuredangle(IAB) \end{cases} \Rightarrow O \in AB \Rightarrow \measuredangle(BCA) = 90^\circ \text{ and } \measuredangle(BAC) = 30^\circ.$$

- Show that $AB = AI\sqrt{2}$
  $$\text{In } \triangle ABC \quad \begin{cases} \cos 15^\circ = \dfrac{AM}{AB} \Rightarrow AM = AB\cos 15^\circ \\ \sin 15^\circ = \dfrac{BM}{AM} \Rightarrow BM = AB\sin 15^\circ \end{cases}$$
  $AI = AM - IM = AM - BM = AB\cos 15^\circ - AB\sin 15^\circ$ so

$$AI = AB\left(\frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}\right) = AB\frac{\sqrt{2}}{2} \Rightarrow AB = AI\sqrt{2}$$

Also solved by Kanav Talwar, Delhi Public School, Faridabad, India; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Kamran Mehdiyev, Baku, Azerbaijan; Nicusor Zlota, Focşani, România; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Todor Zaharinov, Sofia, Bulgaria.

S569. Find all perfect squares written in base 10 with one digit of 6, and n digits of 1, for some positive integer n.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy

We claim that the only such square is 16.

We can write such a number as

$$N = \underbrace{11\ldots1}_{n+1} + 5\underbrace{00\ldots0}_{a} = \frac{10^{n+1}-1}{9} + 5\cdot 10^a$$

where a represents the position of the digit 6 (i.e. $a = 0$ if 6 is the units' digit, $a = 1$ if 6 is the tenths' digit and so on). Clearly $a \le n+1$.

Notice that N is a perfect square if and only if $9N$ is a perfect square, so it suffices to look for the values of n and a for which

$$9N = 10^{n+1} - 1 + 45\cdot 10^a$$

is a perfect square. We distinguish 3 cases:

- If $a = 0$, then $9N = 10^{n+1} + 44 \equiv (-1)^{n+1} \pmod{11}$. So n must be odd, since -1 is a nonresidue modulo 11. We write $n = 2k-1$, so that we have:

  $$9N = 10^{2k} + 44 = x^2$$

  for some positive integer x, which is equivalent to $(x-10^k)(x+10^k) = 44$. This implies that

  $$\begin{cases} x - 10^k = d \\ x + 10^k = \dfrac{44}{d} \end{cases}$$

  for some positive divisor d of 44 (in particular, since $x + 10^k > x - 10^k$, we must have $d^2 < 44$). The only acceptable solution of this system occurs when $d = 2$, for which we have $x = 12$ and $k = 1$. In other words $n = 1$ and therefore $N = 16 = 4^2$.

- If $a = 1$, then

  $$9N = 10^{n+1} + 449 \equiv (-1)^{n+1} + 9 \equiv 8, 10 \pmod{11}$$

  However, both 8 and 10 are not perfect squares modulo 11. So we have no solutions.

- If $a \ge 2$, then

  $$9N = 10^{n+1} - 1 + 45\cdot 10^a \equiv 0 - 1 + 45\cdot 0 \equiv -1 \pmod{4}$$

  However, -1 is not a perfect square modulo 4, so we have no solutions in this case.

Also solved by Kanav Talwar, Delhi Public School, Faridabad, India; Nicusor Zlota, Focşani, România; Corneliu Mănescu-Avram, Ploieşti, Romania; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA.

S570. Let a, b, c, d be positive numbers satisfying the equality

$$abc + abd + acd + bcd = ab + ac + ad + bc + bd + cd$$

and such that no two of them are less than 1 and the other two are greater than 1. Prove that

$$a + b + c + d - abcd \geq \frac{15}{16}.$$

Proposed by Marian Tetiva, România

Solution by the author

The given condition can be also written in the form

$$bc(a-1) + ad(b-1) + bd(c-1) + ac(d-1) = ab + cd,$$

thus we cannot have all the numbers less than 1. Since the case of two numbers being less than 1, and the other two being greater than 1 is not allowed from the very statement of the problem, we remain with the following possibilities:

- one number is less than 1 and the other three are greater than 1;
- one number is greater than 1 and the other three are less than 1;
- all the numbers are greater than 1.

Taking account of the given condition, the inequality that we want to prove is equivalent to

$$(a-1)(b-1)(c-1)(d-1) \leq \frac{1}{16}$$

thus it is obvious in the first two cases (when the product from the left-hand is negative). So we remain with the case when the four numbers are all greater than 1, meaning that

$$x = a-1,\ y = b-1,\ z = c-1,\ t = d-1$$

are all positive. The condition for a, b, c, d yields

$$\sum (x+1)(y+1)(z+1) = \sum (x+1)(y+1)$$

(the sums being symmetric); after some simple calculations this becomes

$$\sum xyz + \sum xy = 2.$$

Now, according to the AM-GM inequality, we have

$$2 = \sum xyz + \sum xy \geq 4\sqrt[4]{x^3y^3z^3t^3} + 6\sqrt{xyzt},$$

or, equivalently,

$$2u^3 + 3u^2 - 1 \leq 0,$$

for $u = \sqrt[4]{xyzt}$. But $u > 0$ hence

$$2u^3 + 3u^2 - 1 \leq 0 \Leftrightarrow (2u-1)(u+1)^2 \leq 0$$

implies

$$u \leq \frac{1}{2} \Leftrightarrow xyzt \leq \frac{1}{16} \Leftrightarrow (a-1)(b-1)(c-1)(d-1) \leq \frac{1}{16}$$

and the proof is done.

Also solved by Nicusor Zlota, Focşani, România; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA.

U565. Let p be a prime and let $b_1, \ldots, b_{p-1}$ be integers such that they are congruent (in some order) to $1, \ldots, p-1$ modulo p. Also, let $a_1, \ldots, a_{p-1}$ be integers such that p divides $a_1 b_1 + \cdots + a_{p-1} b_{p-1}$. Prove that there is a permutation $i_1, \ldots, i_{p-1}$ of $1, \ldots p-1$ such that the determinant of the circulant matrix

$$\begin{pmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_{p-1}} \\ a_{i_{p-1}} & a_{i_1} & \cdots & a_{i_{p-2}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_2} & a_{i_3} & \cdots & a_{i_1} \end{pmatrix}$$

is also divisible by p.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

Solution by Li Zhou, Polk State College, USA
Denote the given matrix by A. Let $f(x) = a_{i_1} + a_{i_2} x + \cdots + a_{i_{p-1}} x^{p-2}$, and let U be the $(p-1) \times (p-1)$ matrix whose (i,j)-entry is j^{i-1}. We take all equivalences below modulo p.

Since $j^{p-1} \equiv 1$ for all $j = 1, \ldots, p-1$, we see that the (i,j)-entry of AU is equivalent to $j^{i-1} f(j)$, thus $|A| \equiv f(1) f(2) \cdots f(p-1)$. Now let $g \in \{1, 2, \ldots, p-1\}$ be a primitive root of 1 modulo p, then $\{1, g, \ldots, g^{p-2}\} \equiv \{1, 2, \ldots, p-1\} \equiv \{b_1, \ldots, b_{p-1}\}$.

Hence, there is a permutation $i_1, \ldots, i_{p-1}$ of $1, \ldots p-1$ such that $f(g) \equiv b_1 a_1 + \cdots + b_{p-1} a_{p-1} \equiv 0$, completing the proof.

U566. Solve in real numbers the equation

$$125^x + 64^{1/x} + 81 \cdot 5^x \cdot 4^{1/x} = 27^3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Li Zhou, Polk State College, USA

Let $y = 5^x + 4^{1/x}$, then the equation becomes $(y - 27)(y^2 + 27y + 27^2 - 3 \cdot 5^x \cdot 4^{1/x}) = 0$. By the AM-GM inequality, $y^2 \geq 4 \cdot 5^x \cdot 4^{1/x}$, thus we must have $y = 27$. This forces $x > 0$. Now if $x > \sqrt{\log_5 4}$, then $5^x > 4^{1/x}$, so

$$\frac{dy}{dx} = 5^x \ln 5 - \frac{1}{x^2} 4^{1/x} \ln 4 > \left(5^x - 4^{1/x}\right) \ln 5 > 0.$$

Likewise, if $0 < x < \sqrt{\log_5 4}$ then $dy/dx < 0$. Therefore, $x = 2$ is the only solution for $x \geq \sqrt{\log_5 4}$ and $x = \log_5 2$ is the only solution for $0 < x < \sqrt{\log_5 4}$.

Also solved by Arkady Alt, San Jose, CA, USA; G. C. Greubel, Newport News, VA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Kanav Talwar, Delhi Public School, Faridabad, India; Nicusor Zlota, Focşani, România; Le Hoang Bao, Tien Giang, Vietnam; Prajnanaswaroopa S, Bangalore, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Stets; SUNY Brockport, NY, USA; Joe Simons, Utah Valley University Orem, UT, USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.

U567. Let d be an even positive integer and C be a complex number. Prove that there are no polynomials $R(x), Q(x)$ with complex coefficients of degree at least 2 such that

$$(x-1^2)(x-3^2)...(x-(d-1)^2)+C=Q(R(x)).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let us denote the polynomial $(x-1^2)(x-3^2)...(x-(d-1)^2)$ by $S(x)$, further, assume for contradiction that $S(x)+C=Q(R(x))$ for some complex C and some polynomials $R(x), Q(x)$ with complex coefficients of degree at least 2. Note that $S'(x)=Q'(R(x))\cdot R'(x)$. Assume that r is a root of $Q'(x)$. Then the polynomial $R(x)-r$ divides $S'(x)$. Further, the polynomial $Q(R(x))-Q(r)$ is divisible by polynomial $R(x)-r$. Hence, $R(x)-r$ divides $\gcd(S'(x), S(x)+C-Q(r))$. This shows that

$$\deg(\gcd(S'(x), S(x)+C-Q(r)))\geq \deg(R(x)-r)\geq 2.$$

Now, we shall prove following lemma.

Lemma:

Let $P(x)=x(x+1)...(x+m-1)$. Then, for each complex number a,

$$\deg(\gcd(P'(x), P(x)-a))\leq 2.$$

Moreover, if m is odd, then $\deg(\gcd(P'(x), P(x)-a))\leq 1$.

Proof:

Let $r_1, ..., r_{m-1}$ be the roots of $P'(x)$, it is known that

$$-(m-1)<r_{m-1}<-(m-2)<r_{m-2}<...<-1<r_1<0.$$

Note that $|P(x)|$ has the axis of symmetry at $x=-\dfrac{m-1}{2}$, therefore, $|P(r_i)|=|P(r_{m-i})|$, for each i. It follows that $|P(x)|$ assumes its unique maximal value on the interval $[-i,-i+1]$ at r_i. Hence,

$$\frac{|P(r_{i-1})|}{|P(r_i)|}\geq\frac{|P(r_i+1)|}{|P(r_i)|}=\frac{|r_i+1||r_i+2|...|r_i+m|}{|r_i||r_i+1|...|r_i+m-1|}=\frac{|r_i+m|}{|r_i|}.$$

Since for $i<\dfrac{m}{2}$ we have $|r_i-(-m)|>|r_i-0|$ and, therefore, $|P(r_{i-1})|>|P(r_i)|$. By symmetry, for $i\geq\dfrac{m+2}{2}$ we have $|P(r_{i-1})|<|P(r_i)|$. Thus, for each complex a, the equation $P(x)=a$ can have at most 2 roots in the set $\{r_1, ..., r_{m-1}\}$. Further, if m is odd, then $P(r_i)=-P(r_{m-i})$. Hence, $\deg(\gcd(P'(x), P(x)-a))\leq 1$. This completes the proof.

Now we shall prove that for each complex C, $\deg(\gcd(S(x)-C, S'(x)))\leq 1$. This will result in contradiction. Note that if $P(x)=x(x+1)...(x+d-1)$ then $P(x)=2^{-d}S((2x+d-1)^2)$. Let $s_1, ..., s_{\frac{d}{2}-1}$ be roots of $S'(x)$ and they are all simple. Thus,

$$1^2<s_{\frac{d}{2}-1}<3^2<...<(d-3)^2<s_1<(d-1)^2.$$

Then $s_i=(2r_i+d-1)^2, i=1, ..., \dfrac{d}{2}-1$. Hence, $S(s_i)=2^dP(r_i)$. It follows that

$$|S(s_1)|>|S(s_2)|>...>|S(s_{\frac{d}{2}-1})|.$$

and we are done.

U568. Let $a > 2$ be a real number. Evaluate

$$\int_0^a \frac{\arctan x}{ax^2 - ax + a - 1} dx.$$

Proposed by Nicusor Zlota, Focşani, România

Solution by Li Zhou, Polk State College, USA

Denote the integral by I. Let $x = (a-t)/(1+at)$, then $dx/dt = -(a^2+1)/(1+at)^2$ and

$$I = -\int_a^0 \frac{(a^2+1)\arctan\frac{a-t}{1+at}}{a(a-t)^2 - a(a-t)(1+at) + (a-1)(1+at)^2} dt = \int_0^a \frac{\arctan a - \arctan t}{at^2 - at + a - 1} dt,$$

thus

$$\begin{aligned} I &= \frac{\arctan a}{2} \int_0^a \frac{1}{ax^2 - ax + a - 1} dx = \frac{\arctan a}{\sqrt{a(3a-4)}} \left[\arctan \frac{(2x-1)\sqrt{a}}{\sqrt{3a-4}}\right]_0^a \\ &= \frac{\arctan a}{\sqrt{a(3a-4)}} \left[\arctan \frac{(2a-1)\sqrt{a}}{\sqrt{3a-4}} + \arctan \frac{\sqrt{a}}{\sqrt{3a-4}}\right] = \frac{\arctan a}{\sqrt{a(3a-4)}} \left[\pi - \arctan \frac{a\sqrt{a(3a-4)}}{a^2 - 2a + 2}\right]. \end{aligned}$$

Also solved by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA.

U569. Let d be a positive integer and let $P(X) = a_0+a_1X+...+a_dX^d$ be a polynomial with positive coefficients. Prove that for any monic polynomial $f \in \mathbb{R}[X]$ taking positive values on $(0,\infty)$ there is a positive integer m such that all coefficients of $P(X)^m f(X)$ are nonnegative.

Proposed by Titu Andreescu, USA, and Navid Safaei, Iran

Solution by the authors

Let us call a polynomial positive if all of its coefficients are nonnegative. Clearly a product of positive polynomials is positive and so is a sum of positive polynomials. The first key observation is that

$$P(X) = a_dX^d + ... + a_1X + a_0 = \sum_{i=0}^{d-1} X^i P_i(X)$$

with $P_0(X) = a_0 + \frac{a_1}{2}X$, $P_i(X) = \frac{a_{i+1}X+a_i}{2}$ for $1 \le i < d-1$ and $P_{d-1}(X) = a_dX + \frac{a_{d-1}}{2}$.

Suppose that we can find positive integers m_i such that $P_i(X)^{m_i} f(X)$ is positive for $0 \le i \le d-1$. Let $m = m_0 + ... + m_{d-1}$. We claim that $P(X)^m f(X)$ is positive. It suffices to prove that $P(X)^m$ is of the form $\sum_{i=0}^{d-1} A_i(X)P(X)^{m_i}$ with $A_i(X)$ positive. But this is clear from the multinomial formula:

$$P(X)^m = \sum_{k_0+...+k_{d-1}=m} C_{k_0,...,k_{d-1}} X^{k_1+...+(d-1)k_{d-1}} P_1(X)^{k_1}...P_{d-1}(X)^{k_{d-1}}$$

for some positive numbers $C_{k_0,...,k_{d-1}}$. Moreover, if $k_0 + ... + k_{d-1} = m$ then at least one k_i is greater than or equal to m_i and so $P_i(X)^{k_i} = P_i(X)^{m_i} \cdot P_i(X)^{k_i-m_i}$. The previous formula exhibits therefore $P(X)^m$ as a linear combination $\sum_{i=0}^{d-1} A_i(X)P(X)^{m_i}$ with $A_i(X)$ positive.

Combining the previous two observations, it suffices to prove that for any $r > 0$ there is m such that $(X+r)^m f(X)$ is positive. This result is relatively classical for $r = 1$, which we may of course assume by a change of variable. The proof proceeds by observing that f is a product of polynomials of the form $X-x$ with $x \le 0$ (for which the result is clear) and X^2+bX+c with $b^2 < 4c$. The coefficient of X^d in $(X+1)^m(X^2+bX+c)$ is, for $2 \le d \le m+1$ (the cases $d = 0$, $d = 1$ and $d = m+2$ are easily handled) $\binom{m}{d-2} + \binom{m}{d-1}b + \binom{m}{d}c$. This coefficient is positive if and only if

$$c(m-d+1)(m-d+2) + (m-d+2)db + d(d-1) > 0.$$

Consider this as a quadratic inequality in d. It suffices to make sure that the discriminant is negative. A brutal expansion shows that this happens if and only if

$$4(1-b+c)c(m+1)(m+2) - ((m+2)b - 1 - (2m+3)c)^2 > 0.$$

This is a quadratic polynomial in m, with leading coefficient $4c - b^2$, thus it takes positive values for m large enough, which finishes the proof.

U570. Solve the following differential equation

$$\frac{dy}{dx} = \tan(x-y) + \cot(x-y).$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

Solution by Li Zhou, Polk State College, USA

The equation can be written as $\frac{dx}{dy} = \frac{1}{2}\sin(2x-2y)$. Let $v = y$ and $u = x - y$, then $\frac{dx}{dy} = \frac{du}{dv} + 1$, so we need to solve $\frac{du}{dv} = \frac{1}{2}\sin 2u - 1$. Separating variables we get

$$v + C = \int dv = \int \frac{2}{\sin 2u - 2} du.$$

Using the standard substitution $t = \tan u$, we see that the integral equals (up to a constant)

$$-\int \frac{1}{t^2 - t + 1} dt = -\frac{2}{\sqrt{3}} \arctan \frac{2t-1}{\sqrt{3}} = -\frac{2}{\sqrt{3}} \arctan \frac{2\tan u - 1}{\sqrt{3}}.$$

Therefore, the solutions are

$$\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}y + C\right) = 1 - 2\tan(x-y).$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Rajyavardhan, NIT Jamshedpur, Jharkhand, India; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Olympiad problems

O565. Let a, b, c be the sidelengths of a triangle, $s = (a+b+c)/2$ its semiperimeter, and r its inradius. We denote

$$x = \sqrt{\frac{s-a}{s}}, \quad y = \sqrt{\frac{s-b}{s}}, \quad \text{and} \quad z = \sqrt{\frac{s-c}{s}}.$$

Let $S = x + y + z$ and $Q = xy + xz + yz$. Prove that

$$\frac{r}{s} \leq \frac{2S - \sqrt{4-Q}}{9} \leq \frac{1}{3\sqrt{3}}.$$

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

Solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

Let k, m, n be the lengths of the tangents from the vertices A, B, C, respectively, to the incircle of the triangle. Then $(a, b, c) = (m+n, n+k, k+m)$, $s = k+m+n$, and $(s-a, s-b, s-c) = (k, m, n)$. Finally, $r = \sqrt{\dfrac{kmn}{k+m+n}}$. Therefore,

$$S = \frac{1}{\sqrt{k+m+n}}(\sqrt{k} + \sqrt{m} + \sqrt{n}), \quad \text{and} \quad Q = \frac{1}{k+m+n}(\sqrt{km} + \sqrt{kn} + \sqrt{mn}).$$

The inequality on the left-hand-side is now equivalent to

$$\frac{9\sqrt{kmn}}{k+m+n} \leq 2(\sqrt{k} + \sqrt{m} + \sqrt{n}) - \sqrt{4(k+m+n) - (\sqrt{km} + \sqrt{kn} + \sqrt{mn})}.$$

Rearranging, squaring, and simplifying we get

$$\sum k^2(\sqrt{km} + \sqrt{kn}) + 2\sum km\sqrt{km} + 9kmn$$
$$\geq 3\sqrt{kmn}\sum k\sqrt{k} + 2\sqrt{kmn}\sum k(\sqrt{m} + \sqrt{n}).$$

The above is obvious because by AM-GM we have

$$k^2(\sqrt{km} + \sqrt{kn}) + kmn \geq 3\sqrt{kmn} \cdot k\sqrt{k},$$

and its cyclical permutations, and by Schur we have

$$\sum km\sqrt{km} + 3kmn \geq \sqrt{kmn}\sum k(\sqrt{m} + \sqrt{n}).$$

The inequality on the right-hand-side is now equivalent to

$$2(\sqrt{k} + \sqrt{m} + \sqrt{n}) - \sqrt{4(k+m+n) - (\sqrt{km} + \sqrt{kn} + \sqrt{mn})} \leq \sqrt{3(k+m+n)}.$$

Rearranging, squaring, and simplifying we get

$$3(k+m+n) + 9(\sqrt{km} + \sqrt{kn} + \sqrt{mn}) \leq 4\sqrt{3}(\sqrt{k} + \sqrt{m} + \sqrt{n})\sqrt{k+m+n}.$$

Setting $p = \sum\sqrt{k}$ and $q = \sum\sqrt{km}$, the above is equivalent to

$$3(p^2 - 2q) + 9q \leq 4\sqrt{3} \cdot p\sqrt{p^2 - 2q} \iff \sqrt{3}(p^2 + q) \leq 4p\sqrt{p^2 - 2q}.$$

Squaring we get

$$13p^4 - 38qp^2 - 3q^2 \geq 0 \iff (13p^2 + q)(p^2 - 3q) \geq 0,$$

which is obvious because $p^2 \geq 3q$ from AM-GM.

Equality for both inequalities occurs for an equilateral triangle.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

O566. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{(a+b+1)^2}{a^3+b^3+1} + \frac{(b+c+1)^2}{b^3+c^3+1} + \frac{(c+a+1)^2}{c^3+a^3+1} \le 9.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Theo Koupelis, Broward College, Pembroke Pines, FL, USA

Using Hölder's inequality we get $(a^3+b^3+1)(1+1+1)(1+1+1) \ge (a+b+1)^3$ and thus

$$\frac{(a+b+1)^2}{a^3+b^3+1} \le \frac{9}{a+b+1}.$$

Adding similar expressions we see that it is sufficient to show that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \le 1.$$

After clearing denominators, the above is equivalent to

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b \ge 2(a+b+c).$$

Using AM-GM we have $a^2b + a^2c = a^2b + \frac{a}{b} \ge 2a^{3/2}$, and therefore it is sufficient to show that $a^{3/2} + b^{3/2} + c^{3/2} \ge a + b + c$, which after homogenizing is equivalent to

$$a^{3/2} + b^{3/2} + c^{3/2} \ge (abc)^{1/6}(a+b+c).$$

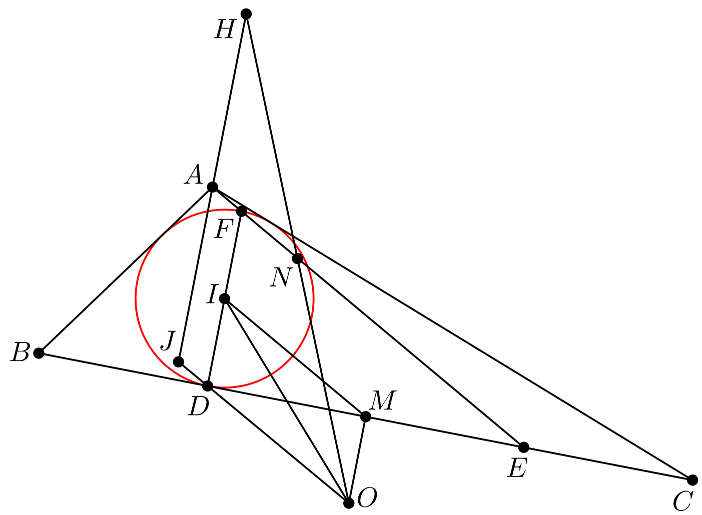
The above is obvious due to Muirhead's inequality because $(\frac{3}{2}, 0, 0) \succ (\frac{7}{6}, \frac{1}{6}, \frac{1}{6})$.

Also solved by Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Nandan Sai Dasireddy, Hyderabad, Telangana, India; Kanav Talwar, Delhi Public School, Faridabad, India; Jiang Lianjun, No.2 Middle School of Quan Zhou, Gui-Lin, China; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O567. Let ABC be a scalene triangle with incenter I and circumcenter O. Let N be the center of the nine-point circle of $\triangle ABC$ and M be the midpoint of BC. Knowing that the midpoint of OI lies on side BC, prove that IM is parallel to AN.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution by Li Zhou, Polk State College, USA



Suppose that the incircle of $\triangle ABC$ is tangent to BC at D and has a diameter DF. It is well known that AF intersects BC at E, the point where the A-excircle of $\triangle ABC$ is tangent to BC. Since M is the midpoint of DE, $MI \parallel EF$. If the midpoint of OI lies on BC, then A is obtuse and $OMID$ is a parallelogram. Thus,

$$-\cos A = \cos \angle BOM = \frac{OM}{OB} = \frac{r}{R},$$

where r and R are the inradius and circumradius of $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$ and suppose that OD intersects AH at J, then $AJ = 2r = -2R\cos A = AH$. Hence, AF passes through the midpoint N of OH, completing the proof.

Also solved by Kanav Talwar, Delhi Public School, Faridabad, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Kamran Mehdiyev, Baku, Azerbaijan; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.

O568. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 + 11xyz \geq 20.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

We use the following inequalities:

1.(see also in [1]) If $a, b, c > 0$ then

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 + \frac{15}{2} \geq \frac{11}{4}\left(\frac{a^2 + b^2}{ab} + \frac{b^2 + c^2}{bc} + \frac{c^2 + a^2}{ca}\right) \tag{1}$$

If we denote $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$, $xyz = 1$, the inequality becomes $f(x, y, z) \geq 0$, where

$$f(x, y, z) = (x + y + z)^2 + \frac{15}{2} - \frac{11}{4}(x + y + z + xy + yz + zx).$$

Without loss of generality, we may assume that $z = \min(x, y, z)$, so it follows that $z \leq 1$, $xy \geq 1$. We notice that

$$\begin{aligned}
&f(x, y, z) - f\left(\sqrt{xy}, \sqrt{xy}, z\right) \\
&= (\sqrt{x} - \sqrt{y})^2\left[\left(\sqrt{x} + \sqrt{y}\right)^2 + 2z\right] - \frac{11}{4}\left(\sqrt{x} - \sqrt{y}\right)^2(1 + z) \\
&= \left(\sqrt{x} - \sqrt{y}\right)^2\left[\left(\sqrt{x} + \sqrt{y}\right)^2 - \frac{11 + 3z}{4}\right] \\
&\geq \left(\sqrt{x} - \sqrt{y}\right)^2\left(4 - \frac{11 + 3z}{4}\right) \geq 0.
\end{aligned}$$

Therefore, it remains to prove that $f\left(\sqrt{xy}, \sqrt{xy}, z\right) \geq 0$ or, if we denote

$$z = \frac{1}{t^2},\ t \geq 1,\ f\left(t, t, \frac{1}{t^2}\right) \geq 0,$$

that is

$$\frac{(t - 1)^2(5t^4 - 12t^3 + t^2 + 8t + 4)}{4t^4} \geq 0,$$

which is true. The equality holds when $x = y = z = 1$.

2. If $x, y, z > 0$ such that $x^2 + y^2 + z^2 + xyz = 4$, then

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(2 + \sqrt{4 - 3xyz}).$$

We the following substitutions $x = \frac{2\sqrt{bc}}{(a+b)(c+a)}$, $y = \frac{2\sqrt{ca}}{(a+b)(b+c)}$, $z = \frac{2\sqrt{ab}}{(c+a)(b+c)}$, the inequality becomes

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} - 1 \geq \sqrt{1 - \frac{6abc}{(a + b)(b + c)(c + a)}},$$

or, after squaring both sides and a few calculations,

$$\frac{a^6 + b^6 + c^6 + 3a^2b^2c^2 - a^2b^2(a^2 + b^2) - b^2c^2(b^2 + c^2) - c^2a^2(c^2 + a^2)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} \geq 0,$$

true due to Schur's inequality.

Returning to the proposed inequality and using these two results we can write the following

$$\begin{aligned}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2 &\geq \frac{11}{4}\left(\frac{x^2+y^2}{xy}+\frac{y^2+z^2}{yz}+\frac{z^2+x^2}{zx}\right)-\frac{15}{2}\\ &\geq \frac{33}{4}\sqrt[3]{\frac{(x^2+y^2)(y^2+z^2)(z^2+x^2)}{x^2y^2z^2}}-\frac{15}{2}\\ &= \frac{33}{4}\sqrt[3]{\frac{(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2)-x^2y^2z^2}{x^2y^2z^2}}-\frac{15}{2}\\ &\geq \frac{33}{4}\sqrt[3]{\frac{(4-xyz)(2+\sqrt{4-3xyz})-xyz}{xyz}}-\frac{15}{2}.\end{aligned}$$

Therefore, it remains to prove that

$$3\sqrt[3]{\frac{(4-xyz)(2+\sqrt{4-3xyz})-xyz}{xyz}}+4xyz \geq 10,$$

or, if we denote $t^2 = 4-3xyz \geq 1$,

$$9\sqrt[3]{\frac{t^3+3t^2+8t+12}{4-t^2}} \geq 14+4t^2,$$

or

$$729(t^3+3t^2+8t+12)-(4-t^2)(14+4t^2)^3 \geq 0,$$

that is

$$(t-1)(t+2)(64t^6-64t^5+608t^4-736t^3+1616t^2-2359t+1114) \geq 0,$$

true for $t \geq 1$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O569. Let a, b, c be positive real numbers such that $a+b+c=ab+bc+ca$. Prove that

$$\frac{4abc}{(1+a)(1+b)(1+c)}+4 \le \frac{3a}{1+a}+\frac{3b}{1+b}+\frac{3c}{1+c} \le 5.$$

Proposed by An Zhenping, Xianyang Normal University, China

First solution by Li Zhou, Polk State College, USA

We strengthen the upper bound from 5 to $5-\frac{4}{(1+a)(1+b)(1+c)}$.

Let $p=a+b+c$, $q=ab+bc+ca$, and $r=abc$. Then $p=q$ and these claimed inequalities are equivalent to $4-p \le r \le (p+1)/4$. By the r-lemma in the pqr-method, it suffices to consider $c=0$ or $c=b$.

If $c=0$, then $p=a+b=ab \le (a+b)^2/4=p^2/4$, thus $4-p \le 0=r \le (p+1)/4$.

If $c=b$, then $a+2b=p=q=2ab+b^2$, that is, $b(2-b)=a(2b-1)$. Therefore,

$$b(r+p-4)=b(ab^2+a+2b-4)=ab(b^2+1)-2b(2-b)=ab(b^2+1)-2a(2b-1)=a(b+2)(b-1)^2 \ge 0,$$

and

$$p-4r+1=a+2b-4ab^2+1=a(1-2b)(1+2b)+2b+1=b(b-2)(2b+1)+2b+1=(2b+1)(b-1)^2 \ge 0.$$

Therefore, $4-p \le r \le (p+1)/4$.

Second solution by Theo Koupelis, Broward College, Pembroke Pines, FL, USA

Let $f = t^3 - pt^2 + pt - r$ be the cubic with roots a, b, c, where $p = a + b + c$ and $r = abc$. The roots of $f'(t) = 3t^2 - 2pt + p = 0$ are $t_\pm = \frac{1}{3}(p \pm \sqrt{p^2 - 3p})$. Clearly $p \geq 3$ and we can set $p = 3(1 + \omega^2)$ where $\omega \geq 0$. With a, b, c being positive, we must have $f(t_+) \leq 0$ and $f(t_-) \geq 0$; substituting we get

$$\max\{r_1(\omega), 0\} \leq r \leq r_2(\omega)$$

where

$$r_1(\omega) = (1 + \omega^2)^{3/2}\left[(1 - 2\omega^2)(1 + \omega^2)^{1/2} - 2\omega^3\right]$$

and

$$r_2(\omega) = (1 + \omega^2)^{3/2}\left[(1 - 2\omega^2)(1 + \omega^2)^{1/2} + 2\omega^3\right].$$

It is easy now to show that $r_1(\omega)$ is a decreasing function with $r_1(0) = 1$, and $r_1\left(\frac{\sqrt{3}}{3}\right) = 0$. Also, $r_2(\omega)$ is an increasing function with $r_2(0) = 1$.

Clearing denominators, the given inequalities are equivalent to

$$8abc + 4\sum ab + 4\sum a + 4 \leq 9abc + 6\sum ab + 3\sum a \leq 5abc + 5\sum ab + 5\sum a + 5.$$

(i) After simplifying, the inequality on the left is equivalent to $p + r \geq 4$.

(a) For $\omega \geq \frac{\sqrt{3}}{3}$ it is sufficient to show that $p \geq 4 \Leftrightarrow 3(1 + \omega^2) \geq 4$, which is obvious. Equality occurs at $\omega = \frac{\sqrt{3}}{3}$, that is when one of the variables is zero and the other two equal 2.

(b) For $0 \leq \omega < \frac{\sqrt{3}}{3}$, it is sufficient to show that $p + r_1 \geq 4$. We get

$$\begin{aligned}3(1 + \omega^2) + (1 - 2\omega^2)(1 + \omega^2)^2 - 2\omega^3(1 + \omega^2)^{3/2} \geq 4 \Longleftrightarrow \\ \omega^2(3 - 3\omega^2 - 2\omega^4) \geq 2\omega^3(1 + \omega^2)^{3/2} \Longleftrightarrow \omega^4(3\omega^2 - 1)(5\omega^2 + 9) \leq 0,\end{aligned}$$

which is obvious. Equality occurs when $\omega = 0$, that is when $a = b = c = 1$.

(ii) After simplifying, the inequality on the right is equivalent to $p + 5 \geq 4r$. It is sufficient to show that $p + 5 \geq 4r_2$, and therefore

$$\begin{aligned}3(1 + \omega^2) + 5 &\geq 4(1 - 2\omega^2)(1 + \omega^2)^2 + 8\omega^3(1 + \omega^2)^{3/2} \Longleftrightarrow \\ &8\omega^6 + 12\omega^4 + 3\omega^2 + 4 \geq 8\omega^3(1 + \omega^2)^{3/2} \Longleftrightarrow \\ &(3\omega^2 + 4)(24\omega^4 + 3\omega^2 + 4) \geq 0,\end{aligned}$$

which is obvious.

Also solved by Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jiang Lianjun, No.2 Middle School of Quan Zhou, GuiLin, China.

O570. Find all perfect squares in base 10 with one digit of 4, and n digits of 9, for some positive integer n.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

First solution by the authors

Of course, 49 is such a perfect square; we show further that no other square can be written in base ten with one 4 and a number of 9's.

Suppose N is such a square. Clearly N can end nor with 94 (since it would be divisible by 2, but not by 4), neither with 99 (since then it would be congruent to 3 modulo 4). Thus we only need to investigate when can we have

$$N = \underbrace{9\dots9}_{n-1\ 9's}49 = m^2$$

for some positive integers n and m. This equality can also be put in the form

$$10^{n+1} - 51 = m^2 \Leftrightarrow 10^{n+1} - m^2 = 51.$$

Thus, if $n+1$ is even, we can read it as

$$\left(10^{\frac{n+1}{2}} - m\right)\left(10^{\frac{n+1}{2}} + m\right) = 51.$$

Since $10^{\frac{n+1}{2}} - m < 10^{\frac{n+1}{2}} + m$, $10^{\frac{n+1}{2}} + m > 0$, and $51 = 3 \cdot 17$, we can only have either

$$10^{\frac{n+1}{2}} - m = 1 \text{ and } 10^{\frac{n+1}{2}} + m = 51,$$

or

$$10^{\frac{n+1}{2}} - m = 3 \text{ and } 10^{\frac{n+1}{2}} + m = 17.$$

In the first case we get $10^{\frac{n+1}{2}} = 26$, which is impossible. The second system of equations gives $10^{\frac{n+1}{2}} = 10$, that is, $n = 1$, and $m = 7$; so, we obtain the announced solution $N = 49$.

It remains to consider that $n+1$ is odd, when we have the following situations.

- If $n+1 = 6k+1$ (for some positive integer k), because $10^6 \equiv 1 \bmod 13$, we get

  $$m^2 = 10^{n+1} - 51 = 10^{6k} \cdot 10 - 51 \equiv 11 \bmod 13,$$

  which is impossible, since 11 is not a quadratic residue modulo 13.

- If $n+1 = 6k+3$, we use $10^3 \equiv 1 \bmod 37$. Accordingly, we get

  $$m^2 = 10^{n+1} - 51 = \left(10^3\right)^{2k+1} - 51 \equiv 24 \bmod 37.$$

  However, 24 is not a quadratic residue modulo 37, and the equality is, again, impossible.

- Finally, if $n+1 = 6k+5$, we use $10^3 \equiv -1 \bmod 7$ in order to obtain

  $$m^2 = 10^{n+1} - 51 = \left(10^3\right)^{2k+1} \cdot 10^2 - 51 \equiv (-1) \cdot 2 - 2 \equiv 3 \bmod 7.$$

  This is also contradictory, since 3 is not a quadratic residue modulo 7.

We conclude that 49 is, indeed, the only solution to our problem.

Second solution by Li Zhou, Polk State College, USA

Note that $94 \equiv 2 \pmod 4$ and $99 \equiv 3 \pmod 4$, neither of which is a quadratic residue modulo 4. Thus we only need to consider $m^2 = 9\cdots 949 = 10^{n+1} - 51$.

Consider first $n = 2k$ for some $k \geq 1$. Then $10^{n+1} - 51 = 10 \cdot 100^k - 3 \cdot 17 \equiv 10(-2)^k \pmod{17}$, which yields the complete set of residues $\{\pm 3, \pm 5, \pm 6, \pm 7\}$ modulo 17. Since the complete set of quadratic residues modulo 17 is $\{0, \pm 1, \pm 2, \pm 4, \pm 8\}$, no such m is possible in this case.

Next, consider $n = 2k-1$ for some $k \geq 1$. Then $51 = 10^{2k} - m^2 = (10^k + m)(10^k - m)$, so $(10^k + m, 10^k - m) = (51, 1)$ or $(17, 3)$. Only $(17, 3)$ yields $k = 1$ and $m = 7$. Therefore, 49 is the only such perfect square.

Also solved by Luca Ferrigno, Università degli studi di Roma Tre, Roma, Italy; Kanav Talwar, Delhi Public School, Faridabad, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Theo Koupelis, Broward College, Pembroke Pines, FL, USA.