Junior Problems

J571. Consider the quadratic equation

$$m^5x^2-(m^7+m^6-m^4-m)x+m^8-m^5-m^3+1=0,$$

with roots x_1, x_2, where m is a real parameter. Prove that $x_1=1$ if and only if $x_2=1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J572. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$a+b+c+\frac{3}{ab+bc+ca}\geq 4.$$

Proposed by An Zhenping, Xianyang Normal University, China

J573. Prove that in any triangle ABC

$$\sin\frac{A}{2}+2\sin\frac{B}{2}\sin\frac{C}{2}\leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J574. Let a, b, c be positive real numbers such that $ab+bc+ca=1$ and

$$\left(a+\frac{1}{a}\right)^2\left(b+\frac{1}{b}\right)^2-\left(b+\frac{1}{b}\right)^2\left(c+\frac{1}{c}\right)^2+\left(c+\frac{1}{c}\right)^2\left(a+\frac{1}{a}\right)^2=1.$$

Prove that $a=1$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J575. Through point C lying outside of the circle ω two lines are drawn that are tangent to the circle at points A and B. Point D lies on the segment AB and M is the midpoint of CD. Through point M a line is drawn that is tangent to circle ω at X. Prove that lines CX and DX are perpendicular.

Proposed by Waldemar Pompe, Warsaw, Poland

J576. Let a, b, c, d be positive real numbers such that

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}=1.$$

Prove that

$$ab+ac+ad+bc+bd+cd-3(a+b+c+d)\geq 18.$$

Proposed by An Zhenping, Xianyang Normal University, China

Senior Problems

S571. Let a be a nonzero real number for which there is a real number $b \geq 1$ such that

$$a^3 + \frac{1}{a^3} = b\sqrt{b+3}.$$

Prove that

$$a^2 + \frac{1}{a^2} = b + 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S572. Prove that in any triangle ABC

$$\frac{9}{4}\sqrt{\frac{r}{2R}} \leq \sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq 1 + \frac{r}{4R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S573. Points A, B, C, D lie on a line in that order. Let o_1 and o_2 be the circles with diameters AB and CD, respectively. Circle ω is externally tangent to circles o_1 and o_2. Circle Ω is internally tangent to o_1 and o_2 and intersects ω at points E and F. Prove that $\angle AEB = \angle CFD$.

Proposed by Waldemar Pompe, Warsaw, Poland

S574. Find all real numbers x such that

$$\left\{\frac{6x^2 + 168x + 2022}{x^2 + 24x + 237}\right\} = \frac{6}{7},$$

where $\{x\}$ denotes the fractional part of x.

Proposed by Navid Safaei, Sharif University of Technology ,Tehran, Iran

S575. Let ABC be an acute triangle, with orthocenter H. Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, respectively, and let A_2, B_2, C_2 be points inside segments HA_1, HB_1, HC_1 such that

$$\frac{HA_2}{A_2A_1} = \frac{HB_2}{B_2B_1} = \frac{HC_2}{C_2C_1} = 2.$$

Prove that lines $AA_2,\ BB_2,\ CC_2$ are concurrent.

Proposed by Mihaela Berindeanu, Bucharest, România

S576. .Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = \sqrt{n}$. Prove that

$$\left(a_1 + \frac{1}{a_1}\right)^2 + \left(a_2 + \frac{1}{a_2}\right)^2 + \cdots + \left(a_n + \frac{1}{a_n}\right)^2 \geq (n+1)^2.$$

Proposed by Titu Andreescu, USA and Alessandro Ventullo, Italy

Undergraduate Problems

U571. Let A be a $n \times n$ matrix with real entries and let $\alpha \neq 0$ be a real number. Prove that if $A^3 = I$ and $(A - \alpha I)^3 = 0$, then $AB = BA$, for all $n \times n$ matrices B.

Proposed by Mircea Becheanu, Montreal, Canada

U572. .Evaluate

$$\int \left(x + \frac{1}{4x}\right) \frac{e^x}{\sqrt{x}} dx$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

U573. Prove the inequality

$$\sum_{k=1}^{\infty} \frac{k 2^{\frac{k+1}{2}}}{3^{2^k - 1}} < e$$

Proposed by Mohammed Imran, Chennai, India

U574. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function such that $\int_a^b f(x)dx > 0$. Prove that for all positive numbers $c < (b-a)^2$, there is $\varepsilon \in (a, b)$ such that

$$\int_a^b f(x)dx > \frac{c}{b-a} f(\varepsilon).$$

Proposed by Ovidiu Gabriel Dinu, Bălceşti-Vâlcea, România

U575. Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function with $f(0) = 0$. Prove that there is $c \in (0, 1)$ such that

$$\int_0^c (1-x)^2 f(x)dx = (1-c) \int_0^c f(x)dx.$$

Proposed by Florin Stănescu, Găeşti, România

U576. Find all triples (m, n, p) of positive integers for which there is a real polynomial $P(x)$ such that

$$P(x) + x^m P(1-x) = (x^2 - x + 1)^n (x^2 - x - 1)^p.$$

Determine whether there are infinitely many such polynomials.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Olympiad Problems

O571. .Let a, b, c, d be positive real numbers such that

$$abcd = ab + ac + ad + bc + bd + cd.$$

Prove that

$$\sum abc - 2\sum ab + 3\sum a \geq 36(\sqrt{6} - 2),$$

where all sums are symmetric sums.

Proposed by Marian Tetiva, România

O572. .Let a, b, c, d be positive integers and let C be a nonzero integer. The map $f : \mathbb{Z} \to \mathbb{Z}$ has the property that $f(mn) = f(m)f(n)$ for all integers m, n, and there is N such that, for all $n \geq N$,

$$f(c(an + b) + d) \equiv C \bmod(an + b).$$

Prove that there is an integer e such that $|f(n)| = |n|^e$ for all integers n relatively prime to ac.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O573. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$x + f(x^2 + f(y)) = xf(x) + f(x + y),$$

for all $x, y \in \mathbb{R}$.

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

O574. Segment AB is a chord of circle Γ. Different circles ω_1 and ω_2 are internally tangent to Γ at points P and Q, respectively, and to the segment AB at a common point. Chords AB and PQ meet at D. Let C be the midpoint of arc AB of circle Γ that contains point P. Prove that line CD passes through the center of ω_1.

Proposed by Waldemar Pompe, Warsaw, Poland

O575. Let $(L_n)_{n\geq 1}$ be the Lucas sequence, $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, for $n = 1, 2, 3, \ldots$.
Prove that if $n = \frac{1}{4}(L_{6m+1} - 1)$ for some positive integer m, then

$$\prod_{k=0}^{n} [(4k-1)^4 + 64]$$

is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O576. If m_a, m_b, m_c are the medians of a triangle with side-lengths a, b, c, prove that

$$m_a^3(bc - a^2) + m_b^3(ca - b^2) + m_c^3(ab - c^2) \geq 0.$$

Proposed by Marius Stănean, Zalău, România