Junior Problems

J577. Let a and b be positive real numbers such that $a+b=1$. Prove that

$$\left(\frac{1}{a^2}-b-1\right)\left(\frac{1}{b^2}-a-1\right)\geq\frac{25}{4}$$

Proposed by An Zhenping, Xianyang Normal University, China

J578. Let a be a positive real number and let $f(x)=a^x+\frac{1}{a^x}$. Given that $f\left(\frac{2}{3}\right)=1+2\sqrt{2}$, find $f\left(\frac{3}{2}\right)$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J579. Find all pairs (x,y) of integers such that

$$3x^2+10x+5=9\cdot 2^y.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J580. Solve in real numbers the equation

$$\sqrt[3]{x^2+4}=\sqrt{x^3-4}.$$

Proposed by Alessandro Ventullo, Milan, Italy

J581. Let a,b,c,d be real numbers such that $ab+ac+ad+bc+bd+cd-abcd=11$. Prove that

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1)\geq 100.$$

Proposed by An Zhenping, Xianyang Normal University, China

J582. Let $ABCD$ be a quadrilateral inscribed in a circle Γ with center O. Lines AC and BD intersect at X and Γ_1 is the circumcircle of triangle XAB. Lines AD and BC intersect Γ_1 at N and M, respectively. We know that the area of the concave polygon $ONDXCM$ is $DC\cdot AB$. Find the measure of $\angle AXB$.

Proposed by Mihaela Berindeanu, Bucharest, România

Senior Problems

S577. Let k, m, n be integers such that $k+m+n=1$. Prove that

$$(k^2+m^2+n^2+7)^2+(kmn-4)^2$$

is not the square of an odd integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S578. Let a, b, c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$\frac{1}{1+a^2+b^2}+\frac{1}{1+b^2+c^2}+\frac{1}{1+c^2+a^2} \le \frac{9}{5}$$

Proposed by An Zhenping, Xianyang Normal University, China

S579. Let a, b, c be the side-lengths of a triangle with area S, and let α, β, γ be positive real numbers such that

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}+\frac{1}{\gamma+1}=2.$$

Prove that

$$\alpha b^2c^2+\beta c^2a^2+\gamma a^2b^2 \ge 8S^2.$$

Proposed by Marius Stănean, Zalău, România

S580. Prove that there are no positive integers a, b, c such that

$$a^2b^2+b^2c^2+c^2a^2-3abc=2022.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

S581. Let p be a prime such that $p \equiv 1 \pmod 4$. Prove that there is a positive integer a such that

$$a^2+2^a \equiv 0 \pmod{2p}.$$

Proposed by Nguyen Tanh Chuong, Hanoi, Vietnam

S582. . Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1=1,\ a_2=2$ and

$$a_{n+2}=(n+2)a_{n+1}-na_n+2n+1, \text{ for all } n \ge 1.$$

Find all n for which there exists $m \in \mathbb{N}$ such that $a_n = m!$.

Proposed by Prodromos Fotiadis, Dramma, Greece

Undergraduate Problems

U577. For positive integers n, evaluate

$$\lim_{n\to\infty}\left\{\sqrt{4n^2+3n+2}\right\},$$

where $\{a\}$ denotes the fractional part of a.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U578. Evaluate

$$\iint_D \frac{y\sqrt{x^2+y^2}}{x^2}e^{\sqrt{x^2+y^2}/2}\,dx\,dy,$$

where $D=\{(x-1)^2+y^2\le 1, y\ge 0\}$.

Proposed by Paolo Perfetti, Roma, Italy

U579. Let x_0 be an integer and $P(x)$ be a polynomial with integer coefficients. The sequence $(x_n)_{n\ge 0}$ is defined by

$$x_n = x_{n-1}^n + P(n), \text{ for all } n\ge 1.$$

Prove that for any positive integer $m\ge 2$ there is a positive integer T such that

$$x_{n+T}\equiv x_n (\text{mod } m),$$

for all sufficiently large n.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U580. Two mutually tangent circles of radius 1 lie on a common tangent line. The circle on the left is colored white and the circle on the right is colored gray. A third, smaller circle, is tangent to both of the larger circles and the line, and it is also colored gray. An infinite sequence of gray circles are inserted as follows: Each subsequent circle is tangent to the preceding circle, to the largest gray circle, and to the white circle. What is the total area bounded by the gray circles?

Proposed by Brian Bradie, Christopher Newport University, Virginia, USA

U581. Let $r>s$ be two positive integers which are relatively prime and let $P(x)$ and $Q(x)$ be distinct nonconstant polynomials with complex coefficients such that

$$P(x)^r - P(x)^s = Q(x)^r - Q(x)^s.$$

Prove that $r=2$ and $s=1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U582. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a twice differentiable function and let $u:\mathbb{R}^n\longrightarrow\mathbb{R}$, $u(x)=f(\parallel x\parallel)$. Denote by $\nabla^2(u)$ the Hessian of u. Evaluate in terms of f

$$\det(\nabla^2(u)).$$

Proposed by Michele Caselli, ETH Zurich, Switzerland

Olympiad Problems

O577. Let a, b, c be positive numbers such that $abc = 4$ and $a, b, c > 1$. Prove the inequality

$$(a-1)(b-1)(c-1)\left(\frac{a+b+c}{3}-1\right) \leq (\sqrt[3]{4}-1)^4.$$

Proposed by Marian Tetiva, România

O578. Solve in positive integers the equation

$$\frac{x^3+y^3}{6} = 2022\sqrt{xy-4}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O579. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$(b+c-a)(c+a-b)(a+b-c) + \frac{48}{ab+bc+ca} \geq 17.$$

Proposed by An Zhenping, Xianyang Normal University, China

O580. Prove that in any triangle ABC

$$\frac{R}{r} + \frac{r}{R} + \frac{3}{2} \geq \frac{(a+b)(b+c)(c+a)}{2abc}.$$

Proposed by Marius Stănean, Zalău, România

O581. Let ABC be a scalene, non-right triangle with altitudes AD, BE, CF and midpoints M_a, M_b, M_c of sides BC, CA, AB, respectively. Let G be the centroid of the triangle and let N be the center of the nine-point circle. Let P be the intersection of EF and M_bM_c and let $PG \cap BC = Q$. Knowing that AN and EF are parallel, prove that OQ is parallel to the Euler line of triangle DEF.

Proposed by Todor Zaharinov, Sofia, Bulgaria

O582. Let P be a point in the interior of a triangle ABC and let Q be its isogonal conjugate with respect to ABC. Let M and N be the projections of P and Q on BC, and P' and Q' be the second intersections of lines AP and AQ with the circumcircle of ABC. Prove that lines MP' and NQ' intersect on PQ.

Proposed by Marius Stănean, Zalău, România