Junior problems

J571. Consider the quadratic equation

$$m^5x^2-(m^7+m^6-m^4-m)x+m^8-m^5-m^3+1=0,$$

with roots x_1, x_2, where m is a real parameter. Prove that $x_1=1$ if and only if $x_2=1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

We see that

$$m^8-m^5-m^3+1=(m^3-1)(m^5-1)$$

and

$$m^7+m^6-m^4-m=m^4(m^3-1)+m(m^5-1),$$

implying

$$x_1+x_2=\frac{m^3-1}{m}+\frac{m^5-1}{m^4}$$

and

$$x_1x_2=\frac{m^3-1}{m}\cdot\frac{m^5-1}{m^4}.$$

Hence x_1 and x_2 are $\dfrac{m^3-1}{m}$ and $\dfrac{m^5-1}{m^4}$ in some order. Without loss of generality assume that $x_1=\dfrac{m^3-1}{m}$. The equality $x_1=1$ is equivalent to $m^3-m-1=0$ which, because m^2-m+1 cannot be 0, is further equivalent to $(m^3-m-1)(m^2-m+1)=0$. This means $m^5-m^4-1=0$, that is, $x_2=1$.

Second solution by the author

The given equation has solution 1 if and only if $m^8-m^7-m^6+m^4-m^3+m+1=0$, i.e.

$$(m^5-m^4-1)(m^3-m-1)=0,$$

or

$$(m^2-m+1)(m^3-m-1)^2=0.$$

Since m is a real number, then $m^2-m+1>0$ and so $m^3-m-1=0$. Setting

$$f(x)=m^5x^2-(m^7+m^6-m^4-m)x+m^8-m^5-m^3+1,$$

we have

$$f'(1)=-m^7-m^6+2m^5+m^4+m=-m(m^3-m-1)(m^3+m^2-m+1)=0,$$

which means that 1 is double root for $f(x)$.

Third solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
Suppose one of the roots of the quadratic equation

$$m^5x^2 - (m^7 + m^6 - m^4 - m)x + m^8 - m^5 - m^3 + 1 = 0$$

is $x = 1$. Then

$$m^8 - m^7 - m^6 + m^4 - m^3 + m + 1 = 0,$$

or

$$(m^2 - m + 1)(m^3 - m - 1)^2 = 0.$$

The factor $m^2 - m + 1$ does not have real roots, so m must satisfy $m^3 = m + 1$. A graphical analysis of the curves $y = x^3$ and $y = x + 1$ demonstrates that $m^3 = m + 1$ has a unique root between 1 and 2. Now, the equation $m^3 = m + 1$ implies that

$$\begin{aligned}
m^4 &= m^2 + m, \\
m^5 &= m^3 + m^2 = m^2 + m + 1, \\
m^6 &= m^2 + 2m + 1, \\
m^7 &= m^3 + 2m^2 + m = 2m^2 + 2m + 1, \text{ and} \\
m^8 &= 2m^3 + 2m^2 + m = 2m^2 + 3m + 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
m^7 + m^6 - m^4 - m &= 2m^2 + 2m + 1 + m^2 + 2m + 1 - m^2 - m - m \\
&= 2(m^2 + m + 1), \\
m^8 - m^5 - m^3 + 1 &= 2m^2 + 3m + 2 - m^2 - m - 1 - m - 1 + 1 \\
&= m^2 + m + 1,
\end{aligned}$$

and the quadratic equation

$$m^5x^2 - (m^7 + m^6 - m^4 - m)x + m^8 - m^5 - m^3 + 1 = 0$$

reduces to

$$(m^2 + m + 1)(x^2 - 2x + 1) = 0.$$

Because $m^2 + m + 1 \neq 0$, it follows that one root of the quadratic equation

$$m^5x^2 - (m^7 + m^6 - m^4 - m)x + m^8 - m^5 - m^3 + 1 = 0$$

is equal to 1 if and only if the other root is also equal to 1.

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Le Hoang Bao, Tien Giang, Vietnam; Corneliu Mănescu-Avram, Ploieşti, Romania; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Vishwesh Ravi Shrimali, Jaipur, Rajasthan, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Polyahedra, Polk State College, USA; Kanav Talwar, Delhi Public School, Faridabad, India; Fred Frederickson, Utah Valley University, UT, USA.

J572. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$a + b + c + \frac{3}{ab + bc + ca} \geq 4.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Henry Ricardo, Westchester Area Math Circle

Letting $p = a + b + c$ and $q = ab + bc + ca$, our given condition becomes $p^2 - 2q = 3$, or $q = (p^2 - 3)/2$, and we must prove that $p + 3/q \geq 4$.

Now

$$\begin{aligned} p + \frac{3}{q} \geq 4 &\iff p + \frac{6}{p^2 - 3} \geq 4 \\ &\iff p(p^2 - 3) + 6 \geq 4(p^2 - 3) \iff p^3 - 4p^2 - 3p + 18 \geq 0 \\ &\iff (p + 2)(p - 3)^2 \geq 0, \end{aligned}$$

which is obviously true. Equality holds if and only if $a = b = c = 1$.

Also solved by Polyahedra, Polk State College, USA; UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Kanav Talwar, Delhi Public School, Faridabad, India; Fred Frederickson, Utah Valley University, UT, USA; Duy Quan Tran, University of Medicine and Pharmacy at Ho Chi Minh City, Ho Chi Minh City, Vietnam; Corneliu Mănescu-Avram, Ploieşti, Romania; Ejaife ogheneobukome, Warri. Nigeria; G. C. Greubel, Newport News, VA, USA; Jiang Lianjun, Quanzhou, Second Middle School, Gui-Lin, China; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romaina; Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Prajnanaswaroopa S, Bangalore, Karnataka, India; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

J573. Prove that in any triangle ABC

$$\sin\frac{A}{2}+2\sin\frac{B}{2}\sin\frac{C}{2}\leq 1$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by the author

We have known that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1.$$

On the other hand, the AM-GM inequality gives us

$$\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\geq 2\sin\frac{B}{2}\sin\frac{C}{2}.$$

Combining these two relations we get

$$\sin^2\frac{A}{2}+2\sin\frac{B}{2}\sin\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq 1.$$

This is equivalent to

$$2\sin\frac{B}{2}\sin\frac{C}{2}(1+\sin\frac{A}{2})\leq 1-\sin^2\frac{A}{2},$$

or

$$2\sin\frac{B}{2}\sin\frac{C}{2}\leq 1-\sin\frac{A}{2}.$$

The conclusion follows.

Second solution by Daniel Văcaru, Pitești, Romania
We use $2\sin x \sin y = \cos(x-y) - \cos(x+y)$.

We obtain

$$2\sin\frac{B}{2}\sin\frac{C}{2} = \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{\pi-A}{2}\right) = \cos\left(\frac{B-C}{2}\right) - \sin\frac{A}{2}.$$

Consequently, we obtain

$$\sin\frac{A}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2} = \sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right) - \sin\frac{A}{2} = \cos\left(\frac{B-C}{2}\right) \leq 1.$$

Also solved by Polyahedra, Polk State College, USA; Telemachus Baltsavias, Keramies Junior High School, Kefalonia,Greece; UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Kanav Talwar, Delhi Public School, Faridabad, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Arighna Pan, Nabadwip Vidyasagar College, India; Kerameies Junior High School, Cephalonia, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Jiang Lianjun, Quanzhou, Second Middle School, GuiLin, China; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Mihaly Bencze, Brasov, Romania; Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Vishwesh Ravi Shrimali, Jaipur, Rajasthan, India; Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

J574. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$ and

$$\left(a+\frac{1}{a}\right)^2\left(b+\frac{1}{b}\right)^2-\left(b+\frac{1}{b}\right)^2\left(c+\frac{1}{c}\right)^2+\left(c+\frac{1}{c}\right)^2\left(a+\frac{1}{a}\right)^2=1.$$

Prove that $a = 1$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

First solution by the author

The conditions $ab+bc+ca = 1$ and $a, b, c > 0$ allow us to make the substitution $a = \tan\dfrac{X}{2}$, $b = \tan\dfrac{Y}{2}$, $c = \tan\dfrac{Z}{2}$, where XYZ is a triangle. Then

$$\left(\frac{2}{\sin X}\right)^2\left(\frac{2}{\sin Y}\right)^2-\left(\frac{2}{\sin Y}\right)^2\left(\frac{2}{\sin Z}\right)^2+\left(\frac{2}{\sin Z}\right)^2\left(\frac{2}{\sin X}\right)^2=0,$$

implying

$$\sin^2 Z-\sin^2 X+\sin^2 Y=0.$$

Denoting by x, y, z the side-lengths corresponding to vertices X, Y, Z, respectively, it follows that $z^2+y^2=x^2$. Hence $X = 90^\circ$ and so $a = \tan\dfrac{90^\circ}{2} = 1$, as desired.

Second solution by Polyahedra, Polk State College, USA
The 1 on the right side of the second equation should be 0. Notice that

$$a+\frac{1}{a}=\frac{a^2+ab+bc+ca}{a}=\frac{(a+b)(a+c)}{a}.$$

Therefore,

$$\begin{aligned}0&=\left(a+\frac{1}{a}\right)^2\left(b+\frac{1}{b}\right)^2-\left(b+\frac{1}{b}\right)^2\left(c+\frac{1}{c}\right)^2+\left(c+\frac{1}{c}\right)^2\left(a+\frac{1}{a}\right)^2\\&=\frac{(a+b)^2(b+c)^2(c+a)^2}{a^2b^2c^2}\left[c^2(a+b)^2-a^2(b+c)^2+b^2(c+a)^2\right],\end{aligned}$$

thus

$$0=c^2(a+b)^2-a^2(b+c)^2+b^2(c+a)^2=2bc(ab+bc+ca-a^2)=2bc(1-a^2),$$

so $a=1$.

Also solved by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Arkady Alt, San Jose, CA, USA.

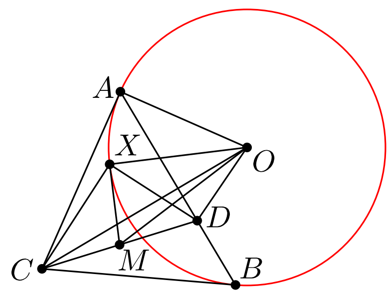
J575. Through point C lying outside of circle ω two lines are drawn that are tangent to the circle at points A and B. Point D lies on the segment AB and M is the midpoint of CD. Through point M a line is drawn that is tangent to circle ω at X. Prove that lines CX and DX are perpendicular.

Proposed by Waldemar Pompe, Warsaw, Poland

First solution by the author

Observe that the line joining the midpoints of segments AC and BC is the radical axis of circle ω and the circle with center C and radius 0. Moreover, M lies on this radical axis. This implies $MC = MX$, which gives the result.

Second solution by Polyahedra, Polk State College, USA



Let O and R be the center and radius of ω. Applying Stewart's theorem in triangles CBA and OCD, we get $CD^2 = CA^2 - AD \cdot BD$ and $OM^2 = (OC^2 + OD^2 - 2MD^2)/2$. By the power of a point, $AD \cdot BD = R^2 - OD^2$. Therefore,

$$MX^2 = OM^2 - R^2 = \frac{CA^2 + 2R^2 - AD \cdot BD - 2MD^2}{2} - R^2 = \frac{CD^2 - 2MD^2}{2} = MD^2,$$

thus $MX = MD = MC$, from which the claim follows.

Also solved by Kanav Talwar, Delhi Public School, Faridabad, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Titu Zvonaru, Comănești, Romania.

J576. Let a, b, c, d be positive real numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+d} = 1.$$

Prove that

$$ab + ac + ad + bc + bd + cd - 3(a + b + c + d) \geq 18.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Polyahedra, Polk State College, USA

Let $x = 1/(1+a)$, $y = 1/(1+b)$, $z = 1/(1+c)$, and $w = 1/(1+d)$. Then $x + y + z + w = 1$ and

$$ab + ac + ad + bc + bd + cd - 3(a + b + c + d) - 18 = \frac{1}{xy} + \frac{1}{xz} + \frac{1}{xw} + \frac{1}{yz} + \frac{1}{yw} + \frac{1}{wz} - 6\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right).$$

By the Cauchy-Schwarz inequality,

$$\frac{1}{xy} + \frac{1}{xz} + \frac{1}{xw} = \left(1 + \frac{y + z + w}{x}\right)\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \geq \frac{1}{y} + \frac{1}{z} + \frac{1}{w} + \frac{9}{x}.$$

Summing this with the other three analogous inequalities completes the proof.

Also solved by Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Zlota, Traian Vuia Technical College, Focşani, Romania.

S571. Let a be a nonzero real number for which there is a real number $b \geq 1$ such that

$$a^3 + \frac{1}{a^3} = b\sqrt{b+3}.$$

Prove that

$$a^2 + \frac{1}{a^2} = b + 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Clearly, $a > 0$. Squaring both sides of the the given relation yields

$$a^6 + 2 + \frac{1}{a^6} = b^2(b+3),$$

implying $a^6 + \frac{1}{a^6} = b^3 + 3b^2 - 2$. Then

$$\left(a^2 + \frac{1}{a^2}\right)^3 - 3\left(a^2 + \frac{1}{a^2}\right) = (b+1)^3 - 3(b+1).$$

The function $f(x) = x^3 - 3x$ is increasing for $x \geq 2$. Indeed, $f(x) - 2 = (x-2)(x+1)^2$, so if $x > y$, then $f(x) > f(y)$. Hence, if u and v are real numbers greater than or equal to 2 such that $u^3 - 3u = v^3 - 3v$, then $u = v$. It follows that $a^2 + \frac{1}{a^2} = b + 1$, as desired.

Second solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

Let $x = a^2 + \frac{1}{a^2}$. Then

$$x^3 = a^6 + \frac{1}{a^6} + 3x = \left(a^3 + \frac{1}{a^3}\right)^2 - 2 + 3x = b^2(b+3) - 2 + 3x,$$

and thus $x^3 - (b+1)^3 = 3[x - (b+1)]$ or

$$[x - (b+1)]\left[\left(x + \frac{b+1}{2}\right)^2 + 3 \cdot \frac{(b+1)^2 - 4}{4}\right] = 0.$$

Because $b \geq 1$, the only acceptable solution is $x = b + 1$.

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Le Hoang Bao, Tien Giang, Vietnam; Kanav Talwar, Delhi Public School, Faridabad, India; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Mihaly Bencze, Brasov, Romania; Daniel Văcaru, Pitești, Romania; G. C. Greubel, Newport News, VA, USA; Jiang Lianjun, Quanzhou, Second Middle School, GuiLin, China; Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Vishwesh Ravi Shrimali, Jaipur, Rajasthan, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

S572. Prove that in any triangle ABC

$$\frac{9}{4}\sqrt{\frac{r}{2R}} \le \sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le 1+\frac{r}{4R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by the author

The left inequality is equivalent to

$$\frac{9}{4}\sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \le \sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

$$\frac{27}{8}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2},$$

$$\frac{27\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}} \le 8.$$

By the AM-GM inequality we have

$$\begin{aligned}
\frac{27\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}} &\le \left(\sum_{\text{cyc}}\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)^3 \\
&= \left(\sum_{\text{cyc}}\frac{\cos(\frac{B}{2}+\frac{C}{2})}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)^3 \\
&= \left(\sum_{\text{cyc}}\frac{\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)^3 \\
&= \left(3-\sum_{\text{cyc}}\tan\frac{B}{2}\tan\frac{C}{2}\right)^3 \\
&= 8.
\end{aligned}$$

Now we have

$$\begin{aligned}
\sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} &= \frac{\sqrt{3}}{4}(\sin A+\sin B+\sin C) \\
&= \frac{\sqrt{3}(a+b+c)}{8R} \\
&= \frac{\sqrt{3s^2}}{4R} \quad (\text{where } s=\frac{a+b+c}{2}) \\
&= \frac{\sqrt{3(r_ar_b+r_br_c+r_cr_a)}}{4R} \\
&\le \frac{r_a+r_b+r_c}{4R} \\
&= \frac{4R+r}{4R} \\
&= 1+\frac{r}{4R}.
\end{aligned}$$

The proof is completed.

Second solution by Daniel Văcaru, Pitești, Romania

We have

$$\prod \cos\frac{A}{2} = \frac{sF}{abc} = \frac{sF}{4RF} = \frac{s}{4R}$$

We must prove that

$$\frac{9}{4}\sqrt{\frac{r}{2R}} \leq \frac{\sqrt{3}s}{4R} \leq 1 + \frac{r}{4R}$$

From $\frac{9}{4}\sqrt{\frac{r}{2R}} \leq \frac{\sqrt{3}s}{4R}$ we obtain $27Rr \leq 2s^2$ (Coșniță inequality [1]) and $\frac{\sqrt{3}s}{4R} \leq 1 + \frac{r}{4R}$ is equvalent to $\sqrt{3}s \leq 4R + r$, Elemente der Mathematik, Basel, 1963 [1]

Bibliography

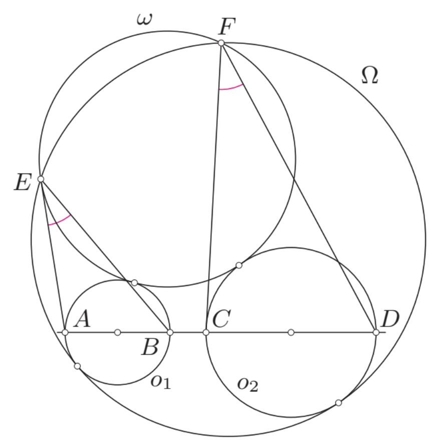
[1] Marin Chirciu, Inequalities with sides and radius in triangle: from initiation to performance (in Romanian), Paralela 45 Publishing House, 2017

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Arighna Pan, Nabadwip Vidyasagar College, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romaina; Nandan Sai Dasireddy, Hyderabad, Telangana, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

S573. Points A, B, C, D lie on a line in that order. Let o_1 and o_2 be the circles with diameters AB and CD, respectively. Circle ω is externally tangent to circles o_1 and o_2. Circle Ω is internally tangent to o_1 and o_2 and intersects ω at points E and F. Prove that $\angle AEB = \angle CFD$.

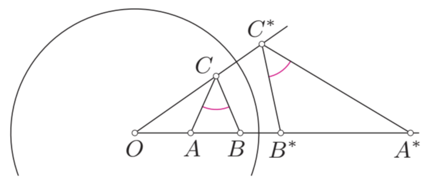
Proposed by Waldemar Pompe, Warsaw, Poland

Solution by the author



We use the following observation regarding inversion.

Suppose points O, A, B lie in a line in that order and denote by A^*, B^* the images of points A, B under an inversion with center O. Moreover, let C be any point that is mapped to C^* by the inversion. Then $\angle ACB = \angle A^*C^*B^*$.



To prove this equality observe that points A, C, A^*, C^* are concyclic, as well as points B, C, B^*, C^*. Thus

$$\angle ACB = \angle OCB - \angle OCA = \angle OB^*C^* - \angle OA^*C^* = \angle A^*C^*B^* .$$

We proceed to the solution to the problem.

Consider an inversion that maps circle o_1 to o_2. Then the touching points of circle Ω with circle o_1 and o_2 are swapped by the inversion, so Ω is mapped to Ω. Similarly, ω is mapped to ω. Thus points E and F are swapped by the inversion. The conculsion follows now from the above observation.

Also solved by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA.

S574. Find all real numbers x such that

$$\left\{\frac{6x^2+168x+2022}{x^2+24x+237}\right\}=\frac{6}{7},$$

where $\{x\}$ denotes the fractional part of x.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

Let $I=(6x^2+168x+2022)/(x^2+24x+237)$. First we show that $5<I<10$. Indeed, $5<I \Longleftrightarrow x^2+48x+837>0$, and $I<10 \Longleftrightarrow 4x^2+72x+348>0$; in both cases the discriminant of the corresponding quadratic is negative.

Therefore, from the given condition we get $I=n+\frac{6}{7}$, where $n=5,6,7,8,9$.
(i) For $I=41/7$, we get $x^2+192x+4437=0$, and thus $x=-96\pm 9\sqrt{59}$.
(ii) For $I=48/7$, we get $x^2-4x-463=0$, and thus $x=2\pm\sqrt{467}$.
(iii) For $I=55/7$, we get $13x^2+144x-1119=0$, and thus $x=\frac{1}{13}(-72\pm\sqrt{19731})$.
(iv) For $I=62/7$, we get $5x^2+78x+135=0$, and thus $x=\frac{1}{5}(-39\pm 3\sqrt{94})$.
(v) For $I=69/7$, we get $9x^2+160x+733=0$, which has no real roots.

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Kanav Talwar, Delhi Public School, Faridabad, India; Fred Frederickson, Utah Valley University, UT, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S575. Let ABC be an acute triangle, with orthocenter H. Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, respectively, and let A_2, B_2, C_2 be points inside segments HA_1, HB_1, HC_1 such that

$$\frac{HA_2}{A_2A_1} = \frac{HB_2}{B_2B_1} = \frac{HC_2}{C_2C_1} = 2.$$

Prove that lines AA_2, BB_2, CC_2 are concurrent.

Proposed by Mihaela Berindeanu, Bucharest, România

First solution by the author

ABC circumcircle is notadet Γ, with circumcenter O and circumradius R.
HBC circumcircle is notated Γ_1, with circumradius R_1

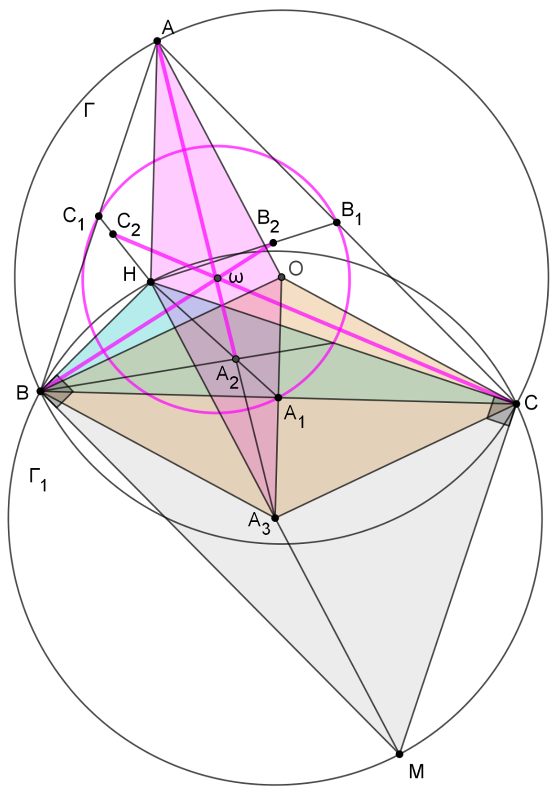


Figura 1:

- Show that AA_2 is the Euler line in $\triangle HBC$.

$$\left.\begin{array}{l} HA_1 - \text{ median in } \triangle BHC \\ A_2 \in HA_1 \\ A_2A_1 = \dfrac{1}{3}HA_1 \end{array}\right\} \Rightarrow A_2 \text{ is the centroid of } \triangle HBC.$$

$$\left.\begin{array}{l} AB \perp HC \\ AC \perp HB \end{array}\right\} \Rightarrow A \text{ is the orthocenter of } \triangle HBC.$$

The lines $\left.\begin{array}{l} BM \parallel AC \\ CM \parallel AB \end{array}\right\}$ are build for find the center of Γ_1 circle. Results $\measuredangle HCM = \measuredangle HBM = 90°$ so the quadrilater $HBMC$ is circumscribed (like $\triangle HBC$), in the circle Γ_1, with the diameter HM and the center notated A_3, $HA_3 = A_3M$.

- Show that $OA_3 \perp BC$

Applying the Sine Rule $\begin{cases} \text{in } \triangle HBC: & \dfrac{BC}{\sin \measuredangle BHC} = 2R_1 \\ \text{in } \triangle ABC: & \dfrac{BC}{\sin \measuredangle A} = 2R \end{cases}$ where $\sin \measuredangle BHC = \sin(180 - A) = \sin A \Rightarrow$

$R = R_1 \Rightarrow BA_3 = A_3H = A_3C = A_3M = R \Rightarrow OBA_3C =$ rhombus because $OB = OC = A_3C = A_3B = R$

$$OBA_3C = \text{ rhombus } \Rightarrow OA_3 \perp BC$$

- Show that AHA_3O is an parallelogram

$\left.\begin{array}{l} \measuredangle BOC \text{ central angle in } \Gamma \\ \measuredangle BAC \text{ inscribed angle in } \Gamma_1 \end{array}\right\} \Rightarrow \measuredangle BOC = 2\measuredangle A \Rightarrow \left.\begin{array}{l} OA_3 = 2R\cos A \\ AH = 2R\cos A \end{array}\right\} \Rightarrow AHA_3O$ parallelogram

- Show that $AA_2,\ BB_2,\ CC_2$ are concurent lines.

$$AA_3 \cap HO = \{\omega\} \text{ and } H\omega = \omega O$$

AA_3 coincides with AA_2 (Euler line in triangle HBC) and passes through the middle point of the segment OH which is the Nine-point circle center.
Analogously, BB_2 and CC_2 pass through the point ω, so $AA_2,\ BB_2,\ CC_2$ are concurent lines.

Second solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

Let G be the intersection point of the medians AA_1, BB_1, CC_1. Then from $AG = 2GA_1$ and $HA_2 = 2A_2A_1$ we get that $GA_2 \parallel AH$ and triangles A_1GA_2, A_1AH are similar. Let K be the intersection point of AA_2 and HG. By construction, point K is on the segment HG, because the segments AA_2 and HG are internal to the triangle AHA_1. Becuse $GA_2 \parallel AH$, triangles AKH, A_2KG are similar. Therefore,

$$\frac{HK}{GK} = \frac{AH}{GA_2} = \frac{AA_1}{GA_1} = 3.$$

Similarly, if K' is the intersection point of the segments BB_2 and HG, and K'' is the intersection point of the segments CC_2 and HG, we find that $HK' = 3GK'$ and $HK'' = 3GK''$. Therefore $K = K' = K''$ and the lines AA_2, BB_2, CC_2 are concurrent at K. We note that this holds true for any triangle ABC.

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Kanav Talwar, Delhi Public School, Faridabad, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S576. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = \sqrt{n}$. Prove that

$$\left(a_1 + \frac{1}{a_1}\right)^2 + \left(a_2 + \frac{1}{a_2}\right)^2 + \cdots + \left(a_n + \frac{1}{a_n}\right)^2 \geq (n+1)^2.$$

Proposed by Titu Andreescu, USA and Alessandro Ventullo, Italy

First solution by the authors

By the QM-AM Inequality, we have

$$\begin{aligned}\left(a_1 + \frac{1}{a_1}\right)^2 + \left(a_2 + \frac{1}{a_2}\right)^2 + \ldots + \left(a_n + \frac{1}{a_n}\right)^2 &\geq n\left(\frac{a_1 + \frac{1}{a_1} + a_2 + \frac{1}{a_2} + \ldots + a_n + \frac{1}{a_n}}{n}\right)^2 \\ &= \frac{\left(\sqrt{n} + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)^2}{n}.\end{aligned}$$

By the AM-HM Inequality, we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \geq \frac{n^2}{a_1 + a_2 + \ldots + a_n} = n\sqrt{n},$$

which gives the desired conclusion. The equality holds if and only if $a_1 = a_2 = \ldots = a_n = \dfrac{\sqrt{n}}{n}$.

Second solution by Henry Ricardo, Westchester Area Math Circle
The arithmetic mean-harmonic mean inequality and the given constraint give us

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2, \text{ or } \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq \frac{n^2}{\sqrt{n}} = n\sqrt{n}.$$

Now we use the last inequality and the arithmetic mean-quadratic mean inequality to see that

$$\begin{aligned}\sum_{i=1}^{n}\left(a_i + \frac{1}{a_i}\right)^2 &\geq n\left(\frac{a_1 + 1/a_1 + \cdots a_n + 1/a_n}{n}\right)^2 \\ &= n\left(\frac{\sqrt{n} + 1/a_1 + \cdots + 1/a_n}{n}\right)^2 \\ &\geq n\left(\frac{\sqrt{n} + n\sqrt{n}}{n}\right)^2 = (n+1)^2.\end{aligned}$$

Equality holds if and only if $a_i = 1/\sqrt{n},\ i = 1, 2, \ldots, n$.

Also solved by Arkady Alt, San Jose, CA, USA; UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Kanav Talwar, Delhi Public School, Faridabad, India; Fred Frederickson, Utah Valley University, UT, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Arighna Pan, Nabadwip Vidyasagar College, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Nandan Sai Dasireddy, Hyderabad, Telangana, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Jiang Lianjun, Quanzhou, Second Middle School, GuiLin, China; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Mihaly Bencze, Brasov, Romania.

U571. Let A be a $n \times n$ matrix with real entries and let $\alpha \neq 0$ be a real number. Prove that if $A^3 = I$ and $(A - \alpha I)^3 = 0$, then $AB = BA$, for all $n \times n$ matrices B.

Proposed by Mircea Becheanu, Montreal, Canada

First solution by the author

We will show that $\alpha = 1$ and $A = I$. Consider the real polynomials $P(X) = X^3 - 1$ and $Q(X) = (X - \alpha)^3$. It is clear that $P(A) = 0$ and $Q(A) = 0$. If $\alpha \neq 1$ we have $\mathrm{GCD}(P(X), Q(X)) = 1$. Then, there exists real polynomials $F(X)$ and $G(X)$ such that $P(X)F(X) + Q(X)G(X) = 1$. In this equality we plug the matrix A and one obtains $I = 0$, which is a contradiction. If $\alpha = 1$, then $\mathrm{GCD}(F(X), G(X)) = X - 1$. Then we have linear combination $P(X)F(X) + Q(X)G(X) = X - 1$. Evaluate in A and obtain $A - I = 0$.

Second solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Since $(\det(A - \alpha I))^3 = \det(A - \alpha I)^3 = 0$, α is an eigenvalue of A, with an eigenvector x. Then $x = A^3 x = \alpha^3 x$, so $\alpha = 1$. Therefore, $0 = (A - I)^3 = -3A^2 + 3A$, so $A^2 = A$, thus $I = A^3 = A^2 = A$, from which the claim follows.

Third solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

Expanding $(A - \alpha I)^3$ and using $A^3 = I$ we get

$$0 = (A - \alpha I)^3 = A^3 - 3\alpha A^2 + 3\alpha^2 A - \alpha^3 I = (1 - \alpha^3)I - 3\alpha A^2 + 3\alpha^2 A, \tag{1}$$

and thus

$$3\alpha A^2 = (1 - \alpha^3)I + 3\alpha^2 A. \tag{2}$$

Multiplying (1) by A we using (2) we get

$$0 = (1 - \alpha^3)A - 3\alpha I + 3\alpha^2 A^2 = (1 - \alpha^3)A - 3\alpha I + \alpha[(1 - \alpha^3)I + 3\alpha^2 A],$$

and therefore,

$$(2\alpha^3 + 1)A = \alpha(\alpha^3 + 2)I \Longrightarrow A = \frac{\alpha(\alpha^3 + 2)}{2\alpha^3 + 1} I,$$

because $(a^3 + 2)(2a^3 + 1) \neq 0$. Using $A^3 = I$ we now get $\alpha(\alpha^3 + 2) = 2\alpha^3 + 1$ or $(\alpha + 1)(\alpha - 1)^3 = 0$. For $\alpha = -1$ we get $A = I$, but this solution does not satisfy the condition $(A - \alpha I)^3 = 0$. Therefore $\alpha = 1$ and $A = I$, and thus $AB = BA$ for all $n \times n$ matrices B.

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Le Hoang Bao, Tien Giang, Vietnam; Fred Frederickson, Utah Valley University, UT, USA; Prajnanaswaroopa S, Bangalore, Karnataka, India; Mihaly Bencze, Brasov, Romania.

U572. Evaluate

$$\int \left(x + \frac{1}{4x}\right) \frac{e^x}{\sqrt{x}}\, dx.$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Given the structure of the integrand, we seek constants a and b such that

$$a\sqrt{x} + \frac{b}{\sqrt{x}} + \frac{d}{dx}\left(a\sqrt{x} + \frac{b}{\sqrt{x}}\right) = \sqrt{x} + \frac{1}{4\sqrt{x^3}}.$$

Now,

$$\begin{aligned} a\sqrt{x} + \frac{b}{\sqrt{x}} + \frac{d}{dx}\left(a\sqrt{x} + \frac{b}{\sqrt{x}}\right) &= a\sqrt{x} + \frac{b}{\sqrt{x}} + \frac{a}{2\sqrt{x}} - \frac{b}{2\sqrt{x^3}} \\ &= a\sqrt{x} + \left(b + \frac{a}{2}\right)\frac{1}{\sqrt{x}} - \frac{b}{2\sqrt{x^3}}, \end{aligned}$$

so $a = 1$ and $b = -1/2$. In other words,

$$\frac{d}{dx}\left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)e^x = \left(\sqrt{x} + \frac{1}{4\sqrt{x^3}}\right)e^x = \left(x + \frac{1}{4x}\right)\frac{e^x}{\sqrt{x}},$$

so

$$\int \left(x + \frac{1}{4x}\right)\frac{e^x}{\sqrt{x}}\, dx = \left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)e^x + C.$$

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Henry Ricardo, Westchester Area Math Circle; G. C. Greubel, Newport News, VA, USA; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Arighna Pan, Nabadwip Vidyasagar College, India; Ankush Kumar Parcha, Indira Gandhi National Open University, Indian; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U573. Prove the inequality

$$\sum_{k=1}^{\infty} \frac{k \cdot 2^{\frac{k+1}{2}}}{3^{2^k-1}} < e.$$

Proposed by Mohammed Imran, Chennai, India

Solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

We consider the series $a_n = \dfrac{n \cdot 2^{\frac{n+1}{2}}}{3^{2^n-1}}$, with first term $a_1 = \frac{2}{3}$, and the geometric series $b_n = \dfrac{2}{3} \cdot \dfrac{1}{3^{n-1}}$, with first term $b_1 = \frac{2}{3}$ and common ratio $r = \frac{1}{3}$. Clearly,

$$\sum_{n=1}^{\infty} b_n = \frac{2}{3} \cdot \frac{1}{1 - \frac{1}{3}} = 1.$$

We will show that $a_n \le b_n$ for all positive integers n, and therefore prove the stronger inequality

$$\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n = 1 < e.$$

Indeed, for $n = 1$ we have $a_1 = b_1 = \frac{2}{3}$. For $n = 2$ we have $a_2 = \frac{4\sqrt{2}}{27}$ and $b_2 = \frac{2}{9}$, and thus $a_2 < b_2$. For $n \ge 3$ we have

$$a_n \le b_n \Longleftrightarrow \frac{n \cdot 2^{\frac{n+1}{2}}}{3^{2^n-1}} \le \frac{2}{3} \cdot \frac{1}{3^{n-1}} \Longleftrightarrow n \le \frac{3^{2^n-n-1}}{2^{\frac{n-1}{2}}}.$$

We proceed by induction. For $n = 3$ we have $3 \le \frac{3^4}{2}$, which is obvious. Assuming that the inequality holds for $n = m \ge 3$, we will show that it holds for $n = m + 1$ too. Indeed,

$$\frac{3^{2 \cdot 2^m - m - 2}}{2^{\frac{m}{2}}} = \frac{3^{2^m-m-1}}{2^{\frac{m-1}{2}}} \cdot \frac{3^{2^m-1}}{2^{\frac{1}{2}}} \ge m \cdot \frac{3^{2^m-1}}{2^{\frac{1}{2}}} = m \cdot \frac{3\sqrt{2}}{2} \cdot 3^{2^m-2} > m \cdot \frac{3}{2} > m + 1.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{k \cdot 2^{\frac{k+1}{2}}}{3^{2^k-1}} < 1.$$

Also solved by Arkady Alt, San Jose, CA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

U574. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $\int_a^b f(x)\,dx > 0$. Prove that for all positive numbers $c < (b-a)^2$, there is $\epsilon \in (a, b)$ such that

$$\int_a^b f(x)\,dx > \frac{c}{b-a} f(\epsilon).$$

Proposed by Ovidiu Gabriel Dinu, Bălceşti-Vâlcea, România

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Because $\int_a^b f(x)\,dx > 0$,

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx > 0,$$

and

$$\int_a^b f(x)\,dx = (b-a) f_{\text{avg}} = \frac{(b-a)^2}{b-a} f_{\text{avg}} > \frac{c}{b-a} f_{\text{avg}}$$

for any $0 < c < (b-a)^2$. Next, because f is continuous on $[a, b]$, by the Extreme Value Theorem, there exist real numbers $c_1, c_2 \in [a, b]$ such that $f(c_1) = m$ and $f(c_2) = M$ where

$$m \le f(x) \le M$$

for all $x \in [a, b]$. Thus,

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx = f_{\text{avg}} \le M,$$

so by the Intermediate Value Theorem, there exists ϵ between c_1 and c_2 such that $f(\epsilon) = f_{\text{avg}}$. Hence, there exists $\epsilon \in [a, b]$ such that

$$\int_a^b f(x)\,dx > \frac{c}{b-a} f(\epsilon)$$

for all positive numbers $c < (b-a)^2$.

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Corneliu Mănescu-Avram, Ploieşti, Romania; Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U575. Let $f:[0,1]\leftrightarrow\mathbb{R}$ be a continuous functions witho $f(0)=0$. Prove that there is $c\in(0,1)$ such that

$$\int_0^c (1-x)^2 f(x)dx = (1-c)\int_0^c f(x)dx.$$

Proposed by Florin Stănescu, Găeşti, România

Solution by the author

We will use Mean Theorem of T.M. Flett (Mathematical Gazette, vol. 42, No. 339, pp 38-39.)

Theorem: Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function such that $f'(a)=f'(b)$. There exists a point $c\in(a,b)$ such that

$$f(c)-f(a)=(c-a)f'(c).$$

Let $F(x)=\int_0^x f(t)dt$ be a primitive of $f(x)$. We consider the function $g:[0,1]\longrightarrow\mathbb{R}$ defined as

$$g(x)=\int_0^x (1-t)F(t)dt.$$

Then $g'(x)=(1-x)F(x)$ and $g'(0)=g'(1)=0$. So, by Flett Theorem, there exists $c_1\in(0,1)$ such that $g(c_1)-g(0)=c_1g'(c_1)$, that is $g(c_1)=c_1g'(c_1)$.

Consider the function $h:[0,1]\longrightarrow\mathbb{R}$ defined by $h(x)=g(x)/x$ for $x\in(0,1]$ and $h(0)=0$. It is clear that h is continuous. We prove that it is also differentiable. Using l'Hospital rule, we have;

$$\lim_{x\to 0}\frac{h(x)-h(0)}{x-0}=\lim_{x\to 0}\frac{g(x)}{x^2}=\lim_{x\to 0}\frac{g'(x)}{2x}=$$

$$=(1/2)\lim_{x\to 0}\left(-\int_0^x f(t)dt+(1-x)f(x)\right)=\frac{f(0)}{2}=0.$$

Hence, $h(x)$ is differentiable on $[0,1]$, $h'(0)=0$ and $h'(x)=\frac{xg'(x)-g(x)}{x^2}$ for $x\neq 0$.

We prove that $h(x)$ satisfy the conditions of Flatt Theorem. We proved above that there exists a point c_1 such that $g(c_1)=c_1g'(c_1)$. It means that there exists a point $c_1\in(0,1)$ such that $h'(c_1)=0$. Now we apply Flett Theorem for the function $h(x)$ on $(0,c_1)$. There exists a point $c\in(0,c_1)$ such that $h(c)-h(0)=ch'(c)\Leftrightarrow\frac{g(c)}{c}=c\frac{cg'(c)-g(c)}{c^2}\Leftrightarrow 2g(c)=cg'(c)$. The last equality can be written as

$$2\int_0^c(1-t)\left(\int_0^t f(x)dx\right)dt=c(1-c)\int_0^c f(x)dx.$$

We compute the left hand side by using integration by parts formula:

$$\int_0^c(1-t)\left(\int_0^t f(x)dx\right)dt=-\frac{1}{2}(1-t)^2\int_0^t f(x)dx\Big|_0^c+\int_0^c\frac{1}{2}(1-t)^2f(t)dt=$$

$$\frac{1}{2}\left[\int_0^c(1-x)^2f(x)dx-(1-c)^2\int_0^c f(x)dx\right].$$

We plug this result in the above equality and obtain:

$$\int_0^c(1-x)^2f(x)dx=[c(1-c)+(1-c)^2]\int_0^c f(x)dx=(1-c)\int_0^c f(x)dx.$$

Also solved by Prajnanaswaroopa S, Bangalore, Karnataka, India; Mihir Kaskhedikar, India;.

U576. Find all triples (m,n,p) of positive integers for which there is a real polynomial $P(x)$ such that

$$P(x)+x^mP(1-x)=(x^2-x+1)^n(x^2-x-1)^p.$$

Determine whether there are infinitely many such polynomials.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

The quadratics $x(x-1)\pm 1$ remain invariant under the substitution of $1-x$ for x, and thus we get

$$P(1-x)+(1-x)^mP(x)=P(x)+x^mP(1-x),$$

or

$$P(x)\cdot[1-(1-x)^m]=P(1-x)\cdot(1-x^m). \tag{3}$$

The polynomials $1-x^m$ and $1-(1-x)^m$ have roots given by $e^{i2k\pi/m}$ and $1-e^{i2\lambda\pi/m}$, respectively, where k,λ are integers. Their common roots satisfy the equations

$$\cos\left(\frac{2k\pi}{m}\right)+\cos\left(\frac{2\lambda\pi}{m}\right)=1$$
$$\sin\left(\frac{2k\pi}{m}\right)+\sin\left(\frac{2\lambda\pi}{m}\right)=0.$$

It is easy to show that these common roots are $x_{1,2}=\frac{1\pm i\sqrt{3}}{2}$, when $m\equiv 0 \mod 6$; these are also the roots of the quadratic $x^2-x+1=0$.

(i) Case 1: When $m\not\equiv 0 \mod 6$, from (1) we get that $P(x)=(1-x^m)Q(x)$ with $Q(x)=Q(1-x)$. From the given condition we now get

$$Q(x)\cdot[1-x^m(1-x)^m]=(x^2-x+1)^n(x^2-x-1)^p. \tag{4}$$

When $m=1$, we get $Q_1(x)=(x^2-x+1)^{n-1}(x^2-x-1)^p$, and the infinity of solutions $P_1(x)=(1-x)Q_1(x)$. When $m=2$, we get $Q_2(x)=-(x^2-x+1)^{n-1}(x^2-x-1)^{p-1}$, and the infinity of solutions $P_2(x)=(1-x^2)Q_2(x)$. For $m\geq 3$, we rewrite (2) as

$$Q(r)\cdot(1-r^m)=(-1)^p(1-r)^n(1+r)^p, \tag{5}$$

where $r=x(1-x)$. We note that $q(r):=1-r^m=(1-r)(1+r+r^2+\cdots+r^{m-1})$, and thus $r=1$ is a single root of $q(r)$; also, $r=-1$ is a single root of $q(r)$ when m is even. Therefore, for $m\geq 3$, $q(r)$ has a factor that does not divide $(1-r)^n(1+r)^p$. Thus, there is no solution for $m\geq 3$.

(ii) Case 2: When $m\equiv 0 \mod 6$, from (1) we get $P(x)=\frac{1-x^m}{x^2-x+1}\cdot Q(x)$ with $Q(x)=Q(1-x)$. From the given condition we now get

$$Q(r)\cdot(1-r^m)=(-1)^p(1-r)^{n+1}(1+r)^p,$$

where $r=x(1-x)$. As in Case 1 above, there is no polynomial $Q(x)$ satisfying the above equation.

O571. Let a, b, c, d be positive real numbers such that

$$abcd = ab + ac + ad + bc + bd + cd.$$

Prove that

$$\sum abc - 2\sum ab + 3\sum a \geq 36(\sqrt{6} - 2),$$

where all sums are symmetric sums.

Proposed by Marian Tetiva, România

Solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

Let K on AB be where ω_1 is tangent to ω_2. Additionally, let $k = 36(\sqrt{6} - 2)$, $p = b + c + d$, $q = bc + bd + cd$, and $r = bcd$. From the given condition we get $0 < a = q/(r - p)$, and thus $p < r$. Without loss of generality we assume that $a = \min\{a, b, c, d\}$. Then $ap \leq q$, and from the given condition we get $ar \leq 2q$; therefore, $qr/(r - p) \leq 2q$ or $p \leq r/2$. Finally, using AM-GM we get $p \geq 3r^{1/3}$; therefore, $3r^{1/3} \leq p \leq \frac{r}{2}$, and thus $r \geq 6\sqrt{6}$.

Substituting the expression for a, we rewrite the given inequality as a quadratic in q:

$$q^2 - q(2r - 3) + (r - p)(3p + r) - k(r - p) \geq 0. \tag{6}$$

The discriminant of the quadratic is $\Delta(p) = 12p^2 - 8pr - 12r + 9 + 4k(r - p)$ and its roots are $q_- = \frac{1}{2}(2r - 3 - \sqrt{\Delta})$ and $q_+ = \frac{1}{2}(2r - 3 + \sqrt{\Delta})$.

(i) When $\Delta(p) < 0$, then (1) holds true. We note that $\Delta(p)$ is a convex quadratic in p. Also, $\Delta\left(\frac{r}{2}\right) = -r^2 - 2(6 - k)r + 9$, and $\Delta\left(3r^{1/3}\right) = -24r^{4/3} + (4k - 12)r + 108r^{2/3} - 12kr^{1/3} + 9$. Using the intermediate value theorem for $\Delta\left(\frac{r}{2}\right)$ and $\Delta(3r^{1/3})$ we easily find that both take negative values for about $r \geq 22$. Therefore, for $r \geq 22$ we have $\Delta(p) < 0$ for all $p \in [3r^{1/3}, \frac{r}{2}]$, and (1) holds true.

(ii) When $\Delta(p) \geq 0$, to prove (1) is sufficient to show that $3r^{2/3} \geq q_+$ because from AM-GM we have $q \geq 3r^{2/3}$. But

$$3r^{2/3} \geq q_+ \iff 6r^{2/3} - 2r + 3 \geq \sqrt{\Delta}.$$

Using the intermediate value theorem for $6r^{2/3} - 2r + 3$ we easily find that it takes positive values for $6\sqrt{6} \leq r \leq 31$, a range of values that overlaps that for $\Delta(p) < 0$. Squaring and rearranging we get

$$f(p) := 3p^2 - p(2r + k) + kr - r^2 - 3r^{2/3}(3r^{2/3} + 3 - 2r) \leq 0.$$

Making the substitution $t = r^{1/3} \geq \sqrt{6}$, we get

$$\begin{aligned} f\left(\frac{r}{2}\right) \leq 0 &\iff 5t^4 - 24t^3 + 36t^2 - 2kt + 36 \geq 0 \\ &\iff (t - \sqrt{6})[t^2(5t + 5\sqrt{6} - 24) + (59 - 24\sqrt{6})t + (7t - 6\sqrt{6})] \geq 0, \end{aligned}$$

which is obvious. Equality occurs when $t = \sqrt{6}$. Also,

$$\begin{aligned} f\left(3r^{1/3}\right) \leq 0 &\iff t^5 - 6t^4 + 15t^3 - kt^2 - 18t + 3k \geq 0 \\ &\iff (t - \sqrt{6})^2[(t + 2\sqrt{6} - 6)t^2 + (33 - 12\sqrt{6})t + (18\sqrt{6} - 36)] \geq 0, \end{aligned}$$

which is obvious. Equality occurs when $t = \sqrt{6}$. Because $f(p)$ is a convex quadratic in p, we have $f(p) \leq 0$ for all $p \in [3r^{1/3}, \frac{r}{2}]$, and thus (1) holds true.

Therefore, the minimum value of the quantity $\sum abc - 2\sum ab + 3\sum a$ is k, and it is achieved when $r = 6\sqrt{6}$, and thus when $a = b = c = d = \sqrt{6}$.

O572. Let a, b, c, d be positive integers and let C be a nonzero integer. The map $f : \mathbb{Z} \to \mathbb{Z}$ has the property that $f(mn) = f(m)f(n)$ for all integers m, n, and there exists N such that for all $n \geq N$

$$f(c(an+b)+d) \equiv C \pmod{an+b}.$$

Prove that there is an integer e such that $|f(n)| = |n|^e$ for all integers n relatively prime to ac.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Note that f is nonzero, thanks to the congruence it satisfies and to the fact that C is nonzero. Thus $f(1) = 1$ and $f(-1) = \pm 1$. In particular it suffices to prove that there is e such that $|f(n)| = n^e$ for $n > 1$ prime to ac. For such n the integer $n^{acN\varphi(ac)}$ is greater than $1 + acN$ and congruent to 1 modulo ac, and if we know that $|f(n^{N\varphi(ac)})| = (n^{N\varphi(ac)})^e$ (with e independent of n), then $|f(n)| = n^e$. Thus we are reduced to proving that there is e such that $|f(u)| = u^e$ for $u > 1 + acN$ congruent to 1 modulo ac. Fix such u in the sequel.

For any integer $k \geq 1$ we can write $(bc+d)u^k = bc + d + acn_k$ for some $n_k > N$. Using the hypothesis, it follows that $an_k + b$ divides $f((bc+d)u^k) - C = f(bc+d)f(u)^k - C$ for all k. Thus

$$(bc+d)u^k - d \mid cf(bc+d)f(u)^k - cC.$$

In particular $f(bc+d)$ and $f(u)$ are nonzero (since c and C are nonzero, thus cannot be divisible by $(bc+d)u^k-d$ for large enough k). The lemma below implies the existence of an integer e such that $f(u)^2 = u^{2e}$ and $f(bc+d)d^e = (bc+d)^e C$. The second equality determines e uniquely, thus e is independent of u. Moreover $|f(u)| = u^e$, as desired. This finishes the proof.

Lemma: Let u, v be positive integers with $u > 1$, and let a, b, c, d be nonzero integers such that $au^k - b \mid cv^k - d$ for all $k \geq 1$. Then there is an integer e such that $v = u^e$ and $cb^e = da^e$.

The case $v = 1$ being clear, assume that $v > 1$. Let $x_k = \frac{cv^k-d}{au^k-b}$. For k large enough we have

$$x_k = \frac{cv^k - d}{au^k(1 - \frac{b}{au^k})} = \frac{cv^k - d}{au^k}(1 + \frac{b}{au^k} + \frac{b^2}{a^2u^{2k}} + \ldots).$$

Let e be the unique integer such that $u^e \leq v < u^{e+1}$. Using the previous expansion we obtain

$$\lim_{k\to\infty} x_k - \frac{c}{a}(\frac{v}{u})^k - \frac{cb}{a^2}(\frac{v}{u^2})^k - \ldots - \frac{cb^{e-1}}{a^e}(\frac{v}{u^e})^k = 0.$$

Next, we prove by induction on t that if $s_1, \ldots, s_t, r_1, \ldots, r_t$ are rational numbers with $s_i \neq 0$ and r_i pairwise distinct, and if x_k is a sequence of integers such that $\lim_{k\to\infty}(x_k - s_1r_1^k - \ldots - s_tr_t^k) = 0$, then all r_i are integers and so $x_k = s_1r_1^k + \ldots + s_tr_t^k$ for all large enough k. Indeed, if $t = 1$ the sequence $x_{k+1} - s_1r_1^{k+1} - r_1(x_k - s_1r_1^k) = x_{k+1} - r_1x_k$ tends to 0 and consists of rational numbers with bounded denominators, thus $x_{k+1} = r_1x_k$ for large enough k. Since x_k are integers, this immediately implies that r_1 is an integer as well. Now, if the result holds for $t-1$, setting $y_k = x_k - s_1r_1^k - \ldots - s_tr_t^k$ we have

$$y_{k+1} - r_ty_k = x_{k+1} - r_tx_k - \sum_{i=1}^{t-1} s_i(r_i - r_t)r_i^k$$

and $x_{k+1} - r_tx_k$ is a sequence of rational numbers with bounded denominators, thus we can apply the inductive hypothesis to deduce that $r_1, \ldots, r_{t-1}$ are integers. By symmetry, all r_i are integers.

We conclude that for all large enough k we have

$$x_k = \frac{c}{a}(\frac{v}{u})^k + \frac{cb}{a^2}(\frac{v}{u^2})^k + \ldots + \frac{cb^{e-1}}{a^e}(\frac{v}{u^e})^k.$$

Dividing by $\frac{cv^k}{au^k}$ we recognize a geometric progression in the right-hand side. Simple algebra shows then that the previous equality is equivalent to

$$1 - \frac{d}{cv^k} = 1 - (\frac{b}{au^k})^e.$$

Since this is satisfied for all large enough k, we have $v = u^e$ and $\frac{d}{c} = \frac{b^e}{a^e}$, thus $da^e = cb^e$, as desired.

O573. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$x+f(x^2+f(y))=xf(x)+f(x+y),$$

for all $x,y\in\mathbb{R}$.

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

Solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

Let $P(x,y)$ represent the statement that $x+f(x^2+f(y))=xf(x)+f(x+y)$. Then we have

$$P(-x,x):-x+f(x^2+f(x))=-xf(-x)+f(0), \tag{7}$$

$$P(x,x):x+f(x^2+f(x))=xf(x)+f(2x). \tag{8}$$

Subtracting (1) from (2) we get

$$2x=x\left[f(x)+f(-x)\right]+f(2x)-f(0). \tag{9}$$

Substituting $-x$ for x in (3) we get

$$-2x=-x\left[f(-x)+f(x)\right]+f(-2x)-f(0). \tag{10}$$

Adding (3) and (4) we get $f(2x)+f(-2x)=2f(0)$, and therefore

$$f(-x)=-f(x)+2f(0). \tag{11}$$

Using (5), we also have

$$P(-x,y):-x+f(x^2+f(y))=-x\left[-f(x)+2f(0)\right]+f(y-x); \tag{12}$$

$$P(x,y):x+f(x^2+f(y))=xf(x)+f(y+x). \tag{13}$$

Subtracting (6) from (7) we get $2x=2xf(0)+f(y+x)-f(y-x)$; setting $y=0$ we get

$$f(x)-f(-x)=2x\left[1-f(0)\right]. \tag{14}$$

Adding (5) and (8) we get

$$f(x)=\left[1-f(0)\right]\cdot x+f(0).$$

Substituting the above in $P(x,0)$ we find $f(0)^2=f(0)$, and therefore $f(0)\in\{0,1\}$. Thus, the two functions that satisfy $P(x,y)$ for all $x,y\in\mathbb{R}$ are $f(x)=x$ and $f(x)=1$.

Also solved by Kanav Talwar, Delhi Public School, Faridabad, India; Hoang Tuan Dung, Hanoi Natinal University of Education, Ha Noi, Vietnam.

O574. Segment AB is a chord of circle Γ. Different circles ω_1 and ω_2 are internally tangent to Γ at points P and Q, respectively, and to the segment AB at a common point. Chords AB and PQ meet at D. Let C be the midpoint of arc AB of circle Γ that contains point P. Prove that line CD passes through the center of ω_1.

Proposed by Waldemar Pompe, Warsaw, Poland

First solution by the author

Since the tangents at P and Q meet on AB (the radical axes), the tangents at A and B meet at E lying on PQ. A homothety with center P sending Γ to ω_1 maps C to L, and since the tangents at C and L are parallel to AB, KL is a diameter of ω_1. Moreover, KL is parallel to CE. Now project line PQ onto LK using C as the center of the projection. Since $(PQDE) = -1$ and E is projected to the infinity point of line KL, D must be projected to the midpoint of KL, which is the center of ω_1.

Second solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA
Let R, r, ρ be the radii of circles Γ, ω_1, ω_2, respectively, and let O, K, L be their respective centers. Let M be the midpoint of AB; then O, M, C are collinear and OC is the perpendicular bisector of AB, intersecting Γ again at point G. Let E be the common point of ω_1, ω_2, and AB, and let $ME = x$ and $OM = d$. Let F be the intersection point of the external tangents to Γ at points P, Q; then point F has the same power with respect to circles Γ, ω_1, ω_2, and therefore, using the radical axis theorem we get that F is on AB, the common external tangent to ω_1, ω_2. Then $\angle FPO = \angle FMO = \angle FQO = 90^\circ$, and points F, P, M, O, Q are concyclic. Considering the power of point D with respect to the circles Γ and $(FPMOQ)$, we get $FD \cdot MD = DP \cdot DQ = DA \cdot DB$, and thus

$$(FM - MD) \cdot MD = (MA - MD) \cdot (MA + MD) \Longrightarrow MD = \frac{MA^2}{FM} = \frac{R^2 - d^2}{FM}.$$

Considering the power of F with respect to Γ and ω_2 we get

$$FP^2 = FE^2 = FO^2 - R^2 = FM^2 + d^2 - R^2 = (FE + x)^2 + d^2 - R^2,$$

and therefore

$$FE = \frac{R^2 - d^2 - x^2}{2x} \Longrightarrow FM = \frac{R^2 - d^2 + x^2}{2x} \Longrightarrow MD = \frac{2x(R^2 - d^2)}{R^2 - d^2 + x^2}. \tag{15}$$

Let O' be the projection of O on EL. Clearly K, E, L are collinear and $KL \parallel MO$. Then from the right triangles $OO'L$ and $OO'K$ we get

$$x^2 = OL^2 - O'L^2 = (R - \rho)^2 - (\rho - d)^2 \Longrightarrow \rho = \frac{R^2 - d^2 - x^2}{2(R - d)};$$
$$x^2 = OK^2 - O'K^2 = (R - r)^2 - (d + r)^2 \Longrightarrow r = \frac{R^2 - d^2 - x^2}{2(R + d)}.$$

Thus,

$$\frac{r}{R - d} = \frac{\rho}{R + d} \Longrightarrow \frac{KE}{CM} = \frac{EL}{MG}.$$

Therefore, the lines CK and GL intersect at a point D' on AB. Indeed, if D' is the point of intersection of the lines CK and AB, we have

$$\frac{ED'}{MD'} = \frac{r}{R - d} \Longrightarrow \frac{MD' - x}{MD'} = \frac{R^2 - d^2 - x^2}{2(R^2 - d^2)} \Longrightarrow MD' = \frac{2x(R^2 - d^2)}{R^2 - d^2 + x^2}. \tag{16}$$

Similarly, if M'' is the point of intersection of the lines GL and AB, we find $MD'' = MD'$ and thus $D' = D''$. Finally, from (1) and (2) we get $D' = D$. Thst is, the lines CD and GL pass through the centers K, L of the circles ω_1, ω_2, respectively.

Also solved by Mihir Kaskhedikar, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

O575. Let $(L_n)_{n\ge 1}$ be the Lucas sequence, $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, for $n = 1, 2, 3, \ldots$. Prove that if $n = \frac{1}{4}(L_{6m+1} - 1)$ for some positive integer m, then

$$\prod_{k=1}^{n} \left[(4k-1)^4 + 64\right]$$

is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

We have

$$n^4 + 64 = \left((n-2)^2 + 4\right)\left((n+2)^2 + 4\right),$$

so

$$(4k-1)^4 + 64 = \left((4(k-1)+1)^2 + 4\right)\left((4k+1)^2 + 4\right).$$

It suffices to show that $5\left((4n+1)^2 + 4\right)$ is a perfect square, that is, $5(L_{6m+1}^2 + 4)$ is the square of a positive integer. But $L_{2t+1}^2 + 4 = F_{2t+1}^2$, where F_q is the q-th Fibonacci number, so $5(L_{6m+1}^2 + 4) = (5F_{6m+1})^2$ and we are done.

Second solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA

We note that $(4k-1)^4 + 64 = (16k^2 - 24k + 13)(16k^2 + 8k + 5)$. With $f_k := 16k^2 - 24k + 13$ and $g_k := 16k^2 + 8k + 5$, we get that $f_{k+1} = g_k$, where k is a positive integer. Then

$$\prod_{k=1}^{n} \left[(4k-1)^4 + 64\right] = (f_1 \cdot g_1) \cdot (f_2 \cdot g_2) \cdots (f_n \cdot g_n) = f_1 \cdot (g_1 \cdots g_{n-1})^2 \cdot g_n.$$

Therefore, in order for the product to be a perfect square, we need to show that $f_1 \cdot g_n$ is a perfect square when $n = \frac{1}{4}(L_{6m+1} - 1)$. The Lucas numbers are given by $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$. Therefore, we need to show that

$$f_1 \cdot g_n = 5(16n^2 + 8n + 5) = 5\left[(L_{6m+1} - 1)^2 + 2(L_{6m+1} - 1) + 5\right] = 5(L_{6m+1}^2 + 4)$$

is a perfect square, or equivalently that $L_{6m+1}^2 + 4 = 5\lambda_m^2$, where λ_m are positive integers. This is obvious, however, because of the fundamental identity $L_s^2 - 5F_s^2 = 4(-1)^s$ between the Lucas and Fibonacci numbers, the latter given by $F_s = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^s - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^s$, where s is a positive integer; the identity is easy to prove by direct substitution of the expressions for the general terms L_s, F_s. Because $6m+1$ are odd integers, we get $L_{2t+1}^2 + 4 = 5F_{2t+1}^2$ and thus $\lambda_m = F_{6m+1}$. Therefore, for $n = \frac{1}{4}(L_{6m+1} - 1)$ and $g_k = 16k^2 + 8k + 5$, where m, k are positive integers, we get

$$\prod_{k=1}^{n} \left[(4k-1)^4 + 64\right] = (5 \cdot F_{6m+1})^2 \cdot (g_1 \cdot g_2 \cdots g_{n-1})^2.$$

Also solved by UM6P MathClub, Mohammed VI Polytechnic University, Ben Guerir, Morocco; Kanav Talwar, Delhi Public School, Faridabad, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

O576. If m_a, m_b, m_c are the medians of a triangle with side-lengths a, b, c, prove that

$$m_a^3(bc - a^2) + m_b^3(ca - b^2) + m_c^3(ab - c^2) \geq 0.$$

Proposed by Marius Stănean, Zalău, România

Solutiuon by the author

Applying this inequality to the triangle with sides $\left(\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c\right)$ and medians $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$, the proposed inequality becomes

$$\sum_{cyc} a^3(m_b m_c - m_a^2) \geq 0,$$

or

$$4\sum_{cyc} a^3 m_b m_c \geq 2\sum_{cyc} a^3(b^2 + c^2) - (a^5 + b^5 + c^5).$$

Let G be the centroid of a $\triangle ABC$ and let G_a be the feet of the perpendiculars from G to the side BC. We have

$$\frac{2}{3}m_b = \sqrt{BG_a^2 + GG_a^2}, \qquad \frac{2}{3}m_c = \sqrt{CG_a^2 + GG_a^2}.$$

By Minkowski's Inequality, if G_a is between B and C then

$$\frac{2}{3}(m_b + m_c) \geq \sqrt{(BG_a + CG_a)^2 + (GG_a + GG_a)^2} = \sqrt{a^2 + \frac{16S^2}{9a^2}}$$

otherwise

$$\frac{2}{3}(m_b + m_c) \geq \sqrt{(BG_a - CG_a)^2 + (GG_a + GG_a)^2} = \sqrt{a^2 + \frac{16S^2}{9a^2}}.$$

Therefore

$$(m_b + m_c)^2 \geq \frac{9a^2}{4} + \frac{4S^2}{a^2},$$

or

$$8m_b m_c \geq 5a^2 - b^2 - c^2 + \frac{16S^2}{a^2},$$

or

$$8a^3 m_b m_c \geq 5a^5 - a^3(b^2 + c^2) + 16S^2 a.$$

It follows that

$$8\sum_{cyc} a^3 m_b m_c \geq 5(a^5 + b^5 + c^5) - \sum_{cyc} a^3(b^2 + c^2) + 16S^2(a + b + c),$$

so it suffices to prove that

$$7(a^5 + b^5 + c^5) + 16S^2(a + b + c) \geq 5\sum_{cyc} a^3(b^2 + c^2).$$

With Ravi's substitutions, i.e. $a = y + z$, $b = z + x$, $c = x + y$, this becomes

$$7(x+y)^5 + 7(y+z)^5 + 7(z+x)^5 + 32xyz(x+y+z)^2 \geq 5\sum_{cyc}(y+z)^3((x+y)^2 + (z+x)^2),$$

or

$$2[(5,0,0)] + 10[(4,1,0)] + 40[(3,2,0)] \geq 24[(3,1,1)] + 28[(2,2,1)],$$

which follows from Muirhead's Inequality,

$$[(5,0,0)] \geq [(4,1,0)] \geq [(3,2,0)] \geq [(3,1,1)] \geq [(2,2,1)].$$

Also solved by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.