Junior problems

J577. Let a and b be positive real numbers such that $a+b=1$. Prove that

$$\left(\frac{1}{a^2}-b-1\right)\left(\frac{1}{b^2}-a-1\right)\geq\frac{25}{4}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Theo Koupelis, Broward College, Pembroke Pines, FL, USA
Using the condition $a+b=1$ repeatedly we get

$$\left(\frac{1}{a^2}-b-1\right)\left(\frac{1}{b^2}-a-1\right)=\frac{b(1+a-a^2)}{a^2}\cdot\frac{a(1+b-b^2)}{b^2}=\frac{(ab+1)^2}{ab},$$

and therefore

$$\frac{(ab+1)^2}{ab}\geq\frac{25}{4}\iff(ab-4)\left(ab-\frac{1}{4}\right)\geq 0.$$

This is obvious because from AM-GM and $a+b=1$ we get $0<ab\leq\frac{1}{4}$. Equality occurs when $a=b=\frac{1}{2}$.

Also solved by Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Le Hoang Bao, Tien Giang, Vietnam; Arkady Alt, San Jose, CA, USA; G. C. Greubel, Newport News, VA,USA; Daniel Văcaru, Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, USA; Sundaresh H R, Shivamogga, India; Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Aaron Kim, The Bronx High School of Science, Bronx, NY, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Mihaly Bencze, Brașov, Romania; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India.

J578. Let a be a positive real number and let $f(x) = a^x + \frac{1}{a^x}$. Given that $f\left(\frac{2}{3}\right) = 1 + 2\sqrt{2}$, find $f\left(\frac{3}{2}\right)$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Since $(f(1/3))^2 = f(2/3) + 2 = (1+\sqrt{2})^2$, $f(1/3) = 1 + \sqrt{2}$. Therefore,

$$(f(1/2))^2 = f(1) + 2 = (f(1/3))^3 - 3f(1/3) + 2 = 7 + 5\sqrt{2} - 3(1+\sqrt{2}) + 2 = 6 + 2\sqrt{2},$$

so $f(1/2) = \sqrt{6+2\sqrt{2}}$. Thus, $f(3/2) = (f(1/2))^3 - 3f(1/2) = (3+2\sqrt{2})\sqrt{6+2\sqrt{2}}$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Aaron Kim, The Bronx High School of Science, Bronx, NY, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA,USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Henry Ricardo, Westchester Area Math Circle, NY, USA; HyunBin Yoo, South Korea; Sundaresh H R, Shivamogga, India; Arkady Alt, San Jose, CA, USA; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India; Jennifer Lee, GA, USA.

J579. Find all pairs (x, y) of integers such that

$$3x^2 + 10x + 5 = 9 \cdot 2^y.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Văcaru, Pitești, Romania

We have

$$3x^2 + 10x + 5 - 9 \cdot 2^y = 0$$

with $\Delta = 10^2 - 4 \cdot 3 \cdot (5 - 9 \cdot 2^y) = 100 - 60 + 4 \cdot 9 \cdot 2^y = 40 + 4 \cdot 9 \cdot 2^y = 4 \cdot (10 + 9 \cdot 2^y) = 4 \cdot 2 \cdot (5 + 9 \cdot 2^{y-1}) = k^2$, for $k \in \mathbb{Z}$. It follows that $5 + 9 \cdot 2^{y-1}$is even. It follows that $9 \cdot 2^{y-1}$is odd, and, consequently, $y = 1$.We obtain

$$3x^2+10x+5 = 18 \Leftrightarrow 3x^2+10x-13 = 0 \Leftrightarrow 3x^2-3x+13x-13 = 0 \Leftrightarrow 3x\,(x-1)+13\,(x-1) = 0 \Leftrightarrow (x-1)\,(13x+13) = 0,$$

and the integer solution is $x = 1$. We obtain just one pair $(1, 1)$ of integers such that $3x^2 + 10x + 5 = 9 \cdot 2^y$.

Also solved by Arkady Alt, San Jose, CA, USA; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Aaron Kim, The Bronx High School of Science, Bronx, NY, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA,USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Pranjal Jha, Academic Affiliations, Whitefield Global School, Karnataka, India; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; HyunBin Yoo, South Korea; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Sundaresh H R, Shivamogga, India; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Petrakis Emmanouil, 2nd High School, Agrinio, Greece; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India; Jennifer Lee, GA, USA.

J580. Solve in real numbers the equation

$$\sqrt[3]{x^2+4} = \sqrt{x^3-4}.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India

Let $y = \sqrt[3]{x^2+4}$. Clearly y is an increasing function. Also, it implies $x = \sqrt{y^3-4}$. So, $f(x) = f^{-1}(x)$ for an increasing function $f(x)$ (which we called as y). It's solution is therefore given by $\sqrt[3]{x^2+4} = x \implies x^2+4 = x^3$, which has $x = 2$ as the only real solution

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Polyahedra, Polk State College, USA; Arkady Alt, San Jose, CA, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Aaron Kim, The Bronx High School of Science, Bronx, NY, USA; Duy Quan Tran, University of Medicine and Pharmacy at Ho Chi Minh City, Ho Chi Minh City, Vietnam; G. C. Greubel, Newport News, VA,USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Daniel Văcaru, Pitești, Romania; Prodromos Fotiadis, Dramma, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India; Jennifer Lee, GA, USA.

J581. Let a, b, c, d be real numbers such that $ab + ac + ad + bc + bd + cd - abcd = 11$. Prove that

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq 100.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Polyahedra, Polk State College, USA

The given equation implies that

$$z = (a + i)(b + i)(c + i)(d + i) = -10 - (a + b + c + d - abc - bcd - cda - dab)i.$$

Therefore,

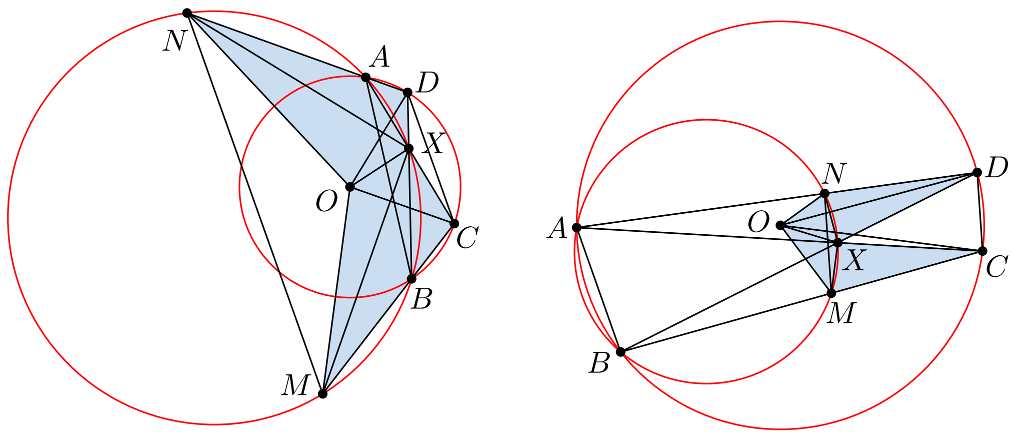
$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) = |z|^2 = 100 + (a + b + c + d - abc - bcd - cda - dab)^2 \geq 100.$$

Also solved by Arkady Alt, San Jose, CA, USA; Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India.

J582. Let $ABCD$ be a quadrilateral inscribed in a circle Γ with center O. Lines AC and BD intersect at X and Γ_1 is the circumcircle of triangle XAB. Lines AD and BC intersect Γ_1 at N and M, respectively. We know that the area of the concave polygon $ONDXCM$ is $DC \cdot AB$. Find the measure of $\angle AXB$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Polyahedra, Polk State College, USA



Let $\alpha = \angle AXB$, $\beta = \angle XAD$, and R, R_1 be the radii of Γ, Γ_1, respectively. We have $AB = 2R_1 \sin\alpha$ and $CD = 2R\sin\beta$. Since $\angle MNX = \angle XBC = \angle XAD = \angle NMX$, $NX = 2R_1 \sin\beta = MX$. Also, $\angle XND = \angle XBA = 90° - \angle ODA$, thus $NX \perp OD$. Likewise, $MX \perp OC$. Therefore,

$$4RR_1 \sin\alpha \sin\beta = CD \cdot AB = [ONDXCM] = [ONDX] + [OXCM]$$
$$= \frac{1}{2}(NX \cdot OD + MX \cdot OC) = 2RR_1 \sin\beta,$$

so $\sin\alpha = 1/2$. Hence, $\alpha = 150°$ or $30°$.

S577. Let k, m, n be integers such that $k + m + n = 1$. Prove that

$$(k^2 + m^2 + n^2 + 7)^2 + (kmn - 4)^2$$

is not the square of an odd integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Theo Koupelis, Broward College, Pembroke Pines, FL, USA
From the condition $k + m + n = 1$ we get that the integers k, m, n are either all odd or two of them are even and one is odd. The latter, however, makes the quantity $S := (k^2+m^2+n^2+7)^2+(kmn-4)^2$ even. Therefore, k, m, n are odd integers. Then

$$\begin{aligned} S &= [(1-n)^2 - 2km + n^2 + 7)^2 + (kmn-4)^2 \\ &= 4[4 + n(n-1) - km]^2 + (kmn-4)^2 = 4(2a-1)^2 + (2b-3)^2, \end{aligned}$$

where a, b are integers and $2a - 1 = [4 + n(n-1) - (km-1)] - 1$ and $2b - 3 = (kmn - 1) - 3$. Thus, $S = 4[4a(a-1) + (b-1)(b-2)] + 5 \equiv 5 \pmod 8 \not\equiv 1 \pmod 8$, and therefore S is not the square of an odd integer.

Also solved by Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Le Hoang Bao, Tien Giang, Vietnam; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Sundaresh H R, Shivamogga, India; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India.

S578. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{1}{1+a^2+b^2}+\frac{1}{1+b^2+c^2}+\frac{1}{1+c^2+a^2}\le\frac{9}{5}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

We rewrite the inequality as

$$\frac{a^2+b^2}{1+a^2+b^2}+\frac{b^2+c^2}{1+b^2+c^2}+\frac{c^2+a^2}{1+c^2+a^2}\ge\frac{6}{5}.$$

The Cauchy-Schwarz inequality gives us

$$\begin{aligned}\sum_{\text{cyc}}\frac{a^2+b^2}{1+a^2+b^2}&=\sum_{\text{cyc}}\frac{(a^2+b^2)^2}{(a^2+b^2)(1+a^2+b^2)}\\&\ge\frac{4(a^2+b^2+c^2)^2}{2(a^2+b^2+c^2)+(a^2+b^2)^2+(b^2+c^2)^2+(c^2+a^2)^2}\\&=\frac{2(a^2+b^2+c^2)^2}{a^2+b^2+c^2+a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}.\end{aligned}$$

It suffices to show that

$$\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}\ge\frac{3}{5}.$$

This is equivalent to

$$5(a^2+b^2+c^2)^2\ge3(a^2+b^2+c^2+a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)$$

The homogeneous form of this inequality is

$$2(a^4+b^4+c^4)+7(a^2b^2+b^2c^2+c^2a^2)\ge3(a^2+b^2+c^2)(ab+bc+ca).$$

After some simple computations this inequality reduces to

$$\sum_{\text{cyc}}(b^2-bc+c^2)(b-c)^2+\frac{3}{2}\sum_{\text{cyc}}a^2(b-c)^2\ge0$$

which is clearly true and we are done.

Also solved by Arkady Alt, San Jose, CA, USA; Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Mihaly Bencze, Brașov, Romania; Nandan Sai Dasireddy, Hyderabad, India; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S579. Let a, b, c be the side lengths of a triangle of area S, and let α, β, γ be positive real numbers such that

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}+\frac{1}{\gamma+1}=2.$$

Prove that

$$\alpha b^2c^2+\beta c^2a^2+\gamma a^2b^2 \geq 8S^2.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Let $a^2=y+z$, $b^2=z+x$, $c^2=x+y$ where $x,y,z\in\mathbb{R}$ and only one of these numbers can be negative. Also, there are positive numbers u,v,w such that $\alpha=\frac{u}{v+w}$, $\beta=\frac{v}{w+u}$, $\gamma=\frac{w}{u+v}$. The inequality becomes

$$\alpha(z+x)(x+y)+\beta(x+y)(y+z)+\gamma(y+z)(z+x)\geq 2(xy+yz+zx),$$

because

$$16S^2=2\sum_{cyc}a^2b^2-\sum_{cyc}a^4=2\sum_{cyc}(y+z)(z+x)-\sum_{cyc}(y+z)^2=4(xy+yz+zx).$$

We need to prove that

$$\alpha x^2+\beta y^2+\gamma z^2+(\alpha+\beta+\gamma-2)(xy+yz+zx)\geq 0.$$

If $\alpha+\beta+\gamma\geq 2$ the inequality is true so it remains to study the case $\frac{3}{2}\leq\alpha+\beta+\gamma<2$.

Without loss of generality, we may assume that $\alpha\geq\beta\geq\gamma$ which means $u\geq v\geq w$. Then, we evaluate the discriminant of the quadratic equation in x, that is

$$\begin{aligned}\Delta_x=&(\alpha+\beta+\gamma-2)^2(y+z)^2-4\alpha(\beta y^2+\gamma z^2+(\alpha+\beta+\gamma-2)yz)\\ =&\left[(\alpha+\beta+\gamma-2)^2-4\alpha\beta\right]y^2-2(\alpha+\beta+\gamma-2)(\alpha-\beta-\gamma+2)yz\\ &+\left[(\alpha+\beta+\gamma-2)^2-4\alpha\gamma\right]z^2.\end{aligned}$$

We have

$$2\sqrt{\alpha\beta}+\alpha+\beta+\gamma\geq 2\sqrt{\alpha\gamma}+\alpha+\beta+\gamma\geq 2\sqrt{\beta\gamma}+\alpha+\beta+\gamma,$$

and we will prove that

$$2\sqrt{\beta\gamma}+\alpha+\beta+\gamma>2,$$

that is

$$\left(\sqrt{\beta}+\sqrt{\gamma}\right)^2+\alpha>2,$$

or

$$\left(\sqrt{\frac{v}{w+u}}+\sqrt{\frac{w}{u+v}}\right)^2+\frac{u}{v+w}>2,$$

By Hölder's Inequality, we have

$$\begin{aligned}\sqrt{\frac{v}{w+u}}+\sqrt{\frac{w}{u+v}}&\geq\sqrt{\frac{(v+w)^3}{v^2(w+u)+w^2(u+v)}}=\sqrt{\frac{(v+w)^3}{u(v^2+w^2+\frac{2vw(v+w)}{2u})}}\\ &\geq\sqrt{\frac{(v+w)^3}{u(v^2+w^2+\frac{2vw(v+w)}{v+w})}}=\sqrt{\frac{v+w}{u}}.\end{aligned}$$

Since

$$\frac{v+w}{u}+\frac{u}{v+w}\geq 2,$$

the inequality is proven. Now, it remains to prove that the discriminant of the quadratic equation $\Delta_x=0$ is ≤ 0, i.e.

$$(\alpha+\beta+\gamma-2)^2(\alpha-\beta-\gamma+2)^2-\left[(\alpha+\beta+\gamma-2)^2-4\alpha\beta\right]\left[(\alpha+\beta+\gamma-2)^2-4\alpha\gamma\right]\leq 0,$$

or

$$-(\alpha+\beta+\gamma-2)^2+2\alpha\beta\gamma\geq 0,$$

or

$$\alpha+\beta+\gamma+\sqrt{2\alpha\beta\gamma}\geq 2,$$

$$\frac{u}{v+w}+\frac{v}{w+u}+\frac{w}{u+v}+\sqrt{\frac{2uvw}{(u+v)(v+w)(w+u)}}\geq 2.$$

If we denote $P=(u+v)(v+w)(w+u)$ this can be rewrite as

$$\frac{u^3+v^3+w^3-3uvw}{P}\geq [(u+v)(v+w)(w+u)-8uvw]\left(\frac{1}{2P}+\frac{1}{2P+2\sqrt{8Puvw}}\right)$$

so it suffice to prove that

$$u^3+v^3+w^3-3uvw\geq (u+v)(v+w)(w+u)-8uvw$$

which is Schur's Inequality.

Also solved by Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S580. Prove that there are no positive integers a, b, c such that

$$a^2b^2 + b^2c^2 + c^2a^2 - 3abc = 2022.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Since $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$ it follows that:

$$2022 \geq abc(a + b + c - 3).$$

Thus, if $\min(a, b, c) \geq 5$ it follows that $2022 \geq 625(a + b + c - 3)$. That is;

$$a + b + c \leq 6.$$

This is a contradiction. It follows that $\min(a, b, c) \leq 5$. Without loss of generality, assume that $c = \min(a, b, c)$. It follows that $c \leq 4$ Note that if c is even then ab must be even. Then, the left side is divisible by 4 while the right side is not divisible by 4. On the other hand, if c is divisible by 3 then ab is divisible by 3. Hence, the left side is divisible by 9 while the right side is not divisible by 9.

Thus $c = 1$ is the only possibility. That is,

$$a^2b^2 + a^2 + b^2 - 3ab = 2022,$$

hence;

$$(a - b)^2 + a^2b^2 - ab = 2022.$$

Yielding to

$$(2a - 2b)^2 + (2ab - 1)^2 = 8089.$$

Since 8089 is a prime number of the form $4k + 1$ it can uniquely be written as the sum of two non-zero squares. That is, $8089 = 60^2 + 67^2$ implying;

$$(|2a - 2b|, |2ab - 1|) = (60, 67).$$

We can then assume that $a > b$. Therefore, $2a - 2b = 60, 2ab - 1 = 67$. It follows that $a - b = 30, ab = 34$. But then, $(a + b)^2 = (a - b)^2 + 4ab = 1036$ which is not a perfect square. Thus the equation has no solution in positive integers.

Note: It is interesting to work on the equations of the form

$$a^2b^2 + b^2c^2 + c^2a^2 - nabc = 2022,$$

where n and a, b, c are positive integers to determine $\min(a, b, c)$.

Also solved by Petrakis Emmanouil, 2nd High School, Agrinio, Greece; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Prodromos Fotiadis, Dramma, Greece.

S581. Let p be a prime such that $p \equiv 1 \pmod 4$. Prove that there is a positive integer a such that

$$a^2 + 2^a \equiv 0 \pmod{2p}$$

Proposed by Nguyen Tanh Chuong, Hanoi, Vietnam

Solution by the author

First, since $p = 4k+1$, we have

$$4k \equiv -1 \pmod p, 4k-1 \equiv -2 \pmod p, 4k-2 \equiv -3 \pmod p, ..., 2k+1 \equiv -2k \pmod p.$$

Hence,

$$(2k+1)(2k+2)...(4k-1)(4k) \equiv (2k)! \pmod p$$

Multiplying the two sides by $(2k)!$ we obtain

$$(4k)! \equiv [(2k)!]^2 \pmod p \tag{1}$$

By Wilson's theorem,

$$(4k)! + 1 \equiv 0 \pmod p$$

Therefore,

$$[(2k)!]^2 + 1 \equiv 0 \pmod p$$

Now let $x = 4k \cdot (2k)!$. Since $2^{(2k)!}$ and p are coprime, Fermat's little theorem implies that

$$2^x - 1 = [2^{(2k)!}]^{4k} - 1 \equiv 0 \pmod p \tag{2}$$

From (1) and (2) we have

$$x^2 + 2^x = [4k \cdot (2k)!]^2 + [2^{(2k)!}]^{4k} = 16k^2[(2k)!]^2 + [2^{(2k)!}]^{4k}$$

$$= [[(2k)!]^2 + 1]16k^2 - (16k^2 - 1) + [2^{(2k)!}]^{4k} - 1$$

$$[[(2k)!]^2 + 1]16k^2 - (4k-1)(4k+1) + [2^{(2k)!}]^{4k} - 1 \equiv 0 \pmod p$$

It is obvious to see that $x^2 + 2^x \equiv 0 \pmod 2$, $x^2 + 2^x \equiv 0 \pmod{2p}$. This completes our proof. In the special case where $p = 2017$, the problem is solved by choosing $k = 504$ and $x = 4k \cdot (2k)! = 2016 \cdot 1008!$

Also solved by Anish Ray, University of Muenster, Germany; Corneliu Mănescu-Avram, Ploieşti, Romania; Machiel van Frankenhuijsen, Utah Valley University, Orem, UT, USA; Prodromos Fotiadis, Dramma, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S582. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$, $a_2 = 2$, and

$$a_{n+2} = (n+2)a_{n+1} - na_n + 2n + 1,$$

for all $n \geq 1$. Find all n for which there exists $m \in \mathbb{N}$ such that $a_n = m!$.

Proposed by Prodromos Fotiadis, Dramma, Greece

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Rewrite the recurrence relation for a_n as

$$a_{n+2} - (n+1)a_{n+1} = a_{n+1} - na_n + 2n + 1$$

and then define $b_n = a_{n+1} - na_n$. It follows that $b_1 = 1$ and

$$b_{n+1} = b_n + 2n + 1$$

for $n \geq 1$. Thus, $b_n = n^2$ for all n, and $a_{n+1} = na_n + n^2$ for $n \geq 2$. With the obsrevation $a_1 = 1!$ and $a_2 = 2!$, let $a_n = n! + \delta_n$. Then

$$a_{n+1} = n \cdot n! + n\delta_n + n^2 = (n+1)! + n\delta_n + n^2 - n!,$$

so the sequence $\{\delta_n\}$ satisfies $\delta_1 = \delta_2 = 0$ and

$$\delta_{n+1} = n\delta + n^2 - n!$$

for $n \geq 2$. From here, we find $\delta_3 = 2$, $\delta_4 = 9$, $\delta_5 = 28$, $\delta_6 = 45$, and $\delta_7 = -414$. Thus,

$$a_3 = 3! + 2 = 8, a_4 = 4! + 9 = 33, a_5 = 5! + 28 = 148, \text{ and } a_6 = 6! + 45 = 765,$$

none of which are factorials. For $n \geq 7$, $n^2 < n!$, so $\delta_n < 0$ implies $\delta_{n+1} < 0$; therefore, $\delta_n < 0$ for $n \geq 7$. Additionally, note $\delta_7 > -6 \cdot 6!$. Suppose $n > 7$ and $\delta_n > -(n-1) \cdot (n-1)!$. Then,

$$\delta_{n+1} > -(n-1) \cdot n! + n^2 - n! = -n \cdot n! + n^2 > -n \cdot n!.$$

It follows that $\delta_n > -(n-1) \cdot (n-1)!$ for all $n \geq 7$. Lastly, for $n \geq 7$,

$$(n-1)! = n! - (n-1) \cdot (n-1)! < a_n < n!,$$

so a_n is not a factorial for any $n \geq 7$. The only n for which there exists $m \in \mathbb{N}$ such that $a_n = m!$ are $n = 1$ and $n = 2$.

Also solved by Petrakis Emmanouil, 2nd High School, Agrinio, Greece; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; HyunBin Yoo, South Korea; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India.

Undergraduate problems

U577. For integers n, evaluate

$$\lim_{n\to\infty}\left\{\sqrt{4n^2+3n+2}\right\}$$

where $\{a\}$ denotes the fractional part of a.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

We have

$$2n < \sqrt{4n^2+3n+2} < 2n+1, \quad \forall n > 1.$$

Therefore

$$\lfloor\sqrt{4n^2+3n+2}\rfloor = 2n.$$

This gives us

$$\left\{\sqrt{4n^2+3n+2}\right\} = \sqrt{4n^2+3n+2} - 2n = \frac{3n+2}{\sqrt{4n^2+3n+2}+2n}.$$

From here we get

$$\lim_{n\to\infty}\left\{\sqrt{4n^2+3n+2}\right\} = \lim_{n\to\infty}\frac{3n+2}{\sqrt{4n^2+3n+2}+2n} = \lim_{n\to\infty}\frac{3+\frac{2}{n}}{\sqrt{4+\frac{3}{n}+\frac{2}{n^2}}+2} = \frac{3}{4}.$$

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Aaron Kim, The Bronx High School of Science, Bronx, NY, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Daniel Văcaru, Pitești, Romania; G. C. Greubel, Newport News, VA,USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; HyunBin Yoo, South Korea; Sundaresh H R, Shivamogga, India; Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; P.V.Swaminathan, Vel's Vidhyalaya Group of Schools, Kovilpatti, Tamilnadu, India.

U578. Compute

$$\iint_D \frac{y\sqrt{x^2+y^2}}{x^2} e^{\sqrt{x^2+y^2}/2} dx\, dy$$

$D = \{(x-1)^2 + y^2 \le 1, y \ge 0\}$

Proposed by Paolo Perfetti, Roma, Italy

Solution by the author

The integral is

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \frac{y\sqrt{x^2+y^2}}{x^2} e^{\sqrt{x^2+y^2}/2} dx \doteq \lim_{\varepsilon\to 0^+} f(\varepsilon)$$

Of course $f(\varepsilon) > 0$ and $y^2 \le 1-(x-1)^2 = x(2-x) \le 2x$ thus

$$\begin{aligned}
&\int_\varepsilon^1 dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \frac{y\sqrt{x^2+y^2}}{x^2} e^{\sqrt{x^2+y^2}/2} dx \le \int_\varepsilon^1 dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \frac{\sqrt{2x}\sqrt{x^2+2x}}{x^2} e^{\sqrt{4+1}/2} dx \le \\
&\le \int_\varepsilon^1 dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \frac{\sqrt{2x}\sqrt{2x+2x}}{x^2} e^{\sqrt{4+1}/2} dx = e^{\sqrt{5}/2} \int_\varepsilon^1 dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \frac{2}{x} = \\
&= 2e^{\sqrt{5}/2} \int_\varepsilon^1 \Big(\ln(1+\sqrt{1-y^2}) - \ln(1-\sqrt{1-y^2})\Big) dy \le \\
&\le 2e^{\sqrt{5}/2} \int_\varepsilon^1 \left(\ln 2 - \ln \frac{y^2}{1+\sqrt{1-y^2}})\right) dy \le 2e^{\sqrt{5}/2} \int_\varepsilon^1 \left(\ln 2 - \ln \frac{y^2}{2}\right) dy = \\
&= (1-\varepsilon) 4e^{\sqrt{5}/2} \ln 2 - 4e^{\sqrt{5}/2} (x\ln x - x)\Big|_\varepsilon^1 = \\
&= (1-\varepsilon) 4e^{\sqrt{5}/2} \ln 2 + 4e^{\sqrt{5}/2} \left(1 + \varepsilon\ln\varepsilon + \varepsilon\right) = \\
&= 4e^{\sqrt{5}/2} \left(1 + \ln 2 + \varepsilon(1-\ln 2) + \varepsilon\ln\varepsilon\right) \le 4e^{\sqrt{5}/2} \left(1 + \ln 2 + \varepsilon(1-\ln 2)\right) \le \\
&\le 4e^{\sqrt{5}/2} \left(1 + \ln 2 + (1-\ln 2)\right) = 8e^{\sqrt{5}/2}
\end{aligned}$$

Since $f(\varepsilon)$ increases with ε, the improper integral converges.

To compute the integral let's introduce the following change of coordinates

$$x = 2\cos\alpha\cos\varphi, \quad y = 2\cos\alpha\sin\varphi, \qquad 0 \le \alpha \le \frac{\pi}{2}, 0 \le \varphi \le \alpha$$

The determinant of the Jacobian is $4\cos\alpha\sin\alpha$ and the integral becomes

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} d\alpha \int_0^\alpha \frac{2\cos\alpha\sin\varphi}{4\cos^2\alpha\cos^2\varphi} 2\cos\alpha\ 4\cos\alpha\sin\alpha e^{\cos\alpha} d\varphi = \\
&= 4\int_0^{\frac{\pi}{2}} \cos\alpha\sin\alpha e^{\cos\alpha} d\alpha \int_0^\alpha \frac{\sin\varphi}{\cos^2\varphi} d\varphi = 4\int_0^{\frac{\pi}{2}} \cos\alpha\sin\alpha e^{\cos\alpha} \left(\frac{1}{\cos\alpha} - 1\right) d\alpha = \\
&= 4\int_0^{\frac{\pi}{2}} \sin\alpha e^{\cos\alpha} d\alpha - 4\int_0^{\frac{\pi}{2}} \cos\alpha\sin\alpha e^{\cos\alpha} d\alpha = \\
&= -4e^{\cos\alpha}\Big|_0^{\frac{\pi}{2}} - 4\cdot 2\sin^2\frac{\alpha}{2} e^{\cos\alpha}\Big|_0^{\frac{\pi}{2}} = 4(e-1) - 8\frac{1}{2} = 4(e-2)
\end{aligned}$$

Also solved by Morgan Orr, Ashley Herbig, Eli Lutz, Christopher Newport University Newport News, VA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U579. Let x_0 be an integer and let $P(x)$ be a non-zero polynomial with integer coefficients. If for all $n \geq 1$:

$$x_n = x_{n-1}^n + P(n).$$

Prove that for any positive integer $m \geq 2$ there is a positive integer T such that

$$x_{n+T} \equiv x_n \pmod{m}$$

for all sufficiently large n.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

We shall prove that there is a positive integer T such that for all sufficiently large n;

$$x_{n+T} \equiv x_n (\mathrm{mod} m).$$

Let D_m be set of positive divisors of m and let L_m be the least common multiple of m and numbers $\varphi(d)$ for all $d \in D_m$. Setting $s = \lfloor \log_2 m \rfloor + 1$ and consider x_{s+kL_m} mad m for $k = 0, 1, \ldots..$ Take x_{t+kL_m}, x_t that are congruent mod m, moreover, choose $k \leq m, t \leq s + m = m + \lfloor \log_2 m \rfloor + 1$. We then claim that for each $n \geq t$; the number m divides $x_{n+kL_m} - x_n$. We can then prove that the length of the least period doesn't exceed mL_m and hence its pre-periodic part is of the length less than or equal to $m + \lfloor \log_2 m \rfloor + 1$.

We now prove $x_{n+kL_m} - x_n$ is divisible by m through induction on n. The base is $n = t$ which is obviously true. Assume that it holds for all integers less than or equal to $n - 1$. That is, $x_{n-1+kL_m} \equiv x_{n-1} \equiv r (\mathrm{mod} m)$ where $0 \leq r \leq m - 1$. Note that m divides $P(n + kL_m) - P(n)$ since m divides L_m. Therefore,

$$x_{n+kL_m} = x_{n-1+kL_m}^{n+kL_m} + P(n + kL_m).$$

Hence,

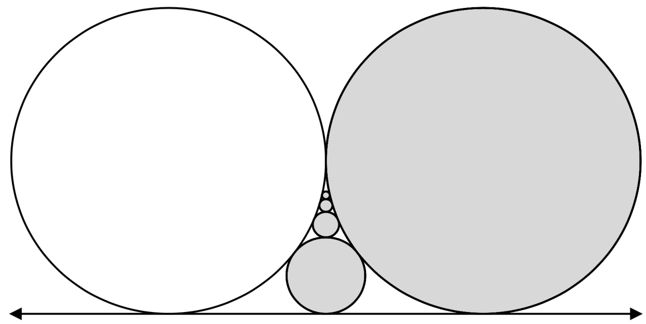
$$x_{n+kL_m} - x_n \equiv x_{n-1+kL_m}^{n+kL_m} - x_{n-1}^n \equiv r^n (r^{nL_m} - 1) (\mathrm{mod} m).$$

Assume that $r > 0$. Let $r = ap_1^{\alpha_1} \cdots p_N^{\alpha_N}, m = bp_1^{\beta_1} \cdots p_N^{\beta_N}$ such that $\alpha_i, \beta_i \geq 1$ and

$$\gcd(ab, p_1 \ldots p_N) = 1, \gcd(a, b) = 1.$$

Since $n \geq t \geq s > \log_2 m$ then $p_i^{n\beta_i} \geq p_i^n \geq 2^n > m \geq p_i^{\alpha_i}$. Thus, r^n is divisible by $p_1^{\beta_1} \ldots p_N^{\beta_N}$. It remains to prove that $r^{nL_m} - 1$ is divisible by b. That is, since $\varphi(b)$ divides L_m then $r^{nL_m} - 1$ is divisible by b. We are done.

U580. Two mutually tangent circles of radius 1 lie on a common tangent line as shown. The circle on the left is colored white and the circle on the right is colored gray. A third, smaller circle, is tangent to both of the larger circles and the line, and it is also colored gray. An infinite sequence of gray circles are inserted as follows: Each subsequent circle is tangent to the preceding circle, to the largest gray circle, and to the white circle. What is the total area bounded by the gray circles?



Proposed by Brian Bradie, Christopher Newport University, VA, USA

Solution by the author

Let r_j denote the radius of the jth gray circle in the sequence. We are given that $r_1 = 1$. Our first step will be to show that

$$r_j = \frac{1}{2(j-1)j} \qquad (1)$$

for all $j \geq 2$. From the diagram in Figure 1, it follows that

$$1^2 + (1 - r_2)^2 = (1 + r_2)^2, \qquad \text{or} \qquad r_2 = \frac{1}{4} = \frac{1}{2(2-1)2}.$$

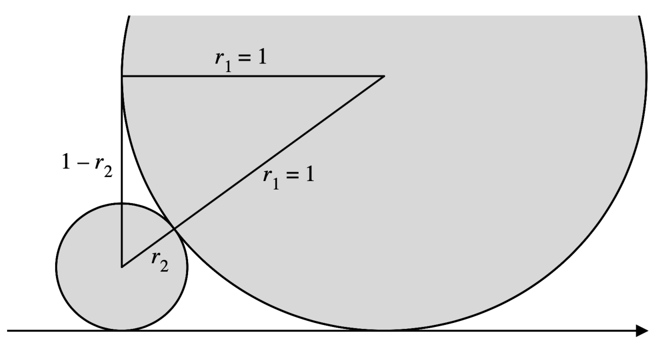


Figure 1

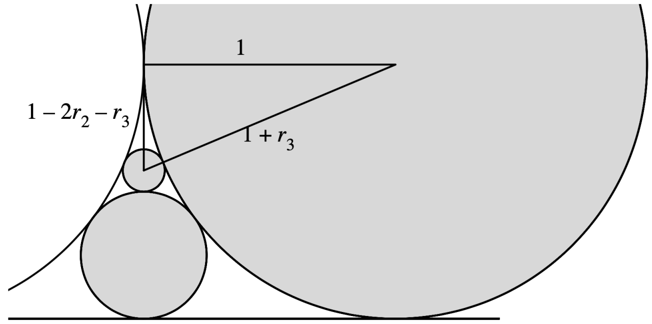


Figure 2

Next, consider the diagram in Figure 2, which shows the first three gray circles in the sequence. With $r_2 = \frac{1}{4}$, the length of the vertical leg of the indicated right triangle is $\frac{1}{2} - r_3$. Then

$$1^2 + \left(\frac{1}{2} - r_3\right)^2 = (1 + r_3)^2 \qquad \text{or} \qquad r_3 = \frac{1}{12} = \frac{1}{2(3-1)3}.$$

Now, suppose that (1) holds for $n = 2, 3, 4, \ldots, j$. Then, using Figure 2 as a template, it follows that

$$1^2 + \left(1 - 2\sum_{n=2}^{j} r_n - r_{j+1}\right)^2 = (1 + r_{j+1})^2. \tag{2}$$

Note, by the induction hypothesis,

$$2\sum_{n=2}^{j} r_n = 2\sum_{n=2}^{j} \frac{1}{2(n-1)n} = 1 - \frac{1}{j}.$$

Substituting this result into the left-hand side of (2) yields

$$1^2 + \left(\frac{1}{j} - r_{j+1}\right)^2 = (1 + r_{j+1})^2,$$

or

$$\frac{1}{j^2} - \frac{2}{j} r_{j+1} = 2r_{j+1}.$$

Thus,

$$r_{j+1} = \frac{1}{2j(j+1)},$$

as desired.

With $r_1 = 1$ and $r_j = \frac{1}{2(j-1)j}$ for all $j \geq 2$, the total area bounded by the gray circles is

$$\pi(1)^2 + \sum_{j=2}^{\infty} \pi r_j^2 = \pi + \frac{\pi}{4} \sum_{j=2}^{\infty} \frac{1}{(j-1)^2 j^2}.$$

For the summation, note

$$\begin{aligned} \sum_{j=2}^{\infty} \frac{1}{(j-1)^2 j^2} &= \sum_{j=2}^{\infty} \frac{1}{(j-1)^2} - 2\sum_{j=2}^{\infty} \frac{1}{(j-1)j} + \sum_{j=2}^{\infty} \frac{1}{j^2} \\ &= \zeta(2) - 2(1) + \zeta(2) - 1 \\ &= 2\zeta(2) - 3 = \frac{\pi^2}{3} - 3. \end{aligned}$$

Therefore, the total area bounded by the gray circles is

$$\pi + \frac{\pi}{4}\left(\frac{\pi^2}{3} - 3\right) = \frac{\pi^3}{12} + \frac{\pi}{4}.$$

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; HyunBin Yoo, South Korea; Sundaresh H R, Shivamogga, India.

U581. Let $r > s$ be two relatively prime positive integers and $P(x), Q(x)$ be two distinct nonconstant polynomials with complex coefficients such that

$$P(x)^r - P(x)^s = Q(x)^r - Q(x)^s.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let ω, ξ be primitive r and s th root of unity, respectively. Rewrite the original equation as follows:

$$\begin{aligned}(P(x) - \omega Q(x))(P(x) - \omega^2 Q(x)) \ldots .(P(x) - \omega^r Q(x)) \\ = (P(x) - \xi Q(x))(P(x) - \xi^2 Q(x)) \ldots (P(x) - \xi^s Q(x)).\end{aligned}$$

Let $D(x)$ be the greatest common divisor of $P(x), Q(x)$ writing $P(x) = D(x)S(x), Q(x) = D(x)R(x)$ where $R(x), S(x)$ are co-prime polynomials. By the degree condition, it follows that $\deg R(x) = \deg S(x)$. Hence

$$\begin{aligned}D(x)^{r-s}(S(x) - \omega R(x))(S(x) - \omega^2 R(x)) \ldots (S(x) - \omega^r S(x)) \\ = (S(x) - \xi R(x))(S(x) - \xi^2 R(x)) \ldots (S(x) - \xi^s R(x)).\end{aligned}$$

If $s = 1$ it follows that the right hand side of the above equation is equal to 1. Thus $D(x) = 1$ and $S(x) - \omega R(x), S(x) - \omega^2 R(x), \ldots$ are all constant. If $r > 2$ then, the polynomials $S(x) - \omega R(x)$ and $S(x) - \omega^2 R(x)$ can not be both constant. Whence $r = 2$.

Assume now, $s > 1$. Clearly at most one of the polynomials $S(x) - \xi R(x), \ldots, S(x) - \xi^s R(x)$ can be constant. Otherwise $R(x), S(x)$ must be constant. Hence right side becomes constant. Therefore $D(x)$ is also a constant polynomial. It follows that $P(x), Q(x)$ are constant. This is impossible,

Take $j \in \{1, 2, \ldots, r-1\}$ such that $S(x) - \omega^j R(x)$ is not a constant polynomial. Take z such that $S(z) = \omega^j R(z)$ then for some $k \in \{1, 2, \ldots, s-1\}$ we must have $S(z) = \xi^k R(z)$. Since $\gcd(r, s) = 1$ it follows that $S(z) = R(z) = 0$. This contradicts the fact that $S(x), R(x)$ are coprime. Hence, $(r, s) = (2, 1)$.

Also solved by Ali Nazarboland, AE High School, Iran.

U582. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and let $u : \mathbb{R}^n \to \mathbb{R}$, $u(x) = f(\|x\|)$. Denote by $\nabla^2(u)$ the Hessian of u. Evaluate in terms of f

$$\det(\nabla^2(u)).$$

Proposed by Michele Caselli, ETH Zurich, Switzerland

First solution by Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, USA

Let

$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \qquad \Longrightarrow \qquad \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Then we compute

$$\frac{\partial}{\partial x_i}(\|x\|) = \frac{x_i}{\|x\|}, \qquad \frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} f(\|x\|) = f'(\|x\|)\,\frac{x_i}{\|x\|}.$$

Here and elsewhere we must assume $x \neq 0$ for derivatives to be defined. Further, let

$$p = \frac{f'(\|x\|)}{\|x\|}, \qquad q = \frac{f''(\|x\|)}{\|x\|^2} - \frac{f'(\|x\|)}{\|x\|^3}.$$

The second order partials are computed to be

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} &= f''(\|x\|)\,\frac{x_i^2}{\|x\|^2} + f'(\|x\|)\,\frac{\|x\| - \frac{x_i^2}{\|x\|}}{\|x\|^2} \\ &= \frac{f'(\|x\|)}{\|x\|} + \left[\frac{f''(\|x\|)}{\|x\|^2} - \frac{f'(\|x\|)}{\|x\|^3}\right] x_i^2 \\ &= p + q x_i^2,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x_j \partial x_i} = \frac{\partial^2 u}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_j}\left(f'(\|x\|)\,\frac{x_i}{\|x\|}\right) \\ &= f''(\|x\|)\,\frac{x_i x_j}{\|x\|^2} + f'(\|x\|)\,\frac{0 - x_i \frac{x_j}{\|x\|}}{\|x\|^2} \\ &= \left[\frac{f''(\|x\|)}{\|x\|^2} - \frac{f'(\|x\|)}{\|x\|^3}\right] x_i x_j = q\, x_i x_j,\end{aligned}$$

for $i \neq j$. Then

$$\det(\nabla^2(u)) = \begin{vmatrix} p + q x_1^2 & q x_1 x_2 & q x_1 x_3 & \cdots & q x_1 x_n \\ q x_2 x_1 & p + q x_2^2 & q x_2 x_3 & \cdots & q x_2 x_n \\ q x_3 x_1 & q x_3 x_2 & p + q x_3^2 & \cdots & q x_3 x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q x_n x_1 & q x_n x_2 & q x_n x_3 & \cdots & p + q x_n^2 \end{vmatrix}$$

Since $x \neq 0$, we may assume, for example, that $x_1 \neq 0$. Subtracting $\frac{x_i}{x_1}$times Row 1 from Row i for each $i = 2, 3, \dots, n$ does not change the value of the determinant, so we have

$$\det(\nabla^2(u)) = \begin{vmatrix} p + q x_1^2 & q x_1 x_2 & q x_1 x_3 & \cdots & q x_1 x_n \\ -\frac{x_2}{x_1} p & p & 0 & \cdots & 0 \\ -\frac{x_3}{x_1} p & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{x_n}{x_1} p & 0 & 0 & \cdots & p \end{vmatrix}$$

Factoring out p from Rows 2 through n and expanding the determinant along Column 1 we have

$$p^{n-1}\left\{(p+qx_1^2)+\frac{x_2}{x_1}\begin{vmatrix} qx_1x_2 & qx_1x_3 & qx_1x_4 & \dots & qx_1x_n \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}\right.$$

$$\left.-\frac{x_3}{x_1}\begin{vmatrix} qx_1x_2 & qx_1x_3 & qx_1x_4 & \dots & qx_1x_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}+\frac{x_4}{x_1}\begin{vmatrix} qx_1x_2 & qx_1x_3 & qx_1x_4 & \dots & qx_1x_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}-\cdots\right\}$$

Let Δ_i be the determinant inside the braces that multiplies $\pm\frac{x_{i+1}}{x_1}$, $i=1,2,\dots,n-1$. Such determinant has the ith column consisting of all zeroes except the entry qx_1x_{i+1}in the top position. Expanding each of Δ_i along Column i we get $\Delta_i=(-1)^{i+1}qx_1x_{i+1}$. Consequently,

$$\begin{aligned}\det(\nabla^2(u)) &= p^{n-1}\left\{p+qx_1^2+\frac{x_2}{x_1}qx_1x_2-\frac{x_3}{x_1}(-qx_1x_3)+\frac{x_4}{x_1}qx_1x_4-\cdots\right\} \\ &= p^{n-1}\left\{p+q\sum_{i=1}^n x_i^2\right\} = p^{n-1}\{p+q\|x\|^2\} \\ &= p^{n-1}\left\{\frac{f'(\|x\|)}{\|x\|}+f''(\|x\|)-\frac{f'(\|x\|)}{\|x\|}\right\} \\ &= \left[\frac{f'(\|x\|)}{\|x\|}\right]^{n-1} f''(\|x\|),\end{aligned}$$

which is the value of the determinant of the Hessian of u in terms of f at a point $x\in\mathbb{R}^n$.

Second solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix}^T \in \mathbb{R}^n.$$

Then

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= f'(\|x\|)\frac{x_j}{\|x\|}, \\ \frac{\partial^2 u}{\partial x_j^2} &= f''(\|x\|)\frac{x_j^2}{\|x\|^2} + f'(\|x\|)\frac{\|x\|^2 - x_j^2}{\|x\|^3}, \end{aligned}$$

and

$$\frac{\partial^2 u}{\partial x_k \partial x_j} = f''(\|x\|)\frac{x_j x_k}{\|x\|^2} - f'(\|x\|)\frac{x_j x_k}{\|x\|^3}.$$

Now, form the Hessian $\nabla^2(u)$, and

- Add $-\dfrac{x_j}{x_1}$ times the first row to row j for $j = 2, 3, \dots, n$;
- Add $\dfrac{x_1 x_j}{\|x\|^2}$ times row j to row 1 for $j = 2, 3, \dots, n$;
- Factor $\dfrac{f''(\|x\|)}{\|x\|^2}$ from row 1 and $\dfrac{f'(\|x\|)}{\|x\|}$ from row j for $j = 2, 3, \dots, n$;
- Add $-x_1 x_j$ times row j to row 1 for $j = 2, 3, \dots, n$.

Then,

$$\begin{aligned} \det(\nabla^2(u)) &= \left(\frac{f'(\|x\|)}{\|x\|}\right)^{n-1} \frac{f''(\|x\|)}{\|x\|^2} \det \begin{vmatrix} \|x\|^2 & 0 & 0 & \dots & 0 \\ -x_2/x_1 & 1 & 0 & & 0 \\ -x_3/x_1 & 0 & 1 & & 0 \\ & & & & \\ -x_n/x_1 & 0 & 0 & & 1 \end{vmatrix} \\ &= \left(\frac{f'(\|x\|)}{\|x\|}\right)^{n-1} f''(\|x\|). \end{aligned}$$

Olympiad problems

O577. Let a,b,c be positive numbers such that $abc = 4$ and $a,b,c > 1$. Prove the inequality

$$(a-1)(b-1)(c-1)\left(\frac{a+b+c}{3}-1\right) \le (\sqrt[3]{4}-1)^4.$$

Proposed by Marian Tetiva, România

Solution by Theo Koupelis, Broward College, Pembroke Pines, FL, USA

Let $(a,b,c) = (1+x,1+y,1+z)$, where $x,y,z > 0$. We want to show that $f(x,y,z) := \frac{1}{3}xyz(x+y+z) \le (\sqrt[3]{4}-1)^4$ when $g(x,y,z) := x+y+z+xy+yz+zx+xyz-3 = 0$. Because f and g are polynomials, they have continuous first partial derivatives. Also, every component of the gradient of g is positive. Therefore, by the theorem of Lagrange multipliers, any extrema occur on the boundary or where $\nabla f = \lambda \nabla g$, for suitable scalars λ. We have

$$\begin{aligned} xyz + yz(x+y+z) &= 3\lambda(1+y)(1+z),\\ xyz + zx(x+y+z) &= 3\lambda(1+z)(1+x),\\ xyz + xy(x+y+z) &= 3\lambda(1+x)(1+y). \end{aligned}$$

Subtracting by pairs we get

$$\begin{aligned} z(y-x)(x+y+z) &= 3\lambda(1+z)(y-x),\\ x(y-z)(x+y+z) &= 3\lambda(1+x)(y-z),\\ y(z-x)(x+y+z) &= 3\lambda(1+y)(z-x). \end{aligned}$$

If $x \ne y \ne z \ne x$, from

$$\frac{3\lambda}{x+y+z} = \frac{z}{1+z} = \frac{x}{1+x} = \frac{y}{1+y}$$

we get a contradiction. Therefore, we need to examine the following cases.

(i) If $x = y = z$, then $a = b = c = \sqrt[3]{4}$, and $f(x) = (\sqrt[3]{4}-1)^4$.

(ii) If $x \ne y = z$, then $a \ne b = c$, and from the given condition we get $a = \frac{4}{b^2}$. Therefore, $f(b) = \frac{1}{3b^4}(b-1)^2(4-b^2)(2b^3-3b^2+4)$. We have $f'(b) = -\frac{2}{3b^5}(b-1)(b^3-4)[3(b-1)^3+(5b^2-13b+11)]$ and thus $f(b)$ obtains its maximum at $b = 2^{2/3}$ where $f(2^{2/3}) = (\sqrt[3]{4}-1)^4$. However, in this case we also get $a = b = c$.

We now check for extrema on the boundary. As $z \to 0^+$, the condition leads to $3 = x+y+xy \ge xy + 2\sqrt{xy}$, and thus $0 < xy \le 1$. Therefore, $f \to 0^+$. As $z \to +\infty$, at least one of the other variables tends to 0^+, because from the condition we get $xyz < 3$. In summary, $f \le (\sqrt[3]{4}-1)^4$, with equality when $a = b = c = 2^{2/3}$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O578. Solve, in positive integers, the equation

$$\frac{x^3 + y^3}{6} = 2022\sqrt{xy - 4}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Theo Koupelis, Broward College, Pembroke Pines, FL, USA

Clearly $4 \mid x^3 + y^3$ and thus x, y are both even. With $(x, y) = (2a, 2b)$, where a, b are positive integers, we rewrite the given equation as $a^3 + b^3 = 9 \cdot 337 \cdot \sqrt{ab - 1}$. Let $a + b = n$ and $ab = 1 + m^2$, where n, m are positive integers. We then get $n^3 - 3n(1 + m^2) = 9 \cdot 337m$, and thus $n = 3k$, where k is a positive integer. Therefore, $k(3k^2 - 1 - m^2) = 337m$, and thus $k \mid 337m$, where 337 is a prime number.

(i) If $k = 1$, then $m^2 + 337m - 2 = 0$, which has no solution in positive integers.

(ii) If $k = 337$, then $m^2 + m + 1 - 3 \cdot 337^2 = 0$, which has no integer solutions.

(iii) If $k = 337\ell$ and $m = \ell t$, where ℓ, t are positive integers, then $\ell^2(3 \cdot 337^2 - t^2) = t + 1$. We must have $t \le 583$ so that $3 \cdot 337^2 - t^2 > 0$, but then $\ell < 1$, and thus there are no integer solutions in this case.

(iv) If $m = k\ell$, where ℓ is a positive integer, then $k^2(3 - \ell^2) = 337\ell + 1$ and thus $\ell = 1$. Therefore, $m = k = 13$, and thus $(x, y) = (68, 10)$ is the only solution of the given equation in positive integers.

Second solution by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

WLOG assume that $x \ge y$ and let $x+y=a, x-y=b$. Now $xy = \dfrac{(x+y)^2-(x-y)^2}{4} = \dfrac{a^2-b^2}{4}$ and $x^3+y^3 = \dfrac{(x+y)^3+3(x+y)(x-y)^2}{4} = a\cdot\dfrac{a^2+3b^2}{4}$. Since $\sqrt{xy-4}\in\mathbb{Q}$ and $x-y\in\mathbb{N}$ we get thgat $xy-4\in\mathbb{N}$.
The given condition implies that $2 \mid x^3+y^3$ so $2\mid a$ and since $a\equiv b \pmod 2$ we have $2\mid b$ as well. The equation now can be written as $a(a^2+3b^2) = 12\cdot 2022\sqrt{a^2+b^2-16}$. Let $a=2c, b=2d$. Now we have $c(c^2+3d^2) = 3\cdot 2022\sqrt{c^2-d^2-4} \Rightarrow 3\mid c$. So let $c=3e$. Note that the prime factorization of $2022 = 2\cdot 3\cdot 337$.
Now, we get $e(3e^2+d^2) = 2\cdot 337\sqrt{9e^2-d^2-4} < 2\cdot 3e\cdot 337 \Rightarrow 3e^2+d^2 < 6\cdot 337$ and $3e^2 < 6\cdot 337 \Rightarrow e < 337$ so $337 \nmid e$.
However, $337 \mid e(3e^2+d^2)$ so in fact $337 \mid 3e^2 < 6\cdot 337 \Rightarrow 3e^2+d^2 = 337k$ where $k\in\{1,2,3,4,5\}$.
This gives $e^2k^2 = 4(9e^2-d^2-4)$. If e is odd then we must have $2\mid k$ so $2\mid d$ and $337k = 3e^2+d^2 \equiv 3+1 \equiv 0 \pmod 4 \Rightarrow 4\mid k$ thus $k=4$. Now, $e=13, d=29 \Rightarrow x=68, y=10$.
If e is even then d must be odd, otherwise $4|3e^2+d^2 = 337k \Rightarrow k=4$ and as before $e=13$, contradiction. So $337k\equiv d^2 \equiv 1 \pmod 4 \Rightarrow k\in\{1,5\}$. Additionally, $337k \equiv d^2 \pmod 3 \Rightarrow k\ne 5$ (because 2 is not a quadratic residue ((mod 3))), thus $k=1$. Now $e^2 = 36e^2-4d^2-16 \pmod 7 \Leftrightarrow 7\mid d^2+4$ which is impossible. Therefore, the solutions for the qiven equation are $(x,y) = (68,10),(10,68)$.

Also solved by G. C. Greubel, Newport News, VA, USA.

O579. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$(b+c-a)(c+a-b)(a+b-c)+\frac{48}{ab+bc+ca}\geq 17.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Arkady Alt, San Jose, CA, USA

In homogeneous form

$$\frac{27(b+c-a)(c+a-b)(a+b-c)}{(a+b+c)^3}+\frac{16(a+b+c)^2}{3(ab+bc+ca)}\geq 17 \iff$$

$$\frac{27\Delta\left(a^2,b^2,c^2\right)}{(a+b+c)^4}+\frac{16(a+b+c)^2}{3(ab+bc+ca)}\geq 17 \qquad (1)$$

where $\Delta\left(a^2,b^2,c^2\right)=2\sum a^2b^2-\sum a^4$.

Let $p:=ab+bc+ca, q:=abc$. Assuming $a+b+c=1$ (due homogeneity of (1)) we obtain

$$\Delta\left(a^2,b^2,c^2\right)=4\left(p^2-2q\right)-(1-2p)^2=4p-8q-1$$

and then inequality (1) becomes $27(4p-8q-1)+\dfrac{16}{3p}\geq 17$.

Since $3p=3\sum ab\leq\left(\sum a\right)^2=1$ then denoting $t:=\sqrt{1-3p}$ we obtain $p=\dfrac{1-t^2}{3}$ where $0\leq t<1 \iff 0<p\leq 1/3$.

Since Vieta′s system $\begin{cases} a+b+c=1 \\ ab+bc+ca=p=\dfrac{1-t^2}{3} \\ abc=q \end{cases}$ is solvable in real $a, b, c \iff$

$$\frac{(1+t)^2(1-2t)}{27}\leq q\leq\frac{(1-t)^2(1+2t)}{27}$$

then

$$27(4p-8q-1)+\frac{16}{3p}-17\geq 27\left(4\cdot\frac{1-t^2}{3}-8\cdot\frac{(1-t)^2(1+2t)}{27}-1\right)+\frac{16}{3\left(\dfrac{1-t^2}{3}\right)}-17=$$

$$\frac{4t^2\left(1+3t^2+4t^3-4t\right)}{(1-t)(t+1)}\geq 0$$

because $1+3t^2+4t^3-4t>0$for $t\in[0,1)$.

For $t\in(0,1)$ we have $1+3t^2+4t^3-4t>0 \iff \dfrac{1}{t}+3t+4t^2>4$. By AM-GM Inequality

$$\frac{1}{t}+3t+4t^2=3\cdot\frac{1}{3t}+3t+4t^2\geq 10\sqrt[10]{\left(\frac{1}{3t}\right)^3\cdot 3t\cdot 4t^2}=10\sqrt[10]{\frac{4}{9}}$$

and

$$10\sqrt[10]{\frac{4}{9}}>4 \iff 5\sqrt[10]{\frac{4}{9}}>2 \iff 5^{10}\cdot\frac{4}{9}>2^{10} \iff 5^{10}>2^8\cdot 9$$

where later inequality holds ($5^{10}>2^8\cdot 10 \iff 5^9>2^9 \iff 5>2$).

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Nandan Sai Dasireddy, Hyderabad, India; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania.

O580. Prove that in any triangle ABC

$$\frac{R}{r}+\frac{r}{R}+\frac{3}{2}\geq\frac{(a+b)(b+c)(c+a)}{2abc}.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

With Ravi's substitutions i.e. $a=y+z$, $b=z+x$, $c=x+y$ the inequality becomes

$$\frac{(x+y)(y+z)(z+x)}{4xyz}+\frac{4xyz}{(x+y)(y+z)(z+x)}+\frac{3}{2}\geq\frac{\prod_{cyc}(2x+y+z)}{2(x+y)(y+z)(z+x)}.$$

Without loss of generality, we may assume that $z=\max(x,y,z)$. Denoting $P=(x+y)(y+z)(z+x)$ and $Q=(2x+y+z)(2y+z+x)(2z+x+y)$ the inequality becomes as follows:

$$\frac{P}{4xyz}-2+\frac{4xyz}{P}-\frac{1}{2}\geq\frac{Q}{2P}-4,$$

$$(P-8xyz)(P-2xyz)\geq 2xyz(Q-8P),$$

$$(P-2xyz)\left[2z(x-y)^2+(x+y)(x-z)(y-z)\right]$$

$$\geq 2xyz\left[2(x+y)(x-y)^2+(2z+x+y)(x-z)(y-z)\right].$$

It remains to prove that

$$P-2xyz\geq 2xy(x+y),$$

$$(P-2xyz)(x+y)\geq 2xyz(2z+x+y).$$

The first inequality is equivalent to

$$z^2(x+y)+z(x^2+y^2)\geq xy(x+y),$$

clearly true. The second inequality is equivalent to

$$z^2(x+y)^2+z(x^2+y^2)(x+y)+xy(x+y)^2\geq 4xyz^2+2xyz(x+y),$$

or

$$z^2(x-y)^2+z(x+y)(x-y)^2+xy(x+y)^2\geq 0,$$

clearly true.

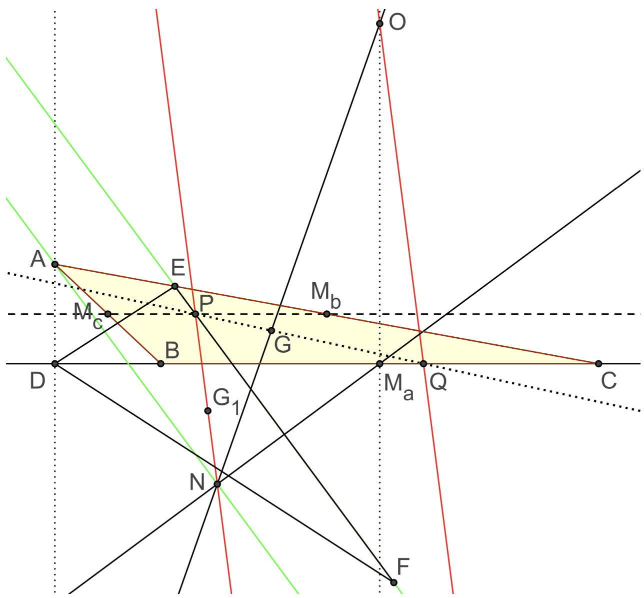
Also solved by Nandan Sai Dasireddy, Hyderabad, India; Theo Koupelis, Broward College, Pembroke Pines, FL, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Mihaly Bencze, Brașov, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Daniel Văcaru, Pitești, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Scott H. Brown, Auburn University Montgomery, AL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

O581. Let ABC be a scalene, non-right triangle with altitudes AD, BE, CF and midpoints M_a, M_b, M_c of BC, CA, AB respectively. Let G be the centroid of the triangle and let N be the center of the nine-point circle. Let P be the intersect point of EF and M_bM_c. Denote $PG \cap BC = Q$. Knowing that AN and EF are parallel , prove that OQ is parallel to the Euler line of the triangle DEF.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution by the author

The conditions ABC be a scalene, non-right triangle, means that the triangle DEF is non-degenerate and the lines EF and M_bM_c are non-parallel, i.e. P exist.



Let h be the homothety with center G and ratio -2. It is well known that the centroid G divides the median BM_b in the ratio $BG : GM_b = 2 : 1$ so $B = \mathsf{h}(M_b)$. Similarly $C = \mathsf{h}(M_c)$, hence $BC = \mathsf{h}(M_bM_c)$

The point $P \in M_bM_c$ and $Q = BC \cap PG$, hence $Q = \mathsf{h}(P)$.

Let O be the circumcenter of $\triangle ABC$. The line containing O, G, N is called the Euler line of triangle ABC. Note that $OG : GN = 2 : 1$ and so $O = \mathsf{h}(N)$. $OQ = \mathsf{h}(NP)$, hence $OQ \| NP$.

The point N is the circumcenter of $\triangle DEF$, a point from the Euler line of the triangle DEF.

It remains to show that P lies on the Euler line of the triangle DEF.

Let $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$. The points E, F are feet of the perpendiculars from B to AC and from C to AB respectively.

$$AF = AC \cos\alpha = b \cos\alpha$$

By the cosine formula,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha; \;\; \cos\alpha = \frac{-a^2 + b^2 + c^2}{2bc}$$

Now,

$$AF = b\frac{-a^2 + b^2 + c^2}{2bc} = \frac{-a^2 + b^2 + c^2}{2c}$$
$$FB = BC\cos\beta = a\frac{a^2 - b^2 + c^2}{2ac} = \frac{a^2 - b^2 + c^2}{2c}$$
$$F = \frac{FB}{AB}A + \frac{AF}{AB}B = \frac{a^2 - b^2 + c^2}{2c^2}A + \frac{-a^2 + b^2 + c^2}{2c^2}B$$

$$\begin{aligned}AE &= AB\cos\alpha = c\frac{-a^2+b^2+c^2}{2bc} = \frac{-a^2+b^2+c^2}{2b}\\ EC &= BC\cos\gamma = a\frac{a^2+b^2-c^2}{2ab} = \frac{a^2+b^2-c^2}{2b}\\ E &= \frac{EC}{AC}A + \frac{AE}{AC}C = \frac{a^2+b^2-c^2}{2b^2}A + \frac{-a^2+b^2+c^2}{2b^2}C\end{aligned}$$

$$\begin{aligned}BD &= AB\cos\beta = c\frac{a^2-b^2+c^2}{2ac} = \frac{a^2-b^2+c^2}{2a}\\ DC &= AC\cos\gamma = b\frac{a^2+b^2-c^2}{2ab} = \frac{a^2+b^2-c^2}{2a}\\ D &= \frac{DC}{BC}B + \frac{BD}{BC}C = \frac{a^2+b^2-c^2}{2a^2}B + \frac{a^2-b^2+c^2}{2a^2}C\end{aligned}$$

The centroid G_1 of triangle DEF is

$$\begin{aligned}G_1 &= \frac{1}{3}(D+E+F) =\\ &= \frac{-a^2(-a^2b^2+b^4-a^2c^2-2b^2c^2+c^4)}{6a^2b^2c^2}A + \frac{-b^2(a^4-a^2b^2-2a^2c^2-b^2c^2+c^4)}{6a^2b^2c^2}B+\\ &+ \frac{-c^2(a^4-2a^2b^2+b^4-a^2c^2-b^2c^2)}{6a^2b^2c^2}C\end{aligned} \tag{3}$$

The orthocenter H and the circumcenter O are:

$$\begin{aligned}H &= \frac{(a^2+b^2-c^2)(a^2-b^2+c^2)}{(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}A + \frac{(-a^2+b^2+c^2)(a^2+b^2-c^2)}{(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}B+\\ &+ \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)}{(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}C\\ O &= \frac{a^2(-a^2+b^2+c^2)}{(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}A + \frac{b^2(a^2-b^2+c^2)}{(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}B+\\ &+ \frac{c^2(a^2+b^2-c^2)}{(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}C\\ N &= \frac{1}{2}(H+O) = \frac{a^2b^2-b^4+a^2c^2+2b^2c^2-c^4}{2(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}A+\\ &+ \frac{-a^4+a^2b^2+2a^2c^2+b^2c^2-c^4}{2(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}B + \frac{-a^4+2a^2b^2-b^4+a^2c^2+b^2c^2}{2(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}C\end{aligned}$$

The vector

$$\begin{aligned}\overrightarrow{EF} &= F - E =\\ &= \frac{(-b^2+c^2)(-a^2+b^2+c^2)}{2b^2c^2}A + \frac{-a^2+b^2+c^2}{2c^2}B + \frac{-(-a^2+b^2+c^2)}{2b^2}C =\\ &= \frac{-a^2+b^2+c^2}{2b^2c^2}((-b^2+c^2)A + b^2B - c^2C)\end{aligned}$$

$$\begin{aligned}\overrightarrow{AN} &= N - A =\\ &= \frac{1}{2(-a+b+c)(a-b+c)(a+b-c)(a+b+c)}((2a^4-3a^2b^2+b^4-3a^2c^2-2b^2c^2+c^4)A+\\ &+ (-a^4+a^2b^2+2a^2c^2+b^2c^2-c^4)B + (-a^4+2a^2b^2-b^4+a^2c^2+b^2c^2)C)\end{aligned}$$

The lines EF and AN are parallel if and only if $k\overrightarrow{AN} = \overrightarrow{EF}$ for some real $k \neq 0$, so

$$\left| \begin{array}{l} kb^2 = -a^4 + a^2b^2 + 2a^2c^2 + b^2c^2 - c^4; \quad (B) \\ -kc^2 = -a^4 + 2a^2b^2 - b^4 + a^2c^2 + b^2c^2; \quad (C) \end{array} \right.$$

$$\left| \begin{array}{l} k = \dfrac{-a^4 + a^2b^2 + 2a^2c^2 + b^2c^2 - c^4}{b^2} \\ -c^2(-a^4 + a^2b^2 + 2a^2c^2 + b^2c^2 - c^4) = b^2(-a^4 + 2a^2b^2 - b^4 + a^2c^2 + b^2c^2) \end{array} \right.$$

From here, the condition $AN \| EF$ is equivalent to

$$q := a^4b^2 - 2a^2b^4 + b^6 + a^4c^2 - 2a^2b^2c^2 - b^4c^2 - 2a^2c^4 - b^2c^4 + c^6 = 0 \tag{4}$$

The point P lies on the line EF and on M_bM_c, so

$$P = xE + (1-x)F = \frac{b^2(a^2 - b^2 + c^2) + (b^2 - c^2)(-a^2 + b^2 + c^2)x}{2b^2c^2}A +$$
$$+ \frac{(-a^2 + b^2 + c^2)(1-x)}{2c^2}B + \frac{(-a^2 + b^2 + c^2)x}{2b^2}C$$
$$P = yM_b + (1-y)M_c =$$
$$= \frac{y}{2}(A + C) + \frac{1-y}{2}(A + B) = \frac{1}{2}A + \frac{1-y}{2}B + \frac{y}{2}C$$

$$\left| \begin{array}{r} \dfrac{b^2(a^2 - b^2 + c^2) + (b^2 - c^2)(-a^2 + b^2 + c^2)x}{2b^2c^2} = \dfrac{1}{2}; \quad (A) \\ \dfrac{(-a^2 + b^2 + c^2)x}{2b^2} = \dfrac{y}{2}; \quad (C) \end{array} \right.$$

$$\left| \begin{array}{l} x = \dfrac{b^2(-a^2 + b^2)}{(b^2 - c^2)(-a^2 + b^2 + c^2)} \\ y = \dfrac{(-a^2 + b^2 + c^2)x}{b^2} = \dfrac{-a^2 + b^2}{b^2 - c^2} \end{array} \right.$$

From here

$$P = \frac{1}{2}A + \frac{a^2 - c^2}{2(b^2 - c^2)}B + \frac{-a^2 + b^2}{2(b^2 - c^2)}C$$

The points P, G_1, N are collinear if $\overrightarrow{NG_1} = m\overrightarrow{NP}$ for some real $m \neq 0$. From (3) and (4) we receive

$$\frac{-a^2(-a^2b^2 + b^4 - a^2c^2 - 2b^2c^2 + c^4)}{6a^2b^2c^2} = \frac{-a^2(-a^2b^2 + b^4 - a^2c^2 - 2b^2c^2 + c^4) + q}{6a^2b^2c^2 - 3q} =$$
$$= \frac{a^2b^4 - b^6 + 4a^2b^2c^2 + b^4c^2 + a^2c^4 + b^2c^4 - c^6}{3(-a^4b^2 + 2a^2b^4 - b^6 - a^4c^2 + 4a^2b^2c^2 + b^4c^2 + 2a^2c^4 + b^2c^4 - c^6)} =$$
$$= \frac{a^2b^4 - b^6 + 4a^2b^2c^2 + b^4c^2 + a^2c^4 + b^2c^4 - c^6}{3(-a+b+c)(a-b+c)(a+b-c)(a+b+c)(b^2+c^2)}$$

$$G_1 = \frac{a^2b^4 - b^6 + 4a^2b^2c^2 + b^4c^2 + a^2c^4 + b^2c^4 - c^6}{3(-a+b+c)(a-b+c)(a+b-c)(a+b+c)(b^2+c^2)}A +$$
$$+ \frac{2a^4b^2 - 3a^2b^4 + b^6 + a^4c^2 - 4a^2b^2c^2 - 2b^4c^2 - 2a^2c^4 + c^6}{3(-a+b+c)(a-b+c)(a+b-c)(a+b+c)(b^2+c^2)}B +$$
$$+ \frac{a^4b^2 - 2a^2b^4 + b^6 + 2a^4c^2 - 4a^2b^2c^2 - 3a^2c^4 - 2b^2c^4 + c^6}{3(-a+b+c)(a-b+c)(a+b-c)(a+b+c)(b^2+c^2)}C$$

$$\begin{aligned}\overrightarrow{NG_1} &= G_1 - N\\ &= \frac{b^2-c^2}{6(-a+b+c)(a-b+c)(a+b-c)(a+b+c)(b^2+c^2)}[(b^2-c^2)(a^2-b^2-c^2)A+\\ &+(a^4-3a^2b^2+2b^4-2a^2c^2+b^2c^2+c^4)B-(a^4-2a^2b^2+b^4-3a^2c^2+b^2c^2+2c^4)C]\end{aligned}$$

$$\begin{aligned}\overrightarrow{NP} &= P - N\\ &= \frac{a^2}{2(-a+b+c)(a-b+c)(a+b-c)(a+b+c)(b^2-c^2)}[(b^2-c^2)(a^2-b^2-c^2)A+\\ &+(a^4-3a^2b^2+2b^4-2a^2c^2+b^2c^2+c^4)B-(a^4-2a^2b^2+b^4-3a^2c^2+b^2c^2+2c^4)C]\end{aligned}$$

$$\overrightarrow{NG_1} = \frac{3a^2(b^2+c^2)}{(b^2+c^2)^2}\overrightarrow{NP}$$

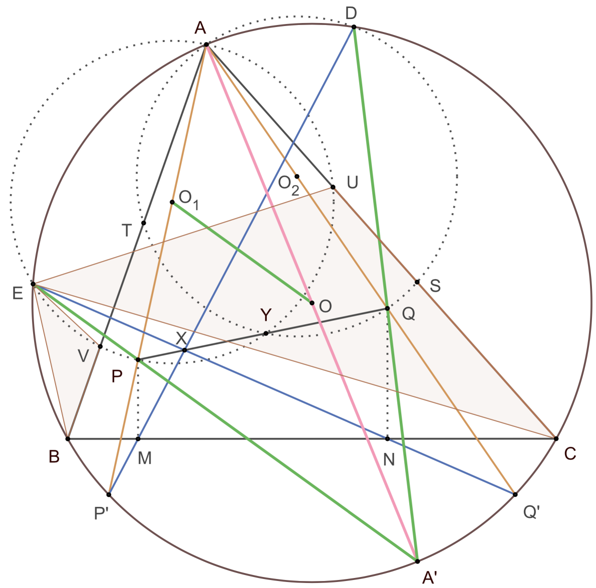
It follows that P lies on the Euler line NG_1 of DEF and $OQ \| NG_1$.

O582. Let P be a point in the interior of a triangle ABC and let Q be its isogonal conjugate with respect to ABC. Let M, N be the projections of P, Q on BC and P', Q' be the second intersections of the lines AP, AQ with the circumcircle of ABC. Prove that the lines MP' and NQ' intersect on PQ.

Proposed by Marius Stănean, Zalău, România

Solution by the author

Let $\Gamma(O)$ be the circumcircle of triangle ABC. Let $\{D\} = MP' \cap \Gamma$, $\{E\} = NQ' \cap \Gamma$, $\{X\} = MP' \cap NQ'$, U, V be the projections of P and S, T be the projections of Q on AC, AB.



We state that $\triangle BEV \sim \triangle CEU$ $(*)$. Indeed, we have $\angle EBV \equiv \angle ECU$ so it remains to be seen $\dfrac{BE}{CE} = \dfrac{BV}{CU}$. We have

$$\triangle BNE \sim \triangle Q'NC \Longrightarrow \frac{BE}{CQ'} = \frac{BN}{Q'N} \tag{1}$$

$$\triangle BNQ' \sim \triangle ENC \Longrightarrow \frac{CE}{BQ'} = \frac{CN}{Q'N} \tag{2}$$

From (1) and (2) it follows that

$$\frac{BE}{CE} = \frac{CQ'}{BQ'} \cdot \frac{BN}{CN} \tag{3}$$

From the Law of Sines, applied twice in triangles $BQ'N$ and $CQ'M$, we get that

$$\frac{BQ'}{\sin\angle BNQ'} = \frac{Q'N}{\sin\angle CBQ'}, \quad \frac{CQ'}{\sin\angle CNQ'} = \frac{Q'N}{\sin\angle BCQ'}$$

$$\Longrightarrow \frac{CQ'}{BQ'} = \frac{\sin\angle CBQ'}{\sin\angle BCQ'} = \frac{\sin\angle CAQ}{\sin\angle BAQ}$$

Now, $BN = NQ\cot\angle CBQ$, $CN = QN\cot\angle BCQ$ so

$$\frac{BN}{CN} = \frac{\sin BCQ \cdot \cos CBQ}{\sin CBQ \cdot \cos BCQ}$$

Returning to (3), we have

$$\frac{BE}{CE} = \frac{\sin\angle CAQ \cdot \sin BCQ \cdot \cos CBQ}{\sin\angle BAQ \cdot \sin CBQ \cdot \cos BCQ} \tag{4}$$

On other hand, in the right triangles BPV, CPU and using the Law of Sines in triangle BPC, we have

$$\frac{BV}{CU} = \frac{BP\cos\angle ABP}{CP\cos\angle ACP} = \frac{\sin\angle BCP\cdot\cos\angle ABP}{\sin\angle CBP\cdot\cos\angle ACP}$$

But the points P, Q are isogonal conjugate with respect to ABC which means $\angle BCP = \angle ACQ$, $\angle ABP = \angle CBQ$, $\angle CBP = \angle ABQ$ and $\angle ACP = \angle BCQ$, so

$$\frac{BV}{CU} = \frac{\sin\angle ACQ\cdot\cos\angle CBQ}{\sin\angle ABQ\cdot\cos\angle BCQ} \tag{5}$$

Comparing (4) with (5) it remains to be shown that

$$\frac{\sin\angle CAQ\cdot\sin ABQ\cdot\sin BCQ}{\sin\angle BAQ\cdot\sin CBQ\cdot\sin ACQ} = 1$$

which is the trigonometric form of Ceva's theorem for the cevians AQ, BQ, CQ.

Let (O_1), (O_2) the circles of diameters AP, AQ and circumcenter O_1, O_2.

From $(*)$ we deduce that $\angle BVE \equiv \angle CUE$ which implies $\angle AVE \equiv \angle AUE$ and because $U, V \in (O_1)$ it follows that the point E is also on the circle (O_1) which means $\angle AEP = 90^\circ$. Let A' be the antipode of A on Γ. We have $OO_1 \perp AE$ and $OO_1 \parallel A'P$ so $A'P \perp AE$. Therefore A', P, E there are three collinear points.

Similarly results that A', Q, D are three collinear points.

Finally, considering the hexagon inscribed $AP'DA'EQ'$, according to Pascal's theorem we obtain that the points $AP' \cap A'E = \{P\}$, $P'D \cap EQ' = \{X\}$ and $AQ' \cap DA' = \{Q\}$ are collinear, as desired.