A New Approach to the Interior Goat and Bird Problems

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1 Introduction

A circle of radius r is cut by another whose center is on the circumference of the given circle. The problem of determining the radius R of this cutting circle such that the area common to both circles is one half of that of the given circle is termed as the interior goat problem. Recently, Ullisch [1] presented a closed-form solution to this interior goat problem. An analogous problem can also be posed in three-dimensions with circles/areas replaced by spheres/volumes, and is referred to as the interior bird problem [2]. Indeed, the problem can be generalized to even n-dimensions [2].

In this work, we present a new approach to solving the interior goat (two-dimensional) and bird (three-dimensional) problems based on bipolar and toroidal coordinates. Although several solution methodologies (including approximate ones) have appeared for these problems, quite surprisingly, none, in their 125 year history, have attempted solving them using the natural setting for these problems based on the above-mentioned coordinate systems. The reason that these coordinate systems are the 'natural setting' for these problems is that the arcs of the two circles in the two-dimensional case, and the surfaces of the spheres in the three-dimensional case coincide with coordinate curves/surfaces in the bipolar and toroidal coordinate systems, respectively, as will be evident in the following sections.

2 The interior goat problem

First consider the goat problem (see Fig. 1). The bipolar coordinate system with curvilinear coordinates (ξ, η) is mapped to the Cartesian coordinates (x, y) via the relations

$$\begin{aligned} x &= \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \\ y &= \frac{c \sinh \xi}{\cosh \xi - \cos \eta}. \end{aligned} \tag{1}$$

The scale factors for this coordinate system are

$$h_1 = h_2 = \frac{c}{\cosh \xi - \cos \eta}.$$

From Eqns. (1), we see that

$$(x - c \cot \eta)^2 + y^2 = (c \csc \eta)^2. \tag{2}$$

Thus, the curve $\eta = \eta_0$ to the right of the y-axis with $0 < \eta_0 < \pi$ (see Fig. 1), represents an arc of a circle with center at $(c \cot \eta_0, 0)$ and radius $r = c \csc \eta_0$. Similarly, the curve $\eta = \eta_1$ with $0 < \eta_1 < \pi$ represents the arc of the cutting circle with center at $(c \cot \eta_1, 0)$ and radius $R = c \csc \eta_1$, so that

$$\frac{R}{r} = \frac{\csc \eta_1}{\csc \eta_0}. \tag{3}$$

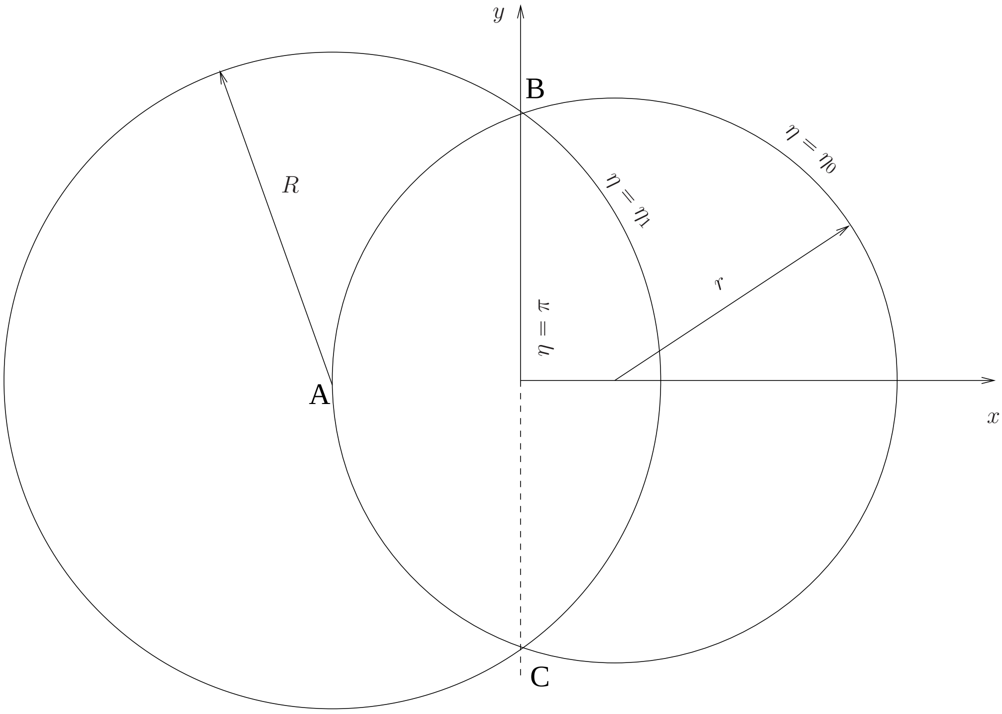

Figure 1: The two-dimensional goat problem.

The coordinates of the points B and C where the circles cut the y-axis are $(c, 0)$ and $(-c, 0)$, respectively, and the segment of the y-axis that lies between B and C (approaching from the positive x-axis) is represented by $\eta = \pi$. The x-coordinate of the point A where the original circle cuts the x-axis is $c(\cot\eta_0 - \csc\eta_0)$, and since this is also the center of the cutting circle, we have

$$\cot\eta_0 - \csc\eta_0 = \cot\eta_1,$$

whose solution under the above-mentioned inequality constraints on η_0 and η_1 is

$$\eta_1 = \frac{\eta_0 + \pi}{2}. \tag{4}$$

Substituting Eqn. (4) into Eqn. (3), we get

$$\frac{R}{r} = 2\sin\frac{\eta_0}{2}. \tag{5}$$

The area bounded by the curve $\eta = \eta_0$ and the y-axis (i.e., the $\eta = \pi$ line) is given by

$$A_0 = \int_{\eta_0}^{\pi}\int_{-\infty}^{\infty} h_1^2\, d\xi\, d\eta = c^2\left[\cot\eta_0 + (\pi - \eta_0)\csc^2\eta_0\right]. \tag{6}$$

Similarly, the area bounded by the curve $\eta = \eta_1$ and the y-axis is

$$A_1 = c^2\left[\cot\eta_1 + (\pi - \eta_1)\csc^2\eta_1\right],$$

so that

$$A_0 - A_1 = c^2\left[\cot\eta_0 - \cot\eta_1 + (\pi - \eta_0)\csc^2\eta_0 - (\pi - \eta_1)\csc^2\eta_1\right].$$

It is stipulated in the problem statement that this area be half the area of the original circle, i.e.,

$$\cot\eta_0 - \cot\eta_1 + (\pi - \eta_0)\csc^2\eta_0 - (\pi - \eta_1)\csc^2\eta_1 = \frac{\pi\csc^2\eta_0}{2},$$

which on simplifying using Eqn. (4) yields

$$\sin\eta_0 + (\pi - \eta_0)\cos\eta_0 - \frac{\pi}{2} = 0, \tag{7}$$

whose approximate solution is $\eta_0 \approx 1.23589692427991$. Alternatively, by substituting $\eta_0 = \pi - \beta$ into Eqn. (7), we get

$$\sin\beta - \beta\cos\beta - \frac{\pi}{2} = 0, \tag{8}$$

which is the same equation for which Ullisch [1] has presented an exact solution. From Eqn. (4), we obtain the approximate value of η_1 as 2.1887447889348513. Finally, from Eqn. (5), we get

$$\frac{R}{r} \approx 1.15872847301812152,$$

which agrees with the previously presented solutions.

3 The interior bird problem

This problem is the three-dimensional analogue of the goat problem discussed in the previous section with spheres and volumes replacing circles and areas, respectively. In this case, we use the toroidal coordinate system (ξ, η, ϕ) whose mapping to the Cartesian system is given by

$$\begin{aligned} x &= \frac{c\sinh\xi\cos\phi}{\cosh\xi - \cos\eta}, \\ y &= \frac{c\sinh\xi\sin\phi}{\cosh\xi - \cos\eta}, \\ z &= \frac{c\sin\eta}{\cosh\xi - \cos\eta}. \end{aligned} \tag{9}$$

The scale factors are

$$\begin{aligned} h_1 = h_2 &= \frac{c}{\cosh\xi - \cos\eta}, \\ h_3 &= \frac{c\sinh\xi}{\cosh\xi - \cos\eta} = h_1\sinh\xi. \end{aligned} \tag{10}$$

The cross section of the domain in the $\hat{r}$-z plane, where $\hat{r} = \sqrt{x^2 + y^2}$ and z denote cylindrical coordinates, is shown in Fig. 2. The coordinates of points B and C in the $\hat{r}$-z plane are $(c, 0)$. In place of Eqn. (2), we now have

$$\hat{r}^2 + (z - c\cot\eta)^2 = (c\csc\eta)^2, \tag{11}$$

so that the surface $\eta = \eta_0$ to the right of the line BC with $0 < \eta_0 < \pi$ (see Fig. 2), represents a portion of a sphere with center at $(0, c\cot\eta_0)$ in the $\hat{r}$-z plane, and radius $r = c\csc\eta_0$. Similar remarks apply to the surface $\eta = \eta_1$. Following the same procedure as in the previous section, we again get Eqns. (4) and (5), which we rewrite here for convenience:

$$\eta_1 = \frac{\eta_0 + \pi}{2}, \tag{12a}$$

$$\frac{R}{r} = 2\sin\frac{\eta_0}{2}. \tag{12b}$$

The volume of the region bounded by the surfaces $\eta = \eta_0$ and $\eta = \pi$ is given by

$$V_0 = \int_0^{2\pi}\int_{\eta_0}^{\pi}\int_0^{\infty} h_1h_2h_3\, d\xi\, d\eta\, d\phi = 2\pi\int_{\eta_0}^{\pi}\int_0^{\infty} h_1^3\sinh\xi\, d\xi\, d\eta = \frac{\pi c^3\sin\eta_0(2 - \cos\eta_0)}{3(1 - \cos\eta_0)^2}. \tag{13}$$

Similarly, the volume of the region bounded by the surfaces $\eta = \eta_1$ and $\eta = \pi$ is given by

$$V_1 = 2\pi\int_{\eta_1}^{\pi}\int_0^{\infty} h_1^3\sinh\xi\, d\xi\, d\eta = \frac{\pi c^3\sin\eta_1(2 - \cos\eta_1)}{3(1 - \cos\eta_1)^2}.$$

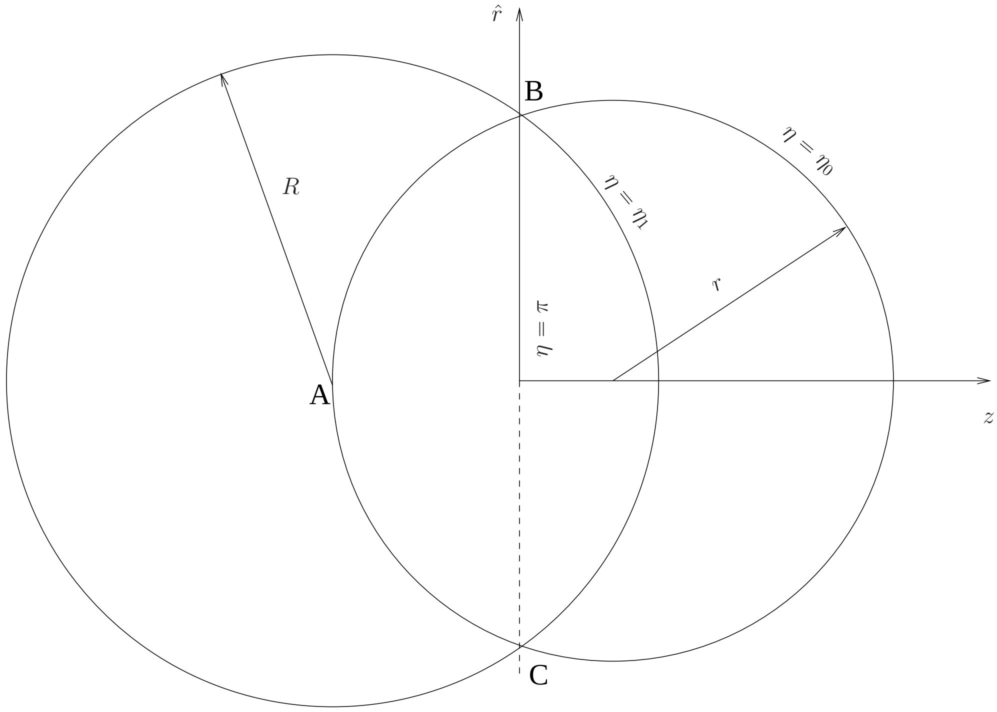

Figure 2: Domain of the bird problem.

It is stipulated that $V_0 - V_1 = 2\pi(c \csc \eta_0)^3/3$, so that in place of Eqn. (7), we now get

$$\begin{aligned} 0 &= \frac{13}{4} - 3\cos\eta_0 + \frac{3}{4}\cos 2\eta_0 - 6\sin\frac{\eta_0}{2} + 2\sin\frac{3\eta_0}{2} \\ &= 1 - 8\sin^3\frac{\eta_0}{2} + 6\sin^4\frac{\eta_0}{2}. \end{aligned} \tag{14}$$

Eqn. (14) is a quartic in $\sin(\eta_0/2)$ for which an exact solution can be given as

$$\begin{aligned} p &:= 1 - \frac{1}{\sqrt{2}}, \\ q &:= \sqrt{4 + 3\left[2^{1/3}p^{-1/3} + 2^{2/3}p^{1/3}\right]}, \\ \sin\frac{\eta_0}{2} &= \frac{1}{6}\left\{q + 2 - \sqrt{12 - q^2 + \frac{16}{q}}\right\} \\ &\approx 0.614272. \end{aligned}$$

The corresponding approximate values of η_0 and η_1 are 1.32293 and 2.23226. Finally, the required ratio in Eqn. (12b) is given by

$$\frac{R}{r} = \frac{1}{3}\left\{q + 2 - \sqrt{12 - q^2 + \frac{16}{q}}\right\} \approx 1.22854, \tag{15}$$

in agreement with the solution in [2].

References

[1] Ullisch I., A closed-form solution to the geometric goat problem, *Math. Intel.*, **42**, 12–16, 2020.

[2] Jameson G. and N. Jameson, Goats and birds, *Math. Gazette*, **101(551)**, 101.17, 2017.