Junior problems

J583. Let m and n be positive integers. Prove that

$$27^{2m+n+1} + 27^{m+2n+1} - 27^{m+n+1} + 1$$

has a factor greater than $6 \cdot 27^{\min(m,n)}$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India

Observe that

$$\begin{aligned} 27^{2m+n+1} + 27^{m+2n+1} - 27^{m+n+1} + 1 &= (3^{2m+n+1})^3 + (3^{m+2n+1})^3 + (1)^3 - 3 \cdot (3^{2m+n+1}) \cdot (3^{m+2n+1}) \cdot 1 \\ &= (3^{2m+n+1} + 3^{m+2n+1} + 1) \cdot M. \end{aligned}$$

Now, observe that $2m + n \geq 3\min(m,n)$ and $m + 2n \geq 3\min(m,n)$.
So, $3^{2m+n+1} + 3^{m+2n+1} + 1 \geq 2 \cdot 3^{3\min(m,n)+1} + 1 = 6 \cdot 27^{\min(m,n)} + 1 > 6 \cdot 27^{\min(m,n)}$ as desired.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Petrakis Emmanouil, 2nd High School, Agrinio, Greece.

J584. Let x, y, z be rational numbers such that:

$$3x^2 + 2022yz - 2016zx, 3y^2 + 2022xz - 2016xy, z^2 + 674xy - 672yz,$$

are all squares of some rational numbers. Prove that $x = y = z = 0$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Writing $(x, y, z) = (\frac{X}{L}, \frac{Y}{L}, \frac{Z}{L})$ for some positive integer L. Multiplying all three expressions by L^2 would change nothing except letting us to assume that (x, y, z) are integers. Letting

$$A^2 = x^2 + 2022yz - 2016zx, B^2 = 3y^2 + 2022xz - 2016xy, C^2 = z^2 + 674xy - 672yz,$$

for some positive integers A, B, C. It follows that:

$$A^2 + B^2 + 3C^2 = 3(x + y + z)^2 = 3D^2.$$

Reducing the equation $A^2 + B^2 = 3(D^2 - C^2)$ modulo 3 it follows that A, B are divisible by 3. Continuing this way, we can find that D, C are divisible by 3,too. Hence, if A, B, C, D satisfy the (), then $(\frac{A}{3}, \frac{B}{3}, \frac{C}{3}, \frac{D}{3})$ also satisfies the same equation. Thus, by the infinite descent it follows that;

$$A = B = C = D = 0.$$

Furthermore, $D = 0$ implies that $x + y + z = 0$. That is, $z = -x - y$ putting it in the equation $z^2 + 674xy - 672yz = 0$ yielding;

$$(x + y)^2 + 674xy + 672y(x + y) = x^2 + 1346xy + 673y^2 = 0.$$

This is a quadratic equation with respect to x. The discriminant is $D = y^2(1346^2 - 4 \cdot 673) = (3 \cdot 7 \cdot 673 \cdot 2^7)y^2$ which, is not a perfect square unless $y = 0$.It then follows that $x = 0$ and hence $z = 0$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J585. Let $a, b, c > 1$ be real numbers such that

$$\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} = 1$$

Prove that

$$abc + 44 \geq 9(a+b+c)$$

Proposed by Marius Stănean, Zalău, Romania

First solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

By writing $x = a-1 > 0$, $y = b-1 > 0$, $z = c-1 > 0$ the inequality becomes

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \implies (x+1)(y+1)(z+1) + 44 \geq 9(x+y+z+3)$$

that is

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \implies (x+1)(y+1)(z+1) + 44 \geq 9(x+y+z+3)$$

Now let's set $X = 1/x$, $Y = 1/y$, $Z = 1/z$ and we get

$$X + Y + Z = 1 \implies (\frac{1}{X} + 1)(\frac{1}{Y} + 1)(\frac{1}{Z} + 1) + 44 \geq 9(\frac{1}{X} + \frac{1}{Y} + \frac{1}{Z} + 3)$$

namely

$$\frac{1}{XYZ} + \sum_{\text{cyc}} \frac{1}{X} + \sum_{\text{cyc}} \frac{1}{XY} + 1 + 44 \geq 9 \sum_{\text{cyc}} \frac{1}{X} + 27$$

and upon simplifications we come to

$$\frac{1}{XYZ} + \sum_{\text{cyc}} \frac{1}{XY} + 18 \geq 8 \sum_{\text{cyc}} \frac{1}{X}$$

$$\frac{1}{XYZ} + \frac{X+Y+Z}{XYZ} + 18 \geq 8\frac{XY+YZ+ZX}{XYZ} \iff 2 + 18XYZ \geq 8 \sum_{\text{cyc}} XY \qquad (1)$$

This is a special version of Schür's inequality $(a, b, c > 0)$

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + b^2a + b^2c + c^2b + a^2c + c^2a \geq (a+b+c)(ab+bc+ca) - 3abc$$

or

$$(a+b+c)^3 - 3(a^2b + b^2a + b^2c + c^2b + a^2c + c^2a) - 3abc \geq (a+b+c)(ab+bc+ca) - 3abc$$

$$(a+b+c)^3 - 3((a+b+c)(ab+bc+ca) - 3abc) - 3abc \geq (a+b+c)(ab+bc+ca) - 3abc$$

If $a+b+c = 1$ it becomes

$$1 - 3(ab+bc+ca) + 9abc \geq ab+bc+ca \iff 1 + 9abc \geq 4(ab+bc+ca)$$

which is (1) and the proof is concluded.

Second solution by the author
Denote $x = \frac{1}{a-1}$, $y = \frac{1}{b-1}$, $z = \frac{1}{c-1}$ then $a = \frac{x+1}{x}, \ldots$. We have $x + y + z = 1$ and we need to prove that

$$\frac{(x+1)(y+1)(z+1)}{xyz} + 44 \geq 9 \sum_{cyc} \frac{x+1}{x},$$

or

$$(2x + y + z)(x + 2y + z)(x + y + 2z) + 44xyz \geq 9 \sum_{cyc} (2x + y + z)yz,$$

or

$$2(x^3 + y^3 + z^3) + 7 \sum_{cyc} xy(x + y) + 16xyz + 44xyz \geq 9 \sum_{cyc} xy(x + y) + 54xyz,$$

that is

$$x^3 + y^3 + z^3 + 3xyz \geq \sum_{cyc} xy(x + y)$$

i.e. Schur's Inequality.

Observation: The inequality is stronger that (Andrei Eckstein site)
https://pregatirematematicaolimpiadejuniori.files.wordpress.com/2021/07/pb-s-258.pdf

$$abc \geq 64.$$

Also solved by Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Piteşti, Romania; Pranjal Jha, Whitefield Global School, Bengaluru, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Jiang Lianjun, GuiLin, China; Le Hoang Bao, Tien Giang, Vietnam; Petrakis Emmanouil, 2nd High School, Agrinio, Greece.

J586. et ABC be a triangle with $AC = BC$ and altitudes AD, BE, CF. The circle with diameter BD cuts AB in M and BE in N. Line MN cuts AC in Q and CF in P. Let S denote the midpoint of segment DC. Show that SQP is an isosceles triangle.

Proposed by Michaela Berindeanu, Bucharest, Romania

Solution by Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA

Let $c = AB$, $a = BC = AC$ be the side lengths of ABC and $\gamma = \angle ACB$, $\alpha = \angle BAC = \angle CBA$ its angles. By Pythagorean theorem on $\triangle ABD$ and $\triangle ADC$ we find

$$BD = \frac{c^2}{2a} \qquad \text{and} \qquad CD = \frac{2a^2 - c^2}{2a}.$$

Since $\angle DMB = \angle DNB = 90^\circ$ as inscribed angles in a semicircle then $DM \parallel CF$ so $\angle MDB = \angle FCB = \gamma/2$ and also

$$\angle NDM = \angle NBM = \angle EBA = 90^\circ - \alpha = \frac{\gamma}{2}$$

$$\angle QMA = \angle NMA = \angle MNB + \angle NBM = 2\frac{\gamma}{2} = \gamma.$$

Then $\angle AQM = \angle QAM = \alpha$ so $\triangle AQM$ is an isosceles triangle similar to $\triangle ABC$. Clearly, E and D have the same distance to AB. We show next that the points E, P, D are collinear, i.e. ED cuts CF perpendicularly at P and is parallel to AB. From the similarity $\triangle DMB \sim \triangle CFB$ we have

$$\frac{BM}{BF} = \frac{BD}{BC} \qquad \Longrightarrow \qquad BM = \frac{c}{2} \cdot \frac{c^2}{2a^2} = \frac{c^3}{4a^2}$$

$$FM = FB - NM = \frac{c}{2} - \frac{c^3}{4a^2} = \frac{(2a^2 - c^2)c}{4a^2}$$

The right triangles PFM and ADC are similar on account of their corresponding angles being equal. Then $PF/FM = AD/CD$ so that

$$PF = \frac{FM \cdot AD}{CD} = \frac{\frac{(2a^2-c^2)c}{4a^2} \cdot AD}{\frac{2a^2-c^2}{2a}} = \frac{c/2}{a} \cdot AD = \left(\sin\frac{\gamma}{2}\right) AD,$$

the last equality following from $\triangle CFB$. Since $\angle BAD = 90^\circ - \alpha = \gamma/2$, from $\triangle AMD$ we get $\sin\frac{\gamma}{2} = \frac{DM}{AD}$. Combining with the previous result we see that $PF = DM$ so the points E, P, D are equidistant from AB, hence they are collinear and $PD \perp CP$. Then S is the midpoint of the hypotenuse in the right triangle CPD so triangle SPC is isosceles with $SP = SC = SD =: r$ and $\angle PSD = 2 \cdot \angle PCS = \gamma$. Then $PS \parallel AC$, triangle SPD is isosceles with $\angle SPD = \angle SDP = \alpha$. Since $EP \parallel AB$, $\angle QPE = \angle QMA = \alpha$ and $\angle PEQ = \angle EQP = \alpha$,

$$\angle SPQ = 180^\circ - (\angle DPS + \angle QPE) = 180^\circ - (\alpha + \gamma) = \alpha.$$

Let $p := EP = PD = QP$ and $q = QE$. From similarity of triangles QEP and PDS we have

$$\frac{EQ}{QP} = \frac{q}{p} = \frac{PD}{PS} = \frac{p}{r} = \frac{PQ}{PS}.$$

Since $\angle EQP = \angle SPQ = \alpha$ and $QE/PQ = PQ/PS$ it follows that triangles QEP and QPS are similar isosceles triangles, implying $SQ = SP$.

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Corneliu Mănescu-Avram, Ploieşti, Romania.

J587. Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab}+\frac{a^3+b^3+c^3+9abc}{(a+b)(b+c)(c+a)} \geq 3.$$

When does the equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

By the Cauchy-Schwarz inequality we have

$$\frac{a^2}{a^2+bc}+\frac{a^2}{a(b+c)} \geq \frac{4a^2}{a^2+bc+a(b+c)}=\frac{4a^2}{(a+b)(a+c)}.$$

Writing similar two inequalities and adding them up we get

$$\begin{aligned}\sum_{\text{cyc}}\frac{a^2}{a^2+bc}+\sum_{\text{cyc}}\frac{a}{b+c} &\geq \frac{4[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{(a+b)(b+c)(c+a)}\\ &=4-\frac{8abc}{(a+b)(b+c)(c+a)}.\end{aligned}$$

Now we note that

$$\begin{aligned}\sum_{\text{cyc}}\frac{a}{b+c} &= \sum_{\text{cyc}}\frac{a(a+b)(a+c)}{(a+b)(b+c)(c+a)}\\ &=\frac{a^3+b^3+c^3+(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}\\ &=\frac{a^3+b^3+c^3+abc}{(a+b)(b+c)(c+a)}+1.\end{aligned}$$

Combining these relations, the desired inequality follows. The equality holds for $a=b=c>0$ or $a=b>0, c=0$ and its permutaions.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J588. Find all nonnegative integers x, y, z such that

$$4^x + 3^y = z^2.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Daniel Văcaru, Pitești, Romania

We can write

$$4^x + 3^y = z^2 \Leftrightarrow 3^y = z^2 - 2^{2x} \Leftrightarrow 3^y = (z - 2^x)(z + 2^x).$$

We obtain $z - 2^x = 3^k, z + 2^x = 3^{y-k}$.We have $k \le y - k$. We obtain $2 \cdot 2^x = 3^{y-k} - 3^k = 3^k \left(3^{y-2k} - 1\right)$. It follows that $k = 0$, and $z = 2^x + 1$. Furthermore, we obtain

$$2^{x+1} + 1 = 3^y$$

For $x = 0$,we obtain $3^y = 2 + 1 = 3 \Rightarrow y = 1$ and $z = 2$. If $x \ge 1$, then $3^y \equiv 1 \pmod{4}$. It follows that $y = 2k$. We obtain $2^{x+1} = 3^{2k} - 1 = \left(3^k - 1\right)\left(3^k + 1\right)$. We have $3^k - 1 = 2^l, 3^k + 1 = 2^{y+1-l}$. We obtain $3^k - 1 \vdots 3^k + 1 \Rightarrow 3^k - 1 \vdots - \left(3^k - 1\right) + \left(3^k + 1\right) = 2 \Rightarrow 3^k = 3 \Rightarrow y = 2$. We obtain $2^{x+1} = 3^2 - 1 \Rightarrow x + 1 = 3 \Rightarrow x = 2$. It follows that $z = 5$. We obtain

$$S = \{(0,1,2);(2,2,5)\}$$

Also solved by Anmol Kumar, IISc Bangalore, India; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Pranjal Jha, Whitefield Global School, Bengaluru, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Senior problems

S583. Solve in integers the equation

$$(2x^2 - 10x + 50)(2y^2 - 10y + 50) = 2022^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA , USA

The equation

$$(2x^2 - 10x + 50)(2y^2 - 10y + 50) = 2022^2$$

is equivalent to

$$(x^2 - 5x + 25)(y^2 - 5y + 25) = 1011^2 = 3^2 \cdot 337^2.$$

It follows that the factors $x^2 - 5x + 25$ and $y^2 - 5y + 25$ must be one of the following pairs:

$$1 \text{ and } 1011^2, \quad 3 \text{ and } 3 \cdot 337^2, \quad 9 \text{ and } 337^2, \quad 337 \text{ and } 3^2 \cdot 337,$$

or 1011 and 1011. Now, the equations

$$x^2 - 5x + 25 = 1, \quad x^2 - 5x + 25 = 3, \quad \text{and} \quad x^2 - 5x + 25 = 9$$

have no real solutions, and the equation $x^2 - 5x + 25 = 337$ has no integer solutions. However, the equation $x^2 - 5x + 25 = 1011$ has solutions $x = -29$ and $x = 34$. Therefore, the equation

$$(2x^2 - 10x + 50)(2y^2 - 10y + 50) = 2022^2$$

has four solutions in integers:

$$(x, y) = (-29, -29), \quad (-29, 34), \quad (34, -29), \quad (34, 34).$$

Also solved by Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Pranjal Jha, Whitefield Global School, Bengaluru, India; Soham Dutta, DPS Ruby Park,West Bengal, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S584. Prove that in any triangle ABC,

$$(b+c)m_a + (c+a)m_b + (a+b)m_c \geq 3\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Let G be the centroid of a $\triangle ABC$ and let D be the feet of the perpendiculars from G to the side BC. We have

$$\frac{2}{3}m_b = \sqrt{BD^2 + GD^2}, \qquad \frac{2}{3}m_c = \sqrt{CD^2 + GD^2}.$$

By Minkowski's Inequality,

$$\frac{2}{3}(m_b + m_c) \geq \sqrt{(BD+CD)^2 + (GD+GD)^2} = \sqrt{a^2 + \frac{16S_1^2}{a^2}},$$

or

$$a(m_b + m_c) \geq \frac{3}{2}\sqrt{a^4 + 16S_1^2},$$

where S_1 is area of the triangle BGC. Similarly, we have

$$b(m_c + m_a) \geq \frac{3}{2}\sqrt{b^4 + 16S_2^2},$$

$$c(m_a + m_b) \geq \frac{3}{2}\sqrt{c^4 + 16S_3^2}.$$

Now, summing these inequalities and using Minkowski's Inequality, we get

$$\begin{aligned}\sum_{cyc}(b+c)m_a &= \sum_{cyc} a(m_b + m_c)\\ &\geq \frac{3}{2}\left(\sqrt{a^4 + 16S_1^2} + \sqrt{b^4 + 16S_2^2} + \sqrt{c^4 + 16S_3^2}\right)\\ &\geq \frac{3}{2}\sqrt{(a^2+b^2+c^2)^2 + (4S_1 + 4S_2 + 4S_3)^2}\\ &= \frac{3}{2}\sqrt{a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 + 16S^2}\\ &= 3\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.\end{aligned}$$

Observation: More general, for any intrior point M of a triangle ABC,

$$(b+c)MA + (c+a)MB + (a+b)MC \geq 2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

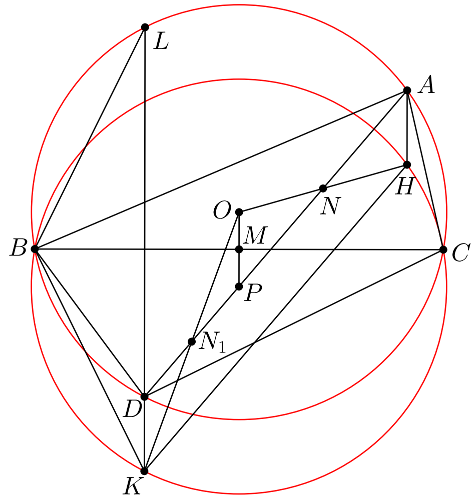
The proof is the same as above.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S585. Let ABC be a scalene triangle with circumcircle Γ, and let N be the center of the nine-point circle. Let D be the intersect point of Γ and AN and let N_1 be the center of the nine-point circle of $\triangle BCD$. Prove that A, N, N_1, D are collinear and $AD = 2NN_1$.

Proposed by Todor Zaharinov, Sofia, Bulgaria

First solution by Li Zhou, Polk State College, USA



Let H and O be the orthocenter and circumcenter of $\triangle ABC$, respectively. Let Ω be the circumcircle of $\triangle HBC$. Since $\sin\angle BHC = \sin\angle BAC$, Ω is of the same size as Γ, thus is a reflection of Γ across BC. Hence, BC is the perpendicular bisector of OP, where P is the center of Ω. Let M be the midpoint of BC, then $OP = 2OM = AH$, so $AOPH$ is a parallelogram with center N. Now suppose that the line through D and perpendicular to BC intersects Ω and Γ at K and L, respectively. Since $\angle KBC = \angle CBL = \angle CDL$, $BK \perp CD$, thus K is the orthocenter of $\triangle BCD$. Therefore, $KD = 2PM = PO$, so $KDOP$ is a parallelogram with center N_1. Finally, since $ADKH$ is a parallelogram, $AD = HK = 2NN_1$.

Second solution by the author

Let O, G are respectively the circumcenter and the centroid of ABC. Let M be the midpoint of side BC. Let G_1 be the centroid of triangle DBC.

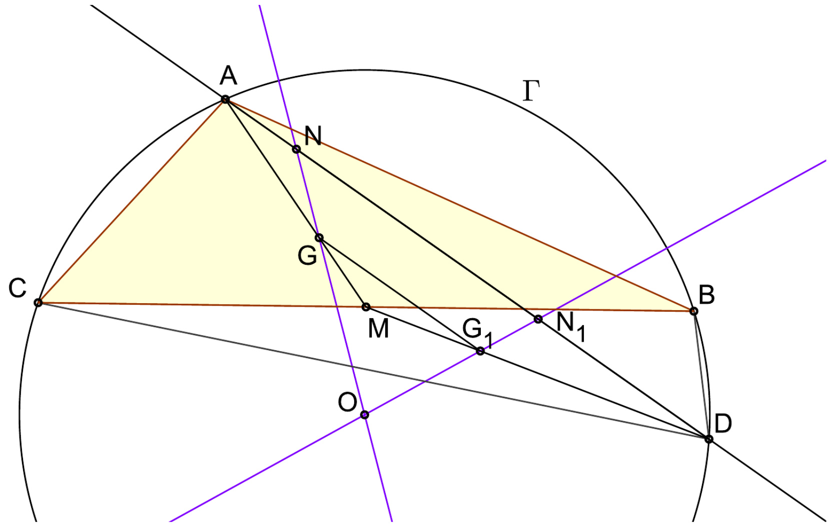


Figure 1

AM is a median in $\triangle ABC$ and DM is a median in $\triangle DBC$.

$$\frac{MG}{MA} = \frac{MG_1}{MD} = \frac{1}{3};\ \triangle MGG_1 \sim \triangle MAD;\ \frac{GG_1}{AD} = \frac{1}{3};\ AD = 3GG_1;\ GG_1 \| AD$$

It is well known that N lies on the Euler line GO of $\triangle ABC$ and $OG = 2GN$. The Euler line of $\triangle DBC$ is G_1O and $OG_1 = 2G_1N_1$.

$$\frac{OG}{ON} = \frac{OG_1}{ON_1} = \frac{2}{3};\ \triangle OGG_1 \sim \triangle ONN_1;\ \frac{GG_1}{NN_1} = \frac{2}{3};\ NN_1 = \frac{3}{2}GG_1;\ GG_1 \| NN_1$$

Hence $NN_1 \| GG_1 \| AD$ but A, N, D are collinear so A, N, N_1, D are collinear.

$$AD = 3GG_1 = 2NN_1.$$

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA.

S586. Prove that in any triangle,

$$(s^2 + r^2 + 10Rr)(4R + 4) \le 8Rs^2.$$

Proposed by Mihaly Bencze and Neculai Stanciu, Romania

First solution by Li Zhou, Polk State College, USA

The inequality $(s^2 + r^2 + 10Rr)(4R + r) \le 8Rs^2$ is equivalent to $rs^2 + 40R^2r + 14Rr^2 + r^3 \le 4Rs^2$. By Gerretsen's inequalities, it suffices to replace the s^2 on the left side of the inequality by $4R^2 + 4Rr + 3r^2$ and the s^2 on the right side by $16Rr - 5r^2$.

The result becomes $0 \le 10R^2 - 19Rr - 2r^2 = (10R + r)(R - 2r)$. The conclusion follows.

Second solution by by the author

Since $s^2 + r^2 + 4Rr = \sum ab$ and $Rr = \dfrac{abc}{2\sum a}$, the given inequality becomes

$$\left(\sum ab + 6 \cdot \frac{abc}{2\sum a}\right)(4R + r) \le 2R\left(\sum a\right)^2$$

$$\Leftrightarrow 2R\left[(\sum a)^2 - 2\sum ab - \frac{6abc}{\sum a}\right] \ge r\left(\sum ab + \frac{3abc}{\sum a}\right)$$

$$\Leftrightarrow \frac{R}{2r} \ge \frac{1}{4} \cdot \frac{\sum a \sum ab + 3abc}{(\sum a)^3 - 2\sum a \sum ab - 6abc},$$

and since $R \ge 2r$, it is enough to prove that $4(\sum a)^3 - 8\sum a \sum ab - 24abc \ge \sum a \sum b + 3abc$

$$\Leftrightarrow 4(\sum a)^3 - 9\sum a \sum ab \ge 27abc$$

$$\Leftrightarrow 4(\sum a)^3 + 12\sum a^2b + 12\sum ab^2 + 24abc - 9\sum a^2b - 9\sum ab^2 - 27abc \ge 27abc$$

$$\Leftrightarrow 3\sum a^3 + 3\sum a^2b + 3\sum ab^2 \ge 30abc$$

which is true because $\sum a^3 \ge 3abc$, $\sum a^2b \ge 3abc$, $\sum ab^2 \ge 3abc$ and we are done.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Perfetti, Università degli studi di Tor Vergata Roma, Italy; Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA; Marin Chirciu, Colegiul National Zinca Golescu, Piteşti, Romania.

S587. Diagonals AC and BD of a convex quadrilateral $ABCD$ meet at point E. Points M and N are the midpoints of sides AB and CD, respectively. Segment MN meets diagonals AC and BD at points P and Q, respectively. Prove that

$$\frac{PQ}{MN} = \frac{|[BCE] - [ADE]|}{[ABCD]}.$$

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by the author

Observe that $[BDM] = \frac{1}{2}[ABD]$ and $[BDN] = \frac{1}{2}[BCD]$, which implies $[MBND] = \frac{1}{2}[ABCD]$. Therefore, we obtain

$$\frac{[ABD]}{[ABCD]} = \frac{[BDM]}{[MBND]} = \frac{MQ}{MN}.$$

Analogously, we have

$$\frac{[ACD]}{[ABCD]} = \frac{NP}{MN}.$$

Since $[BCE] - [ADE] = [ABCD] - [ABD] - [ACD]$, we get

$$\frac{|[BCE] - [ADE]|}{[ABCD]} = \left|1 - \frac{[ABD] + [ACD]}{[ABCD]}\right| = \left|1 - \frac{MQ + NP}{MN}\right| = \frac{PQ}{MN},$$

which completes the proof.

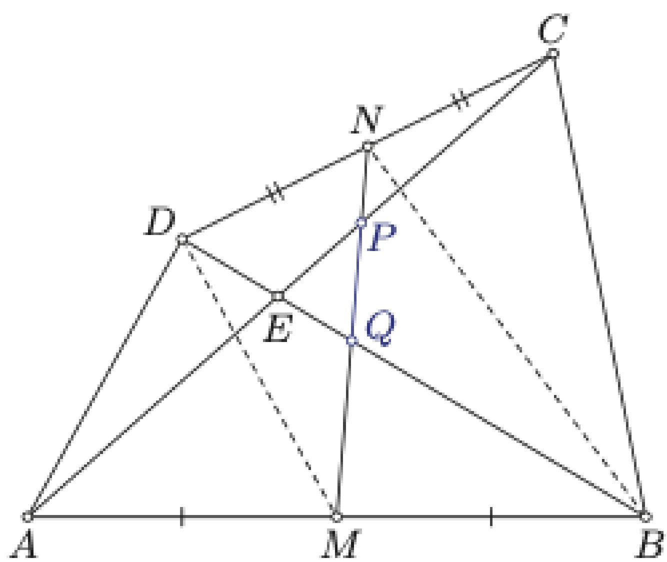


Fig. 1

S588. Find all triples (a,b,c) of nonnegative integers sich that

$$2^a3^b + 7 = c^3.$$

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

Solution by the author

We will show that $(a,b,c) = (0,0,2)$ is the only solution. First, assume that $a = 0$. It's easy to show that $0,1,-1$ are the only possible cube residues (mod 7). Therefore, taking $\pmod 7$ in $3^b + 7 = c^3$ we have $3^b = 0,01,1 \pmod 7$. Now checking the residues of $3^i \pmod 7$ for $i = 1,2,...,6$ we can easily show that 3 must divide b. Let $b = 3d$.

Now, $7 = (c-3^d)(c^2+3^dc+9^d) \Leftrightarrow c-3^d = 1, c^2+3^dc+9^d = 7 \Leftrightarrow c = 2, d = 0$. Hence, in this case $(a,b,c) = (0,0,2)$.

Next, assume that $b = 0$ and $a \neq 0$ i.e. $2^a + 7 = c^3$. Again, we must have $2^a = 0,-1,1 \pmod 7$ and checking $2^i \pmod 7$ for $i = 1,2,3$(since $2^3 = 1 \pmod 7$) we get $3|a$. Doing the same work as before it easily to show that $a = 0$, contradiction.

Now, let's assume that $a,b,c > 0$. It's clear that the only possible cubic residues $\pmod 9$ are $0,-1,1$ and thus if $a \geq 2$ we get $7 = c^3 \pmod 9$ which is impossible. As a result we must have $b = 1$.

$3 \cdot 2^a + 7 = c^3$ and again $\pmod 9 : 3 \cdot 2^a = 1,2,3 \pmod 9$ which means $2^a = 1 \pmod 3 \Rightarrow 2|a$. Let $a = 2e$. It is straightforward to show that $c^3 - 7 = 1,5,6,7,11 \pmod{13}$ and since $3^{-1} = 9 \pmod{13}$ we have $4^e = 2,6,8,9,11 \pmod{13}$ and checking $4^i \pmod{13}$ for $i = 1,...,6$ we get that $e = 4 \pmod 6$. Now that $\pmod{19}$ and perform the same operations: $c^3 - 7 = 1,5,6,7,11,13 \pmod{19} \Rightarrow 3 \cdot 4^e = 0,1,4,5,11,13$ (mod 1)9 and since $3^{-1} = 13 \pmod{19}$ we get $4^e = 0,8,10,13,14,17 \pmod{19}$. Only $4^e = 17 \pmod{19}$ is possible when $e = 5 \pmod 9$. However, it is impossible since $e = 4 \pmod 6$. Hence, $(a,b,c) = (0,0,2)$ is the only solution.

U583. Let $k \geq 1$ be a fixed integer and let

$$P_n(x) = x^n(x^k - x^{k-1} - \ldots - x - 1) - 1$$

Prove that each polynomial $P_n(x)$ has a single positive root, r_n and the sequence $r_1, r_2, \ldots, r_n$ is decreasing

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

$$Q_k(x) = x^k - x^{k-1} - \ldots - x - 1 = x^k - \frac{1 - x^k}{1 - x} = \frac{x^{k+1} - 2x^k + 1}{x - 1}, \quad x \neq 1$$

$Q_k(0) = -1$ and let's assume $Q_k(1) = 1 - k$.

Let $f_k(x) = x^{k+1} - 2x^k + 1, \quad f_k(0) = 1, \quad f_k(1) = 0, \quad \lim\limits_{x\to\infty} f_k(x) = \infty$

$$\left(x^{k+1} - 2x^k + 1\right)' = (k+1)x^k - 2kx^{k-1} \geq 0 \iff x \geq (2k)/(k+1) \geq 1$$

It follows that $f_k(x)$ is positive for $0 \leq x \leq 1$, negative for $1 < x < \overline{x}$ where $\overline{x}$ which is the second solution besides $x = 1$ of the equation $x^{k+1} - 2x^k + 1 = 0$ and positive for $x > \overline{x}$ ($\overline{x} > x_m \doteq (2k)/(k+1)$)

This means that $Q_k(x) \leq 0$ for $0 \leq x \leq \overline{x}$ and $Q_k(x) > 0$ for $x > \overline{x}$. We want to show that $Q_k(x)$ increases for $x \geq \overline{x}$ and it suffices to show that it increases for $x \geq (2k)/(k+1)$.

$$Q_k'(x) = \frac{x^{k+2} - 4x^{k-1} + 4x^k + x - 2}{(x-1)^2} \underbrace{\geq}_{AGM} \frac{4x^{k+1} - 4x^{k-1} + x - 2}{(x-1)^2} \geq 0$$

if and only if $4x^{k+1} - 4x^{k-1} + x - 2 = 4x^{k-1}(x^2 - 1) + x - 2 \geq 0$. Moreover

$$\begin{aligned}
&4x^{k-1}(x^2 - 1) + x - 2 \geq 4x_m^{k-1}(x_m^2 - 1) + x_m - 2 = \\
&= 4\left(\frac{2k}{k+1}\right)^{k-1}\left(\frac{4k^2}{(k+1)^2} - 1\right) + \frac{2k}{k+1} - 2 \geq 4\left(\frac{4k^2}{(k+1)^2} - 1\right) - \frac{2}{k+1} = \\
&\geq \frac{12k^2 - 10k - 6}{(k+1)^2} > 0 \ k \geq 2
\end{aligned}$$

Thus if $k \geq 2$ $Q_k(x)$ is positive and increases for $x \geq (2k)/(k+1)$ and a fortiori for $x > \overline{x}$. It follows that also the function $P_n(x) = x^n Q_k(x)$ is positive and increases and then the equation $P_n(x) = 1$ has the unique solution r_n. Since $(2k)/(k+1) > 1$ for $k \geq 2$, $P_{n+1}(x) > P_n(x)$ and then $r_{n+1} < r_n$ so concluding the proof for $k \geq 2$.

The case $k = 1$ is studied apart. $P_n(x) = x^{n+1} - x^n - 1$ decreases for $0 \leq x \leq n/(n+1)$ and increases for $x > n/(n+1)$ thus there exists the point $\overline{x}$ such that $P_n(\overline{x}) = 0$ with $P_n(x) \geq 0$ if $x \geq \overline{x}$ monotonically increasing. The equation $P_n(x) = 0$ has a unique solution $r_n > \overline{x}$ and $r_{n+1} < r_n$ since $x^{n+1}(x-1) \geq x^n(x-1)$.

Also solved by Anmol Kumar, IISc Bangalore, India.

U584. Let m, n, p be positive integers greater than 1. Let A be a $p\times p$ real matrix such that $A^m B = BA^m$ and $A^n B = BA^n$ for all $p\times p$ real matrices B. Prove that if $\det(A) \neq 0$ and $\gcd(m,n) = 1$ then $AB = BA$ for all $p\times p$ real matrices B.

Proposed by Mircea Becheanu, Canada

Solution by the author

By hypothesis, the matrix A is invertible and it belongs to the center of the ring $M_p(\mathbb{R})$ of $p\times p$ real matrices. The solution of the problem comes from the following more general result.

*Proposition:*Let R be a noncommmutative ring and let $Z(R)$ be its center. Let m, n be positive integers such that $\gcd(m,n) = 1$ and $x \in R$ be an invertibe element such that $x^m, x^n \in Z(R)$. Then $x \in Z(R)$.

Proof: Assume that $m > n$. Consider Euclidean division $m = nq + r$, where $r < n$. Bcause $Z(R)$ is a subring we have $x^m - x^{nq} \in Z(R)$. But

$$x^m - x^{nq} = x^{nq}(x^{m-nq} - 1) = x^{nq}(x^r - 1).$$

Because $x^{-nq} \in Z(R)$ it follows that $x^r - 1 \in Z(R)$ and then $x^r \in Z(R)$.

Now we consider the Euclidean algorithm of the numbers m and n:

$$m = nq_0 + r_0,$$

$$n = r_0 q_1 + r_1,$$

$$r_0 = r_1 q_2 + r_2,$$

$$\cdots\cdots$$

$$r_{k-2} = r_{k-1} q_k + r_k,$$

$$r_{k-1} = r_k q_{k+1} + 0.$$

Here, $r_k = \gcd(m,n) = 1$. Now, using the above argument and going down along Euclidean algorithm we find

$$x^{r_0} \in Z(R),\ x^{r_1} \in Z(R), ..., x^{r_k} \in Z(R).$$

Hence $x \in Z(R)$.

Remark: The hypothesis that A is invertible is necessary. For example, if A is a non-diagonal matrix which is nilpotent, say $A^n = 0$, we have $A^{n+1} = 0$. Then $A^n, A^{n+1} \in Z(M_p(\mathbb{R}))$ but $A \notin Z(M_p(\mathbb{R}))$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA , USA; Anmol Kumar, IISc Bangalore, India.

U585. Evaluate

$$\sum_{n=1}^{\infty}\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)-n+\frac{1}{2}-\frac{1}{6n}\right].$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Cluj-Napoca, Romania

First solution by Brian Bradie, Christopher Newport University, Newport News, VA , USA

By the Stolz-Cesaro theorem,

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sum_{k=n}^{\infty}\frac{1}{k^2}}{1/n} &= \lim_{n\to\infty}\frac{-1/n^2}{1/(n+1)-1/n}=1,\\
\lim_{n\to\infty}\frac{\sum_{k=n}^{\infty}\frac{1}{k^2}-\frac{1}{n}}{1/n^2} &= \lim_{n\to\infty}\frac{-1/n^2-1/(n+1)+1/n}{1/(n+1)^2-1/n^2}=\frac{1}{2},\\
\lim_{n\to\infty}\frac{\sum_{k=n}^{\infty}\frac{1}{k^2}-\frac{1}{n}-\frac{1}{2n^2}}{1/n^3} &= \lim_{n\to\infty}\frac{-1/n^2-1/(n+1)+1/n-1/(2(n+1)^2)+1/(2n^2)}{1/(n+1)^3-1/n^3}=\frac{1}{6},
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\sum_{k=n}^{\infty}\frac{1}{k^2}-\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{6n^3}}{1/n^4}\\
&= \lim_{n\to\infty}\frac{-1/n^2-1/(n+1)+1/n-1/(2(n+1)^2)+1/(2n^2)-1/(6(n+1)^3+1/(6n^3)}{1/(n+1)^4-1/n^4}=0,
\end{aligned}$$

so

$$\sum_{k=n}^{\infty}\frac{1}{k^2}=\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{6n^3}+o\left(\frac{1}{n^4}\right).$$

Now, as $N\to\infty$,

$$\begin{aligned}
&\sum_{n=1}^{N}\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)-n+\frac{1}{2}-\frac{1}{6n}\right]=\sum_{n=1}^{N}\left(n^2\sum_{k=n}^{\infty}\frac{1}{k^2}-n-\frac{1}{2}-\frac{1}{6n}\right)\\
&= \sum_{n=1}^{N}n^2\sum_{k=n}^{N}\frac{1}{k^2}+\sum_{n=1}^{N}n^2\sum_{k=N+1}^{\infty}\frac{1}{k^2}-\frac{N(N+2)}{2}-\frac{1}{6}H_N\\
&= \sum_{k=1}^{N}\frac{1}{k^2}\sum_{n=1}^{k}n^2+\frac{N(N+1)(2N+1)}{6}\left(\frac{1}{N}+\frac{1}{2N^2}+\frac{1}{6N^3}+o\left(\frac{1}{N^4}\right)-\frac{1}{N^2}\right)\\
&\qquad -\frac{N(N+2)}{2}-\frac{1}{6}H_N\\
&= \sum_{k=1}^{N}\frac{(k+1)(2k+1)}{6k}+\frac{N(N+1)(2N+1)}{6}\left(\frac{1}{N}-\frac{1}{2N^2}+\frac{1}{6N^3}+o\left(\frac{1}{N^4}\right)\right)\\
&\qquad -\frac{N(N+2)}{2}-\frac{1}{6}H_N\\
&= \frac{N(N+1)}{6}+\frac{N}{2}+\frac{1}{6}H_N+\frac{N^2}{3}+\frac{N}{3}-\frac{1}{36}+o\left(\frac{1}{N}\right)-\frac{N(N+2)}{2}-\frac{1}{6}H_N\\
&= -\frac{1}{36}+o\left(\frac{1}{N}\right).
\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty}\left[n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)-n+\frac{1}{2}-\frac{1}{6n}\right]=-\frac{1}{36}.$$

Second solution by the authors

The sum of the series is

$$-\frac{1}{36}.$$

We need Abel's summation formula, which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers and $A_n = \sum\limits_{k=1}^{n} a_k$, then $\sum\limits_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum\limits_{k=1}^{n} A_k(b_k - b_{k+1})$, or, the infinite version

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1}). \tag{1}$$

We apply formula (1) with $a_n = n^2$ and $b_n = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3}$ and we have, since $A_n = \frac{1}{6}n(n+1)(2n+1)$ and

$$\begin{aligned} b_n - b_{n+1} &= \frac{1}{(n+1)^2} - \frac{1}{n} + \frac{1}{n+1} + \frac{1}{2n^2} - \frac{1}{2(n+1)^2} - \frac{1}{6n^3} + \frac{1}{6(n+1)^3} \\ &= -\frac{1}{6n^3(n+1)^3}, \end{aligned}$$

that

$$\begin{aligned} &\sum_{n=1}^{\infty}\left[n^2\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) - n + \frac{1}{2} - \frac{1}{6n}\right] \\ &= \sum_{n=1}^{\infty} n^2\left[\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3}\right] \\ &= \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6}\left[\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2} - \frac{1}{n+1} + \frac{1}{2(n+1)^2} - \frac{1}{6(n+1)^3}\right] \\ &\quad - \frac{1}{36}\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2} \\ &= -\frac{1}{36}\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ &= -\frac{1}{36}. \end{aligned}$$

We used in the preceding calculations that

$$\begin{aligned} &\lim_{n\to\infty} n(n+1)(2n+1)\left[\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2} - \frac{1}{n+1} + \frac{1}{2(n+1)^2} - \frac{1}{6(n+1)^3}\right] \\ &= 0. \end{aligned}$$

To prove that the preceding limit holds, first we show that $\lim\limits_{n\to\infty} n^3 b_n = 0$. We use Stolz–Cesaro Theorem, the 0/0 case. We have

$$\lim_{n\to\infty} n^3 b_n = \lim_{n\to\infty}\frac{b_n}{\frac{1}{n^3}} = \lim_{n\to\infty}\frac{b_{n+1} - b_n}{\frac{1}{(n+1)^3} - \frac{1}{n^3}} = \lim_{n\to\infty}\frac{\frac{1}{6n^3(n+1)^3}}{\frac{-3n^2-3n-1}{n^3(n+1)^3}} = 0.$$

This implies that $\lim\limits_{n\to\infty} n(n+1)(2n+1)b_{n+1} = \lim\limits_{n\to\infty}\frac{n(2n+1)}{(n+1)^2}\cdot\lim\limits_{n\to\infty}(n+1)^3 b_{n+1} = 0.$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U586. Find all functions $f, g : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x)f(x+y) = f(y)^2 f(x-y)^2 g(y).$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let $A = \{x \in Q : f(x) \neq 0\}$. Note that $0 \in A$ otherwise, setting $y = 0$ we find that $f(x) = 0$. Let $y \in A$ setting $x = 0$ we find that $f(0)f(y) = f(y)^2 f(-y)^2 g(y)$. Hence, $-y \in A$. Moreover, g doesn't vanish at A. If $x, y \in A$ replacing x by $x+y$ to obtain $f(x+y)f(x+2y) = f(x)^2 f(y)^2 g(y)$. Thus, $x+y \in A$. Letting $x = 0, y \in A$ then

$$f(0) = f(y)f(-y)^2 g(y), y \in A.$$

Replacing y by $-y$ to obtain $f(0) = f(-y)f(y)^2 g(-y), y \in A$. Hence,

$$f(-y) = f(y)\frac{g(-y)}{g(y)} \quad y \in A.$$

There is a sub-group of rational numbers such that $g(x) \neq 0$. Hence,

$$\frac{g(x)g(x+y)}{g(-x)^2 g(-x-y)^2} = f(0)^2 \frac{g(y)^5}{g(-y)^4} \frac{g(x-y)^2}{g(-(x-y))^4}, \quad x, y \in A$$

Letting $x = y, x = -y$ to obtain

$$\frac{g(2x)}{g(-2x)^2} = \frac{f(0)^2}{g(0)^2} \frac{g(x)^4}{g(-x)^4}, \quad x \in A.$$

And,

$$\left(\frac{g(2x)}{g(-2x)^2}\right)^2 = \frac{1}{g(0)f(0)^2} \frac{g(x)^5}{g(-x)^7}, \quad x \in A.$$

That is,

$$g(-x) = \frac{g(0)}{f(0)^2} \frac{1}{g(x)}, \quad x \in A.$$

Hence,

$$g(x)g(x+y) = \sqrt[3]{\frac{f(0)^{10}}{g(0)^4}} g(y)^3 g(x-y)^2, \quad x, y \in A.$$

Changing y by $-y$ to obtain

$$g(x)g(x-y) = \sqrt[3]{\frac{f(0)^{10}}{g(0)^4} \frac{g(x+y)^2}{g(y)^3}}, x, y \in A.$$

Thus, for all $x, y \in A$:

$$g(x)g(y) = Cg(x+y), C = \sqrt[3]{\frac{g(0)^2}{f(0)^2}}$$

Moreover, setting $x = y = 0$ to obtain $g(0)f(0)^2 = 1$. That is,

$$C = g(0)$$

Defining $h(x) = \frac{g(x)}{g(0)}$ we would obtain $h(x+y) = h(x)h(y)$. Hence, $g(x) = bc^x$. Finally,

$$f(x) = \sqrt[3]{\frac{f(0)^5}{g(0)^2}} g(x) = f(0)g(x) = dc^x, bd^2 = 1.$$

Thus, there is a sub-group A of rational numbers such that $f(x) = dc^x$ for all $x \in A$ and otherwise zero, And $g(x) = bc^x$ and g takes arbitrary numbers otherwise. Moreover, $bd^2 = 1$.

U587. For $x, y \geq 5$ show that

$$\left(\frac{1}{x}\right)^{\frac{1}{x}}\left(\frac{1}{y}\right)^{\frac{1}{y}} \leq \left(\frac{4}{x^2+y^2}\right)^{\frac{2}{x+y}}$$

Proposed by Toyesh Prakash Sharma, Agra College, India

Solution by Arkady Alt, San Jose, CA, USA
We have

$$\left(\frac{1}{x}\right)^{\frac{1}{x}}\left(\frac{1}{y}\right)^{\frac{1}{y}} \leq \left(\frac{4}{x^2+y^2}\right)^{\frac{2}{x+y}} \iff x^{\frac{1}{x}}y^{\frac{1}{y}} \geq \left(\frac{x^2+y^2}{4}\right)^{\frac{2}{x+y}}$$

$$\iff \frac{\ln x}{x} + \frac{\ln y}{y} \geq \frac{\ln\left(\frac{x^2+y^2}{4}\right)}{\frac{x+y}{2}} \iff \frac{\frac{\ln x^2}{x} + \frac{\ln y^2}{y}}{2} \geq \frac{\ln\left(\frac{x^2+y^2}{4}\right)}{\frac{x+y}{2}}.$$

Since $\left(\frac{\ln x^2}{x}\right)'' = \frac{2(2\ln x - 3)}{x^3} > 0$ if $x \geq 5$ (because $2\ln 5 > 3 \iff 25 > e^3$) then $\frac{\ln x^2}{x}$ is concave up on $[5, \infty)$ and , therefore,

$$\frac{\frac{\ln x}{x} + \frac{\ln y}{y}}{2} \geq \frac{\ln \frac{x+y}{2}}{\frac{x+y}{2}} \iff \frac{\ln x}{x} + \frac{\ln y}{y} \geq \frac{2\ln\frac{x+y}{2}}{\frac{x+y}{2}} = \frac{\ln\left(\frac{(x+y)^2}{4}\right)}{\frac{x+y}{2}}.$$

Noting that $(x+y)^2 \geq x^2 + y^2$ for $x, y \geq 0$ (with equality iff $xy = 0$) we finally obtain that

$$\frac{\ln x}{x} + \frac{\ln y}{y} > \frac{\ln\left(\frac{x^2+y^2}{4}\right)}{\frac{x+y}{2}}.$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA , USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U588. Prove that

$$\lim_{n\to\infty} \beta^{-\frac{1}{n}}(n\pi, n\pi) = 4^{\pi},$$

where $\beta(x,y)$ is the Euler integral of the first kind.

Proposed by Ankush Kumar Parcha, India

Solution by Brian Bradie, Christopher Newport University, Newport News, VA , USA
In terms of the Gamma function,

$$\beta(n\pi, n\pi) = \frac{\Gamma(n\pi)^2}{\Gamma(2n\pi)}.$$

For large z,

$$\Gamma(z) \sim e^{-z} z^z \left(\frac{2\pi}{z}\right)^{1/2},$$

so

$$\beta(n\pi, n\pi) \sim \frac{e^{-2n\pi}(n\pi)^{2n\pi} \cdot \frac{2}{n}}{e^{-2n\pi}(2n\pi)^{2n\pi} \cdot \frac{1}{\sqrt{n}}} = \frac{2}{\sqrt{n}} \cdot \frac{1}{4^{n\pi}}.$$

Hence,

$$\lim_{n\to\infty} \beta^{-\frac{1}{n}}(n\pi, n\pi) = 4^{\pi} \lim_{n\to\infty} \sqrt[n]{\frac{\sqrt{n}}{2}} = 4^{\pi}.$$

Also solved by Perfetti, Università degli studi di Tor Vergata Roma, Italy; Henry Ricardo, Westchester Area Math Circle, USA.

O583. Let a, b, c be real numbers. Prove that

$$a^3 + b^3 + c^3 - 3abc \leq (a^2 + b^2 + c^2 + 2)^{3/2} - 3(a + b + c)$$

with equality if and only if $ab + bc + ca = 1$.

Proposed by Florin Pop, USA and Gigi Stoica, Canada

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)^3 - 3(a + b + c)(ab + bc + ca)$$

Let's define $a + b + c = 3u$, $ab + bc + ca = 3v^2$, $abc = w^3$. The variable v^2 can be negative. The AGM yields $u^2 \geq v^2$.

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 9u^2 - 6v^2$$

If $a^3 + b^3 + c^3 - 3abc < 0$ there isn't anything to prove. If $a^3 + b^3 + c^3 - 3abc \geq 0$ the inequality reads as

$$27u^3 - 27uv^2 + 9u \leq (2 + 9u^2 - 6v^2)^{3/2} \iff (3v^2 - 1)^2(27u^2 - 24v^2 + 8) \geq 0$$

and this holds true thanks to $u^2 \geq v^2$ with inequality if and only if $v^2 = 1/3$ that is $ab + bc + ca = 1$.

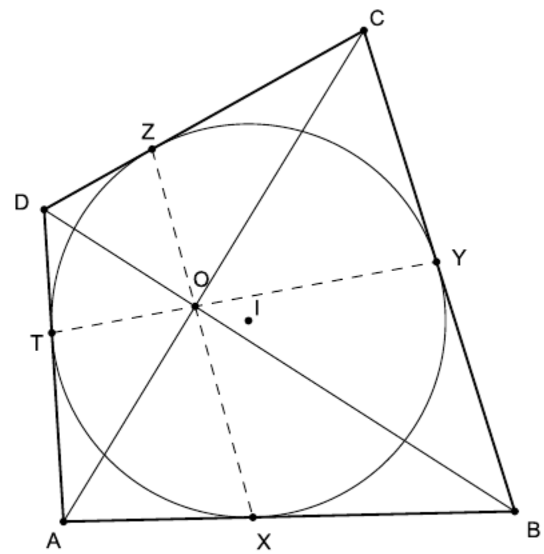
Also solved by Arnab Sanyal, Kolkata,West Bengal,India; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Petrakis Emmanouil, 2nd High School, Agrinio, Greece.

O584. Let $ABCD$ be a circumscriptible quadrilateral and let $\{O\} = AC \cap BD$. Let r_1, r_2, r_3, r_4 be the inradius, respectively R_1, R_2, R_3, R_4 be the radius of O-excircles of triangles AOB, BOC, COD, DOA. Prove that

$$\frac{AB}{1-\frac{r_1}{R_1}} + \frac{CD}{1-\frac{r_3}{R_3}} = \frac{BC}{1-\frac{r_2}{R_2}} + \frac{DA}{1-\frac{r_4}{R_4}}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author



Let X, Y, Z, T be the tangent points of the incircle with the sides AB, BC, CD, DA. According to Brianchon's theorem we have that the lines AC, BD, XZ, YT are concurrent at O. Denote by $a = AX = AT$, $b = BX = BY$, $c = CY = CZ$ and $d = DZ = DT$.

Using Law of Sinus in triangle AOT and COY we have

$$\frac{AT}{\sin \angle AOT} = \frac{AO}{\sin \angle ATO} \Longleftrightarrow \frac{AO}{a} = \frac{\sin \angle ATO}{\sin \angle AOT}$$

and

$$\frac{CY}{\sin \angle COY} = \frac{CO}{\sin \angle CYO} \Longleftrightarrow \frac{CO}{c} = \frac{\sin \angle CYO}{\sin \angle COY}.$$

Therefore, because $\sin \angle ATO = \sin \angle DTO = \sin \angle CYO$, we deduce that

$$\frac{AO}{a} = \frac{CO}{c} \overset{not}{=} x > 0 \Longleftrightarrow AO = ax,\ CO = cx.$$

Similarly we have

$$\frac{BO}{b} = \frac{DO}{d} \overset{not}{=} y > 0 \Longleftrightarrow BO = by,\ DO = dy.$$

Denote by $t = \sin \angle AOB = \sin \angle BOC = \sin \angle COD = \sin \angle DOA$ and we have

$$r_1 = \frac{K_{OAB}}{s_{OAB}} = \frac{OA \cdot OB \cdot t}{OA + OB + AB} = \frac{abxyt}{ax + by + a + b},$$

$$R_1 = \frac{K_{OAB}}{s_{OAB} - AB} = \frac{OA \cdot OB \cdot t}{OA + OB - AB} = \frac{abxyt}{ax + by - a - b},$$

so

$$\frac{r_1}{R_1} = \frac{ax + by - a - b}{ax + by + a + b} \Longleftrightarrow ax + by = \frac{2(a+b)}{1 - \frac{r_1}{R_1}} - a - b,$$

and similarly

$$\frac{r_2}{R_2} = \frac{cx + by - c - b}{cx + by + c + b} \Longleftrightarrow cx + by = \frac{2(c + b)}{1 - \frac{r_2}{R_2}} - c - b,$$

$$\frac{r_3}{R_3} = \frac{cx + dy - c - d}{cx + dy + c + d} \Longleftrightarrow cx + dy = \frac{2(c + d)}{1 - \frac{r_3}{R_3}} - c - d,$$

$$\frac{r_4}{R_4} = \frac{ax + dy - a - d}{ax + dy + a + d} \Longleftrightarrow ax + dy = \frac{2(a + d)}{1 - \frac{r_4}{R_4}} - a - d.$$

Hence, we have

$$(a + c)x + (b + d)y = \frac{2(a + b)}{1 - \frac{r_1}{R_1}} - a - b + \frac{2(c + d)}{1 - \frac{r_3}{R_3}} - c - d,$$

and also

$$(a + c)x + (b + d)y = \frac{2(c + b)}{1 - \frac{r_2}{R_2}} - c - b + \frac{2(a + d)}{1 - \frac{r_4}{R_4}} - a - d,$$

so the conclusion follows.

O585. Prove that in any triangle ABC

$$\frac{9}{16}\left(\frac{12r^2}{R^2}-1\right) \le \sum \cos A \sin B \sin C \le \frac{9}{4}\left(\frac{3}{4}-\frac{2r^2}{R^2}\right)$$

Proposed by Marian Ursărescu, Roman, Romania

Solution by Arkady Alt, San Jose, CA, USA

Note that

$$\frac{9}{16}\left(\frac{12r^2}{R^2}-1\right) \le \sum \cos A \sin B \sin C \le \frac{9}{4}\left(\frac{3}{4}-\frac{2r^2}{R^2}\right) \iff$$

$$\frac{9}{16}\left(\frac{12r^2}{R^2}-1\right)\cdot 4R^2 \le 4R^2 \sum \cos A \sin B \sin C \le \frac{9}{4}\left(\frac{3}{4}-\frac{2r^2}{R^2}\right)\cdot 4R^2$$

$$\iff \frac{9}{4}\left(12r^2-R^2\right) \le \sum bc\cos A \le 9\left(\frac{3}{4}R^2-2r^2\right)$$

$$\iff \frac{9}{4}\left(12r^2-R^2\right) \le \frac{1}{2}\sum a^2 \le 9\left(\frac{3}{4}R^2-2r^2\right)$$

$\iff 9\left(12r^2-R^2\right) \le 2\sum a^2 \le 9\left(3R^2-4r^2\right) \iff 9\left(12r^2-R^2\right) \le 4\left(s^2-4Rr-r^2\right) \le 9\left(3R^2-4r^2\right)$

(because $\sum bc\cos A = \frac{1}{2}\sum c\left(b\cos A + a\cos B\right) = \frac{1}{2}\sum c^2 = \frac{1}{2}\sum a^2$ and $\sum a^2 = 2\left(s^2-4Rr-r^2\right)$)

Since $16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2$ (Gerretsen's Inequalities)
and $R \ge 2r$ (Euler's Inequality) then
$4\left(s^2-4Rr-r^2\right)-9\left(12r^2-R^2\right) \ge 4\left(16Rr-5r^2-4Rr-r^2\right)-9\left(12r^2-R^2\right) = 3\left(R-2r\right)\left(3R+22r\right) \ge 0$
and
$9\left(3R^2-4r^2\right)-4\left(s^2-4Rr-r^2\right) \ge 9\left(3R^2-4r^2\right)-4\left(4R^2+4Rr+3r^2-4Rr-r^2\right) =$

$$11\left(R-2r\right)\left(R+2r\right) \ge 0$$

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Piteşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece.

O586. Diagonals AC and BD of convex quadrilateral $ABCD$ intersect at point E. On the exterior of the quadrilateral triangles ABP and CDQ are constructed, such that

$$\angle PAB = \angle DAE, \quad \angle PBA = \angle CBE$$

$$\angle QDC = \angle ADE, \quad \angle QCD = \angle BCE$$

Prove that points P, E, Q are collinear.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by the author

We prove the result in case lines AD and BC are not parallel and intersect at point S

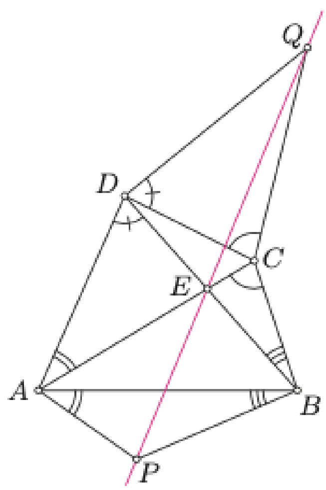


Fig. 2

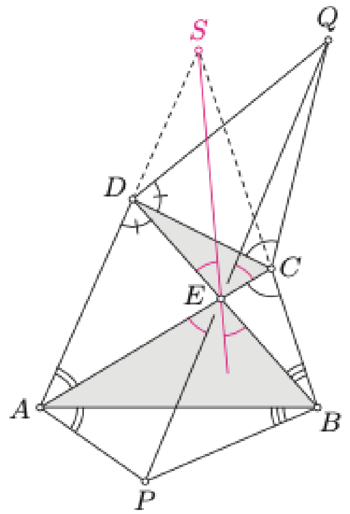


Fig. 3

The proof for $AD \parallel BC$ is analogous.

Since

$$\angle PAB = \angle DAE \quad \text{and} \quad \angle PBA = \angle CBE,$$

it follows that points P and S are isogonal in triangles ABE. So line PE is symmetric to line SE with respect to the angle bisector of AEB.

Similarly, from

$$\angle QDC = \angle ADE \quad \text{and} \quad \angle QCD = \angle BCE$$

we infer that points Q and S are isogonal in triangle CDE. So QE is symmetric to SE with respect to line containing the angle bisector of AEB. It follows that lines PE and QE coincide, which completes the proof.

O587. Let a, b, c, d be positive real numbers. Prove that

$$22a + 25b + 30c + 30d \geq 360\sqrt[3]{\frac{abcd}{2a + 5b + 10c + 30d}}.$$

When does equality hold?

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

$$2a+5b+10c+30s = 10\cdot\frac{a}{5}+20\cdot\frac{b}{4}+30\cdot\frac{c}{3}+60\cdot\frac{d}{2} \geq 120\sqrt[120]{\left(\frac{a}{5}\right)^{10}\cdot\left(\frac{b}{4}\right)^{20}\cdot\left(\frac{c}{3}\right)^{30}\cdot\left(\frac{d}{2}\right)^{60}} = 120\sqrt[12]{\left(\frac{a}{5}\right)\cdot\left(\frac{b}{4}\right)^{2}\cdot\left(\frac{c}{3}\right)^{3}\cdot\left(\frac{d}{2}\right)^{6}}$$

Namely $\sqrt[3]{\frac{abcd}{2a+5b+10c+30d}} \leq \sqrt[3]{\frac{abcd}{120\sqrt[12]{\left(\frac{a}{5}\right)\cdot\left(\frac{b}{4}\right)^{2}\cdot\left(\frac{c}{3}\right)^{3}\cdot\left(\frac{d}{2}\right)^{6}}}} = \sqrt[36]{\left(\frac{a}{5}\right)\cdot\left(\frac{b}{4}\right)^{2}\cdot\left(\frac{c}{3}\right)^{3}\cdot\left(\frac{d}{2}\right)^{6}}$

Now,

$22a + 25b + 30c + 30d = 10\left[11\left(\frac{a}{5}\right) + 10\left(\frac{b}{4}\right) + 9\left(\frac{c}{3}\right) + 6\left(\frac{d}{2}\right)\right]$

$$10\cdot 36\sqrt[36]{\left(\frac{a}{5}\right)^{11}\cdot\left(\frac{b}{4}\right)^{10}\cdot\left(\frac{c}{3}\right)^{9}\cdot\left(\frac{d}{2}\right)^{6}} \geq 360\sqrt[3]{\frac{abcd}{2a+5b+10c+30d}}.$$

Equality holds at $a = 5, b = 4, c = 3, d = 2$.

Also solved by An Zhenping, Xianyang Normal University, China; Jiang Lianjun, GuiLin, China; Petrakis Emmanouil, 2nd High School, Agrinio, Greece.

O588. Let a, b, c, d be positive real numbers. Prove that

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}=1$$

Prove that

$$ab+ac+ad+bc+bd+cd+18\geq 6(a+b+c+d).$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

With substitutions $\frac{1}{1+a}\to a, \frac{1}{1+b}\to b, \frac{1}{1+c}\to c, \frac{1}{1+d}\to d$ it follows that $a+b+c+d=1$ and the inequality is equivalent with

$$\sum_{cyc}\frac{(1-a)(1-b)}{ab}-6\sum_{cyc}\frac{1-a}{a}\geq -18,$$

$$ab+ac+ad+bc+bd+cd-9(abc+bcd+cda+dab)+48abcd\geq 0.$$

We homogenize the inequality and and get the following

$$\left(\sum_{cyc}a\right)^2\sum_{cyc}ab-9\sum_{cyc}a\sum_{cyc}abc+48abcd\geq 0.$$

Without loss of generality assume that $a+b+c+d=4$. The inequality becomes

$$4\sum_{cyc}ab-9\sum_{cyc}abc+12abcd\geq 0.$$

Now, let $x=a-1, y=b-1, z=c-1, t=d-1, x+y+z+t=0, x,y,z,t\in[-1,3]$. We need to prove that

$$-2\sum_{cyc}xy+3\sum_{cyc}xyz+12xyzt\geq 0,$$

$$x^2+y^2+z^2+t^2+3\sum_{cyc}xyz+12xyzt\geq 0$$

or

$$x^2+y^2+z^2+t^2+x^3+y^3+z^3+t^3+12xyzt\geq 0.$$

Assuming that $x\leq y\leq z\leq t$ it is clear $x\leq 0\leq t$. We have the following cases:
Case 1: $0\leq y$, then $y+z+t=-x\leq 1\Rightarrow yzt<1$, we need to prove that

$$x^2(1+x)+(y^2+z^2+t^2)+(y^3+z^3+t^3)\ ge-12xyzt.$$

However, $3(y^2+z^2+t^2)\geq(y+z+t)^2=x^2, 9(y^3+z^3+t^3)\geq(y+z+t)^3=-x^3$ and $27yzt\leq(y+z+t)^3=-x^3$ so it suffices to prove that

$$x^2(1+x)+\frac{x^2}{3}-\frac{x^3}{9}\geq\frac{4x^4}{9},$$

or

$$4x^2(1+x)(3-x)\geq 0,$$

which is true.

Case 2: $y \le 0 \le z$, clearly true because $x^2 + x^3 \ge 0, y^2 + y^3 \ge 0$ and $xyzt \ge 0$.
Case 3: $z \le 0$, then $-x - y - z = t \le 3$ and

$$27(-x)(-y)(-z) \le (-x - y - z)^3 = t^3 \Rightarrow -12xyzt \le \frac{4t^3}{9}$$

so we need to prove that

$$x^2 + y^2 + z^2 + t^2 + x^3 + y^3 + z^3 + t^3 \ge \frac{4t^4}{9},$$

or

$$x^2(1 + x) + y^2(1 + y) + z^2(1 + z) + \frac{t^2(3 - t)(4t + 3)}{9} \ge 0,$$

which is true. The equality holds when $x = y = z = t = 0$ or $x = -1, y = z = t = \frac{1}{3}$ or $x = y = z = -1, t = 3$.

Also solved by Jiang Lianjun, GuiLin, China; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Corneliu Mănescu-Avram, Ploieşti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Petrakis Emmanouil, 2nd High School, Agrinio, Greece.