The missing dominant terms in a Fourier series

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1 Preliminaries

In this work, we show that the dominant terms in standard Fourier representations of a function are often missing in the classical treatments on this subject such as Reference [1]. We first discuss some background material, namely, the solution of a partial differential equation by the method of separation of variables.

Consider the domain to be a rectangular region, and let the governing partial differential equation for some physical variable $T(x, y)$, where (x, y) are Cartesian coordinates, be given by

$$\boldsymbol{\nabla}^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{1}$$

This governing equation is to be solved subject to appropriate Dirichlet or Neumann boundary conditions on the edges of the rectangular domain $[-L, L] \times [0, H]$. By the standard separation of variables method, we write $T = X(x)Y(y)$ which on substituting into Eqn. (1) yields

$$\frac{X''x}{X(x)} = -\frac{Y''(y)}{Y(y)} = -k^2, \tag{2}$$

where k is a constant. By superposing the solutions obtained for the infinite eigenvalues k (see Section (34) of [1]), we get the general solution as

$$\begin{aligned} T &= c_0 + c_1 x + c_2 y + c_3 xy \\ &\quad + \sum_{n=1}^{\infty} \Big\{ A_n \sin(k_{1n}x)\sinh(k_{1n}y) + B_n \sin(k_{2n}x)\cosh(k_{2n}y) \\ &\quad + C_n \cos(k_{3n}x)\sinh(k_{3n}y) + D_n \cos(k_{4n}x)\cosh(k_{4n}y) \\ &\quad + E_n \sinh(k_{5n}x)\sin(k_{5n}y) + F_n \sinh(k_{6n}x)\cos(k_{6n}y) \\ &\quad + G_n \cosh(k_{7n}x)\sin(k_{7n}y) + H_n \cosh(k_{8n}x)\cos(k_{8n}y). \end{aligned} \tag{3}$$

The nonseries terms in Eqn. (3) correspond to $k = 0$ in Eqn. (2), and are *linearly independent* of the series terms which correspond to a nonzero k value. We shall see that these nonseries terms play a key role in what follows.

Similarly, an axisymmetric separable solution in terms of the radial and axial coordinates r and z in a cylindrical coordinate system is

$$\begin{aligned} T = {} & c_0 + c_1 \log r + c_2 z + c_3 z \log r \\ & + \sum_{n=1}^{\infty} \Big\{ A_n I_0(k_{1n} r) \cos(k_{1n} z) + B_n K_0(k_{2n} r) \cos(k_{2n} z) \\ & + C_n I_0(k_{3n} r) \sin(k_{3n} z) + D_n K_0(k_{4n} r) \sin(k_{4n} z) \\ & + E_n J_0(k_{5n} r) \sinh(k_{5n} z) + F_n Y_0(k_{6n} r) \sinh(k_{6n} z) \\ & + G_n J_0(k_{7n} r) \cosh(k_{7n} z) + H_n Y_0(k_{8n} r) \cosh(k_{8n} z) \Big\}, \end{aligned} \tag{4}$$

where J_0 and Y_0 denote Bessel functions of the first and second kind, and I_0 and K_0 denote modified Bessel functions of the first and second kind. Once again, the nonseries terms in Eqn. (4) correspond to a zero separation constant, and will play a key role in what follows.

We see that the solution in Eqn. (3) automatically satisfies the governing equation, and thus the constants are determined using the boundary conditions. If the boundary condition on $y = 0$ is specified as $T|_{y=0} = f(x)$, then from Eqn. (3) and after renaming the constants, we get a representation for $f(x)$ of the form

$$c_0 + c_1 x + \sum_{n=1}^{\infty} [A_n \cos(\alpha_n x) + B_n \sin(\alpha_n x)] = f(x). \tag{5}$$

As an example, if the domain is the rectangular region $[-L, L] \times [0, H]$, if $\partial T/\partial x|_{x=-L} = \partial T/\partial x|_{x=L} = 0$, and if $f(x)$ is specified to be an odd function, then $c_0 = 0$, $A_n = 0$ for all n, and $\alpha_n = (2n-1)\pi/(2L)$.

In a similar manner if on a cylindrical domain the boundary condition on $z = 0$ is specified as $T|_{z=0} = f(r)$, then from Eqn. (4), we obtain

$$c_0 + c_1 \log r + \sum_{n=1}^{\infty} [A_n J_0(\alpha_n r) + B_n Y_0(\alpha_n r)] = f(r). \tag{6}$$

Note the presence of the term $c_0 + c_1 x$ in Eqn. (5), and the presence of the term $c_0 + c_1 \log r$ in Eqn. (6), which are the dominant missing terms in the usual treatments as elaborated in the following sections.

We wish to point out that our goal in this work is not to 'accelerate' convergence of a Fourier series by adding, as often done in the literature, polynomial or other terms that may not all be linearly independent of the Fourier series terms, and in addition, whose choice is not unique (see, for example, the polynomial subtraction technique discussed in Reference [2]), but rather to *mathematically* determine what a correct Fourier representation of a function ought to be; the accelerated convergence that we do obtain as seen from the plots in the following sections is merely a byproduct of the correct representation.

2 The missing dominant terms in a Fourier series

Suppose that we want to represent an odd function[1] $f(x)$ on $[-L, L]$ in terms of functions $\sin(n\pi x/L)$. Then the standard representation is (see, for example, Example (2), page 54 of [1])

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x), \tag{7}$$

with

$$C_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \tag{8}$$

However, based on Eqn. (5), we see that the correct representation is (since we are considering odd functions here, $c_0 = 0$)

$$c_1 x + \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x), \tag{9}$$

with

$$\begin{aligned} c_1 &= \frac{f(L)}{L}, \\ \tilde{f}(x) &= f(x) - c_1 x, \\ C_n &= \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin \frac{n\pi x}{L}\, dx. \end{aligned} \tag{10}$$

As an example, if $f(x) = (x/L)^7$, then from Eqn. (7), we have

$$C_n = \frac{2(-1)^{n+1}\left[(n\pi)^6 - 42(n\pi)^4 + 840(n\pi)^2 - 5040\right]}{(n\pi)^7}, \tag{11}$$

[1]Throughout this work, we assume the functions to be represented to be continuous.

while from Eqn. (9), we have

$$c_1 = \frac{1}{L},$$
$$C_n = \frac{84(-1)^n\left[(n\pi)^4 - 20(n\pi)^2 + 120\right]}{(n\pi)^7}. \tag{12}$$

Thus, the two representations (classical and proposed) are

$$\left(\frac{x}{L}\right)^7 \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\left[(n\pi)^6 - 42(n\pi)^4 + 840(n\pi)^2 - 5040\right]\sin(n\pi x/L)}{(n\pi)^7}, \tag{13a}$$

$$\left(\frac{x}{L}\right)^7 \sim \frac{x}{L} + \sum_{n=1}^{\infty} \frac{84(-1)^n\left[(n\pi)^4 - 20(n\pi)^2 + 120\right]\sin(n\pi x/L)}{(n\pi)^7}. \tag{13b}$$

Fig. 1 clearly shows the rapid convergence of the proposed representation as compared to the classical one besides, of course, removing the error at the endpoints. This figure also shows why convergence is a norm sense (say the L_2 norm) is misleading. For an infinite number of terms, the representation given by Eqn. (13a) converges to the function at all points in the domain except at the endpoints, where the error is one hundred percent. However, since the measure of the endpoints is zero, this error is not captured by the L_2 norm. Thus, it makes more sense to consider pointwise convergence while considering the two representations in Eqns. (13).

Now consider the representation of an even function $f(x)$ over $[-L, L]$ in terms of $\cos[(2n-1)\pi x/(2L)]$. By the classical representation, we have

$$\sum_{n=1}^{\infty} C_n \cos\frac{(2n-1)\pi x}{2L} = f(x), \tag{14}$$

with

$$C_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{(2n-1)\pi x}{2L}. \tag{15}$$

But based on Eqn. (9), the correct representation is

$$c_0 + \sum_{n=1}^{\infty} C_n \cos\frac{(2n-1)\pi x}{2L} = f(x), \tag{16}$$

with

$$c_0 = f(L),$$

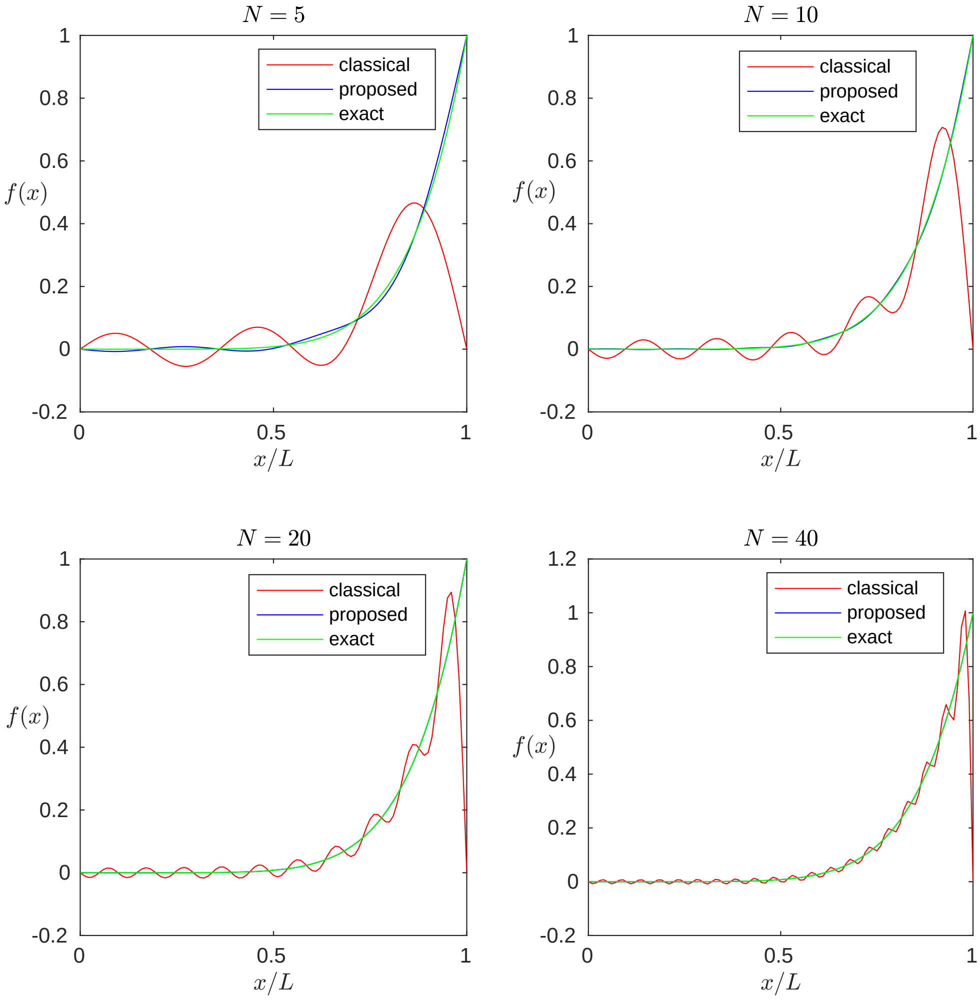

Figure 1: Plots of the representations in Eqns. (13) for different number of terms N in the Fourier series.

$$\tilde{f}(x) := f(x) - f(L),$$
$$C_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\frac{(2n-1)\pi x}{2L}.$$

In the classical framework, the c_0 term is present when the even function is being approximated by $\cos(n\pi x/L)$ but not when it is being approximated by $\cos[(2n-1)\pi x/(2L)]$; e.g., see Eqn. (4.69) of [3].

As an example, if $f(x) = (x/L)^6$, then by Eqn. (14) and with $\beta_n := (2n-1)\pi$, we have

$$C_n = \frac{4(-1)^{n+1}\left(\beta_n^6 - 120\beta_n^4 + 5760\beta_n^2 - 46080\right)}{\beta_n^7},$$

while from Eqn. (16), we have

$$c_0 = 1,$$
$$C_n = \frac{480(-1)^n\left(\beta_n^4 - 48\beta_n^2 + 384\right)}{\beta_n^7}.$$

Thus, the two representation (classical and proposed) are

$$\left(\frac{x}{L}\right)^6 \sim \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}\left(\beta_n^6 - 120\beta_n^4 + 5760\beta_n^2 - 46080\right)\cos[\beta_n x/(2L)]}{\beta_n^7}, \tag{17a}$$

$$\left(\frac{x}{L}\right)^6 \sim 1 + \sum_{n=1}^{\infty}\frac{480(-1)^n\left(\beta_n^4 - 48\beta_n^2 + 384\right)\cos[\beta_n x/(2L)]}{\beta_n^7}. \tag{17b}$$

Once again, Fig. 2 clearly shows the rapid convergence of the proposed representation as compared to the classical one besides, of course, removing the error at the endpoints.

As another example, if $f(x) = |x|/L$ on $[-L, L]$, then the classical and proposed representations are

$$\frac{|x|}{L} \sim \sum_{n=1}^{\infty}\frac{4\left[(-1)^{n+1}\beta_n - 2\right]\cos[\beta_n x/(2L)]}{\beta_n^2}, \tag{18a}$$

$$\frac{|x|}{L} \sim 1 - \sum_{n=1}^{\infty}\frac{8\cos[\beta_n x/(2L)]}{\beta_n^2}. \tag{18b}$$

Inspite of the fact that this is an especially difficult function to represent due to the discontinuity in the slope at $x = 0$, we see from Fig. 3 that the proposed representation converges rapidly.

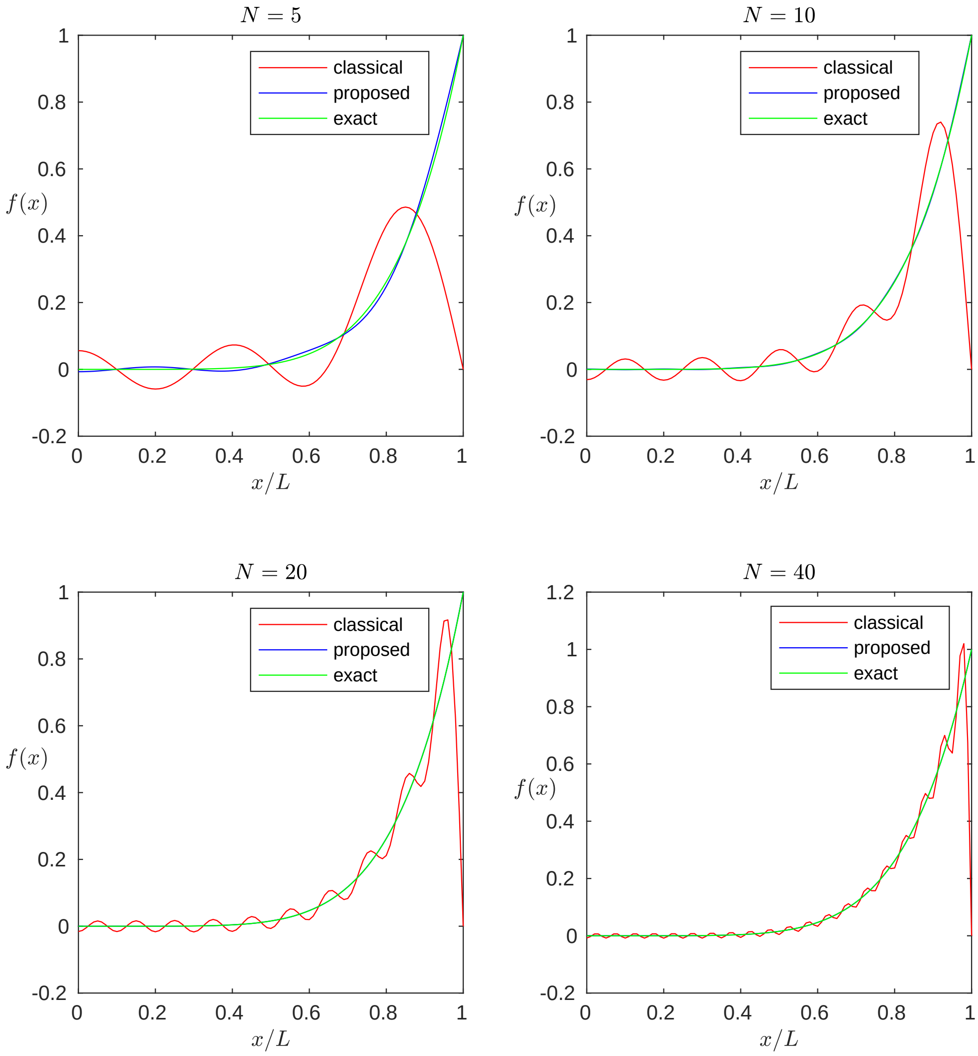

Figure 2: Plots of the representations in Eqns. (17) for different number of terms N in the Fourier series.

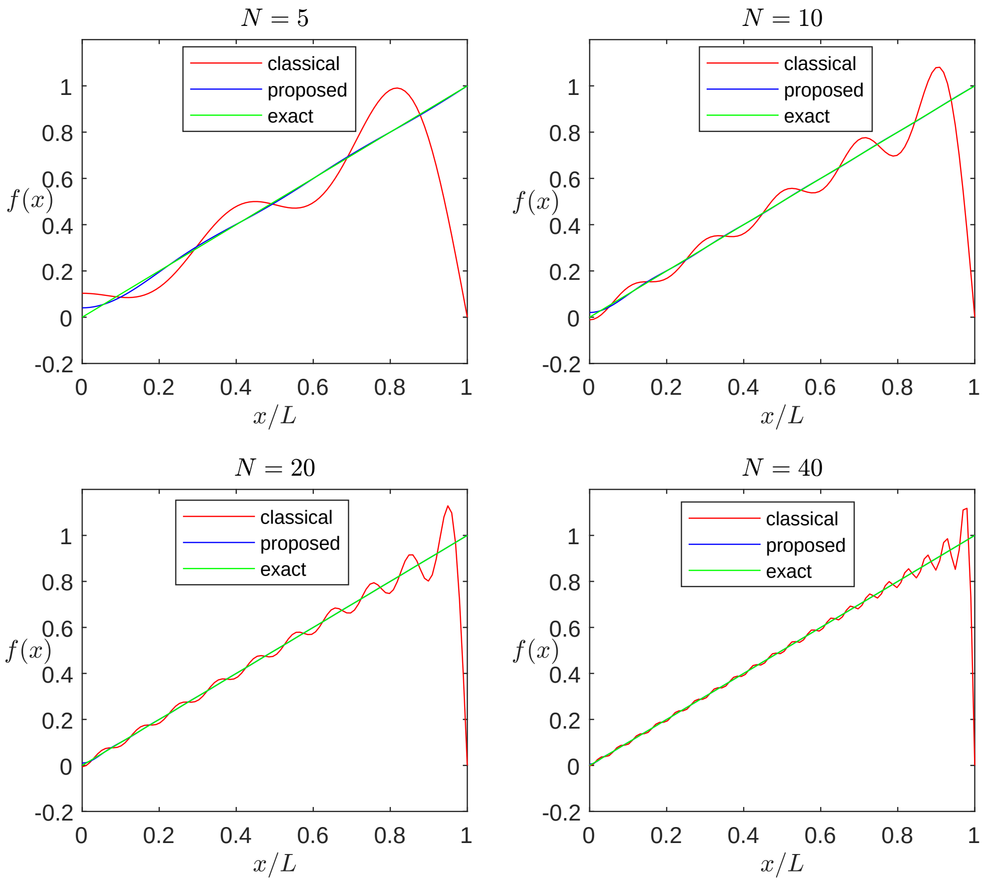

Figure 3: Plots of the representations in Eqns. (18) for different number of terms N in the Fourier series.

Now consider the representation of an odd function $f(x)$ defined on $[-L, L]$ by $\sin[(2n-1)\pi x/(2L)]$. Since we use the derivative of $f(x)$ evaluated at $x = L$, we need to exercise care here. For example, the function

$$f(x) = x\sqrt{1 - \left(\frac{x}{L}\right)^2}, \tag{19}$$

is bounded on $[-L, L]$, but its derivative is not bounded at $x = L$. We do not consider such functions, and restrict ourselves to continuously differentiable functions, i.e., $f(x) \in C^1$. Thus, we now propose

$$C_1 x + \sum_{n=1}^{\infty} C_n \sin\frac{(2n-1)\pi x}{2L} = f(x), \tag{20}$$

with

$$\begin{aligned} C_1 &= f'(L), \\ \tilde{f}(x) &= f(x) - C_1 x, \\ C_n &= \frac{1}{L}\int_{-L}^{L} \tilde{f}(x) \sin\frac{(2n-1)\pi x}{2L}\, dx. \end{aligned} \tag{21}$$

As an example, the classical and proposed approximations for $(x/L)^7$ with $\beta_n := (2n-1)\pi$ are

$$\left(\frac{x}{L}\right)^7 \sim \sum_{n=1}^{\infty} \frac{56(-1)^{n+1}\left(\beta_n^6 - 120\beta_n^4 + 5760\beta_n^2 - 46080\right)\sin[\beta_n x/(2L)]}{\beta_n^8}, \tag{22a}$$

$$\left(\frac{x}{L}\right)^7 \sim \frac{7x}{L} + \sum_{n=1}^{\infty} \frac{6720(-1)^n\left(\beta_n^4 - 48\beta_n^2 + 384\right)\sin[\beta_n x/(2L)]}{\beta_n^8}. \tag{22b}$$

As seen from Fig. 4, the convergence of the proposed approximation is again much faster compared to the classical approximation although not as dramatic as in the previous cases. However, the derivative of the above representations is given by the representation in Eqns. (17) (modulo a multiplicative constant), and, as already pointed out, the difference between the classical and the proposed representations is very significant in this case.

While representing an even function on $[-L, L]$ with $\cos(n\pi x/L)$, the conventional representation given by

$$c_0 + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{L} = f(x), \tag{23}$$

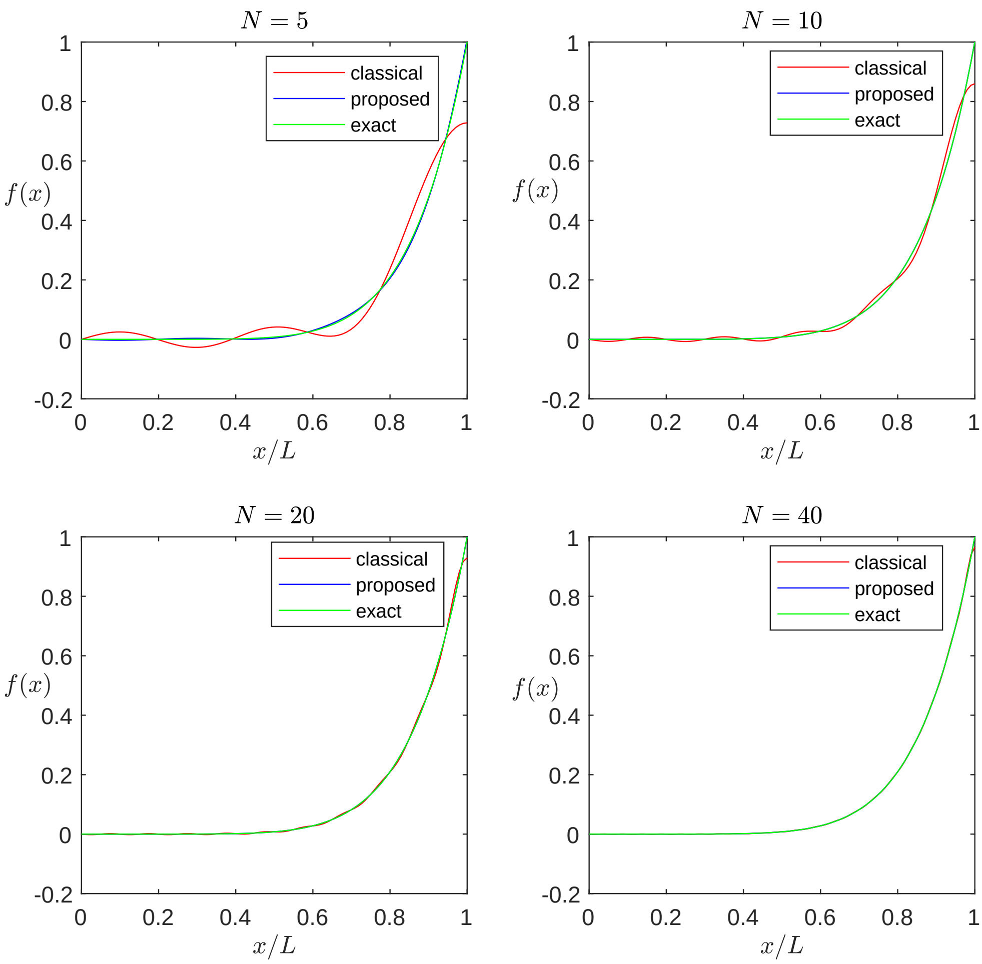

Figure 4: Plots of the representations in Eqns. (22) for different number of terms N in the Fourier series.

with

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$
$$C_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$

already has the constant term c_0, and so in this case, no additional terms need to be added.

If one approximates $|x|/L$ using Eqn. (23), then, with $\beta_n := (2n-1)\pi$, we get

$$\frac{|x|}{L} \sim \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4\cos(\beta_n x/L)}{\beta_n^2}. \tag{24}$$

In this case, as already pointed out, the classical and the proposed representations are the same, and the plots for different values of N are presented in Fig. 5. Once again, the role of the constant term in ensuring rapid convergence is clear from the plots.

As another example, if one wants to approximate the function $f(x)$ on $[0, L]$ using $\sin(n\pi x/L)$, then based on Eqn. (5), we get

$$f(x) \sim f(0) + \frac{[f(L) - f(0)]x}{L} + \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}, \tag{25}$$

where

$$\tilde{f}(x) := f(x) - f(0) - \frac{[f(L) - f(0)]x}{L},$$
$$C_n = \frac{2}{L}\int_0^L \tilde{f}(x)\sin\frac{n\pi x}{L}\,dx.$$

In the classical representation, the nonseries terms would be absent.

As a final example, the representation of a continuously differentiable function on $[0, L]$ with the functions $\cos[(2n-1)\pi x/(2L)]$ is given by

$$C_0 + C_1 x + \sum_{n=1}^{\infty} C_n \cos\frac{(2n-1)\pi x}{2L} = f(x), \tag{26}$$

with

$$\begin{aligned} C_0 &= f(L) - f'(0)L, \\ C_1 &= f'(0), \\ \tilde{f}(x) &= f(x) - C_0 - C_1 x, \\ C_n &= \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\frac{(2n-1)\pi x}{2L}\,dx. \end{aligned} \tag{27}$$

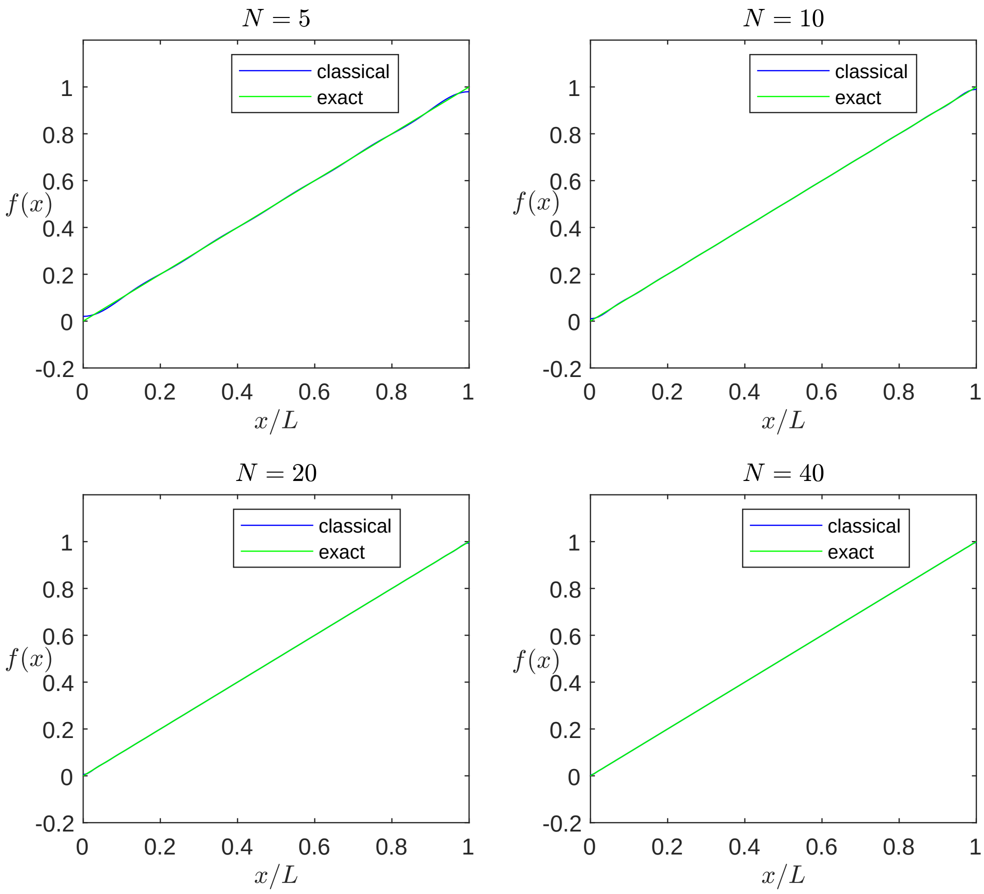

Figure 5: Plots of the representation in Eqn. (24) for different number of terms N in the Fourier series.

3 The missing dominant terms in a Fourier-Bessel series

Consider the domain to be the cross section in the r-z plane of a hollow cylinder of inner and outer radii a and b, respectively, and length L. Let λ_n, $n = 1, 2, \ldots, \infty$, be the positive roots of $J_0(xa)Y_0(xb) - J_0(xb)Y_0(xa)$, and let

$$N_p(r) = J_0(\lambda_n r)Y_0(\lambda_n b) - J_0(\lambda_n b)Y_0(\lambda_n r). \tag{28}$$

Let the prescribed boundary condition on $z = 0$ be given by $T|_{z=0} = f(r)$. We are interested in representing $f(r)$ in terms of $N_p(r)$. The classical representation is

$$\sum_{n=1}^{\infty} A_n N_p(r) = f(r), \tag{29}$$

which, on using the orthogonality of the Bessel functions, leads to

$$A_n = \frac{\pi^2 \lambda_n^2 J_0^2(\lambda_n a)}{2\left[J_0^2(\lambda_n a) - J_0^2(\lambda_n b)\right]} \int_a^b \hat{r} f(\hat{r}) N_p(\hat{r})\, d\hat{r},$$

while the proposed representation, based on a slight rewriting of Eqn. (6), is

$$\frac{c_a \log(b/r)}{\log(b/a)} + \frac{c_b \log(r/a)}{\log(b/a)} + \sum_{n=1}^{\infty} A_n N_p(r) = f(r), \tag{30}$$

where

$$\begin{aligned}
c_a &= f(a), \\
c_b &= f(b), \\
\tilde{f}(r) &:= f(r) - \frac{c_a \log(b/r)}{\log(b/a)} - \frac{c_b \log(r/a)}{\log(b/a)}, \\
A_n &= \frac{\pi^2 \lambda_n^2 J_0^2(\lambda_n a)}{2\left[J_0^2(\lambda_n a) - J_0^2(\lambda_n b)\right]} \int_a^b \hat{r} \tilde{f}(\hat{r}) N_p(\hat{r})\, d\hat{r}.
\end{aligned}$$

As an example, the classical and proposed representations for r^2 are

$$r^2 \sim \sum_{n=1}^{\infty} \frac{\pi J_0(\lambda_n a)\left\{\left[(\lambda_n b)^2 - 4\right] J_0(\lambda_n a) - \left[(\lambda_n a)^2 - 4\right] J_0(\lambda_n b)\right\} N_p(r)}{\lambda_n^2 \left[J_0^2(\lambda_n a) - J_0^2(\lambda_n b)\right]}, \tag{31a}$$

$$r^2 \sim \frac{a^2 \log(b/r) + b^2 \log(r/a)}{\log(b/a)} - \sum_{n=1}^{\infty} \frac{4\pi J_0(\lambda_n a) N_p(r)}{\lambda_n^2 \left[J_0(\lambda_n a) + J_0(\lambda_n b)\right]}. \tag{31b}$$

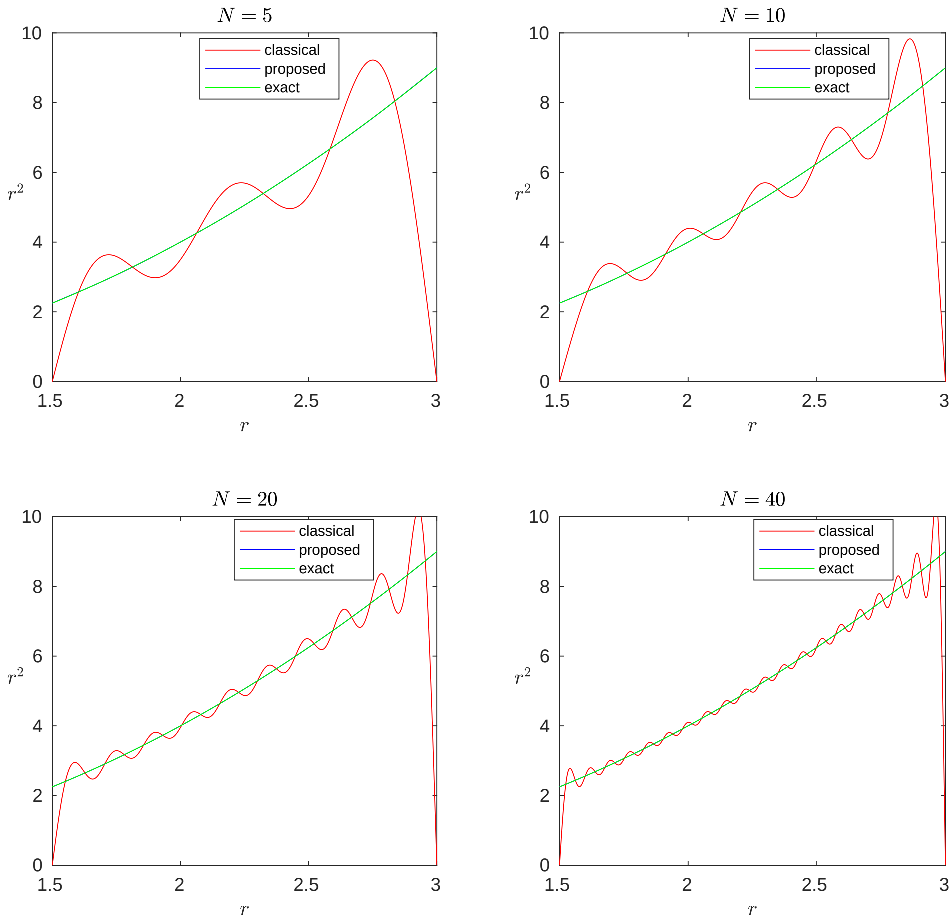

Figure 6: Plots of the representation in Eqn. (31) for different number of terms N in the Fourier–Bessel series.

Once again the plots in Fig. 6 clearly demonstrate the crucial role played by the nonseries terms.

If the domain is a solid cylinder instead of a hollow one, i.e., if $a = 0$, and if we rename b as R, then since $\log r$ and $Y_0(r)$ are both singular at $r = 0$, in place of Eqn. (30), we have

$$c_0 + \sum_{n=1}^{\infty} A_n J_0 \left(\frac{\lambda_n r}{R} \right) = f(r), \tag{32}$$

where λ_n, $n = 1, 2, \ldots, \infty$, are, for example, the positive roots of say $J_0(x) = 0$ or $J_1(x) = 0$. If they are the roots of $J_0(x) = 0$, then using the orthogonality properties of the Bessel functions, we have

$$\begin{aligned} c_0 &= f(R), \\ \tilde{f}(r) &:= f(r) - f(R), \\ A_n &= \frac{2}{R^2 J_1^2(\lambda_n)} \int_0^R \hat{r} \tilde{f}(\hat{r}) J_0 \left(\frac{\lambda_n \hat{r}}{R} \right) d\hat{r}. \end{aligned}$$

Once again, the c_0 term is missing in the classical treatments (see, for example, Eqn. (3), Section (68), Chapter 7 of [1]). As an example, the classical and proposed representations for $(r/R)^2$ would be

$$\left(\frac{r}{R} \right)^2 \sim \sum_{n=1}^{\infty} \frac{2(\lambda_n^2 - 4) J_0(\lambda_n r/R)}{\lambda_n^3 J_1(\lambda_n)}, \tag{33a}$$

$$\left(\frac{r}{R} \right)^2 \sim 1 - \sum_{n=1}^{\infty} \frac{8 J_0(\lambda_n r/R)}{\lambda_n^3 J_1(\lambda_n)}. \tag{33b}$$

The plots in Fig. 7 confirm the need for the missing dominant term c_0.

If, however, λ_n, $n = 1, 2, \ldots, \infty$, are the positive roots of $J_1(x) = 0$, then the c_0 term is present even in the classical treatments (see, for example, Eqns. (5) and (6), Section (68), Chapter 7 of [1]), and from Eqn. (32), we have

$$\begin{aligned} c_0 &= \frac{2}{R^2} \int_0^R \hat{r} f(\hat{r}) \, d\hat{r}, \\ A_n &= \frac{2}{R^2 J_0^2(\lambda_n)} \int_0^R \hat{r} f(\hat{r}) J_0 \left(\frac{\lambda_n \hat{r}}{R} \right) d\hat{r}. \end{aligned}$$

4 No missing terms in a Legendre series!

Let (r, θ, ϕ) represent spherical coordinates. We consider only axisymmetric problems so that there is no dependence of T on ϕ. Let $\xi := \cos\theta$. Then the

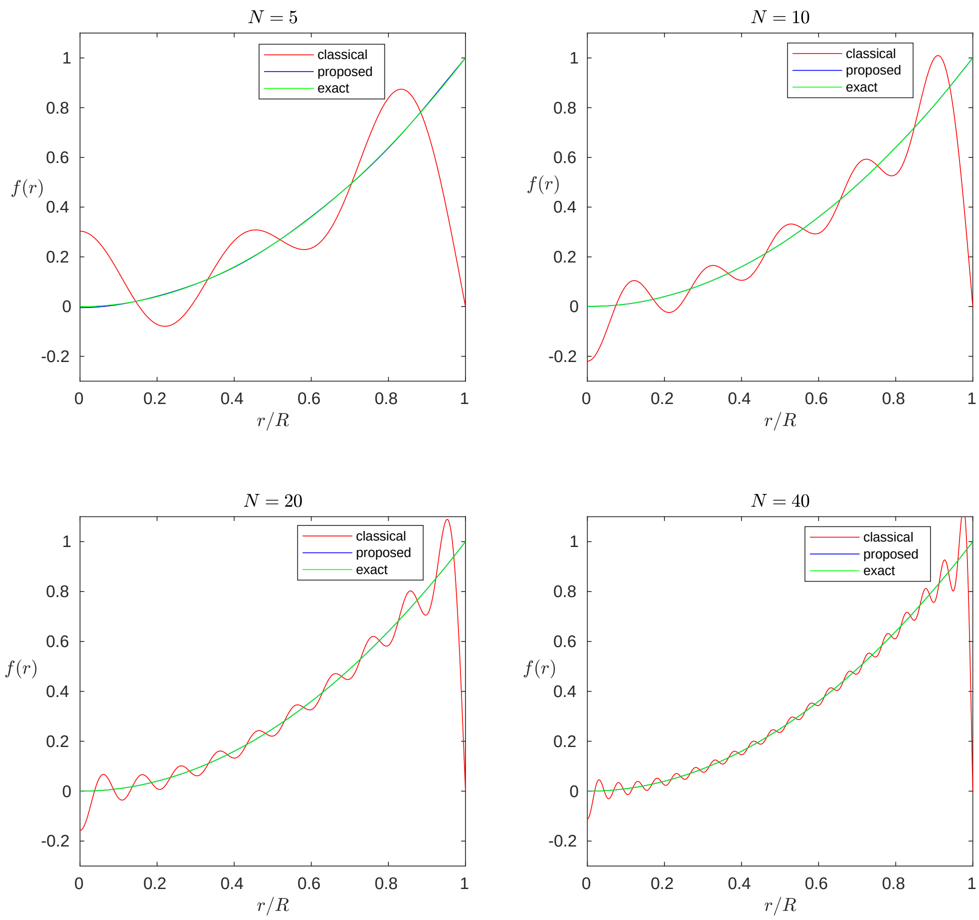

Figure 7: Plots of the representation in Eqn. (33) with $f(r) = (r/R)^2$ for different number of terms N in the Fourier–Bessel series.

most general axisymmetric separable solution to $\nabla^2 T = 0$ on a domain that is a hollow sphere in terms of (r, ξ) is

$$T = \sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{-(n+1)} \right] P_n(\xi), \tag{34}$$

where $P_n(\xi)$ are Legendre polynomials. The Legendre functions of the second kind $Q_n(\xi)$ are ruled out since they are singular at $\xi = \pm 1$ which is part of the domain. The solution corresponding to a zero separation constant is of the form $c_0 + c_1/r$ which is already captured by the $n = 0$ term in Eqn. (34), and thus unlike the ordinary Fourier and Fourier–Bessel series that we considered in the preceding sections, there are no missing terms in the case of a Fourier-Legendre series; this holds true even on domains where the $Q_n(\xi)$ would have to be considered.

From Eqn. (34) and by considering the boundary condition at the inner or outer radius of the sphere, it follows that for an arbitrary continuous function $f(\xi)$ defined on $[-1, 1]$, we have (after renaming the constant)

$$\sum_{n=0}^{\infty} A_n P_n(\xi) = f(\xi),$$

where

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(\hat{\xi}) P_n(\hat{\xi}) \, d\hat{\xi}.$$

As an example, we have

$$\sum_{n=0}^{\infty} \frac{(4n+1) P_{2n}(\xi)}{\Gamma\left(\frac{1}{2} - n\right) \Gamma\left(\frac{3}{2} - n\right) \Gamma(1+n) \Gamma(2+n)} = \frac{8\sqrt{1-\xi^2}}{\pi^2},$$

where $\Gamma(.)$ denotes the Gamma function.

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[3] Arpaci, V. S., *Conduction Heat Transfer*, London: Addison-Wesley Publishing Company, 1966.