Junior Problems

J589. Let $a, b, c \in [0, 1]$ such that $a + b + c = 2$. Prove that

$$a^3 + b^3 + c^3 + 2abc \leq 2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J590. Let p be a positive integer. Evaluate

$$S_p = \sum_{m=1}^{p} \sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2 - 2nk + n^2}.$$

Proposed by Florică Anastase, Lehliu-Gară, Romania

J591. Let D_A, D_B, D_C be disks in the plane with centers O_A, O_B, O_C, respectively. Consider points $A \in D_A$, $B \in D_B$, $C \in D_C$ such that the area of triangle ABC is maximal. Prove that lines AO_A, BO_B, CO_C are concurrent.

Proposed by Josef Tkadlec, Czech Republic

J592. Let M be a point inside triangle ABC. Let D, E, F be the orthogonal projections of M onto sides BC, CA, AB, respectively. Prove that

$$MA \sin\frac{A}{2} + MB \sin\frac{B}{2} + MC \sin\frac{C}{2} \geq MD + ME + MF.$$

When does equality hold?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J593. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{(1+2a)^3} + \frac{1}{(1+2b)^3} + \frac{1}{(1+2c)^3} \geq \frac{1}{3(1+2abc)}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J594. Let a be a positive real number other than 1 and let c, d be real numbers such that

$$a^c + a^d = (a+1)a^{\frac{c+d-1}{2}}.$$

Prove that for all postive real numbers $b \neq 1$,

$$b^c + b^d = (b+1)b^{\frac{c+d-1}{2}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior Problems

S589. Let a, b, c be real numbers such that

$$\cos(a-b) + 2\cos(b-c) \geq 3\cos(c-a).$$

Prove that

$$|3\cos a - 2\cos b + 6\cos c| \leq 7.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S590. Let ABC be an acute triangle and let E be the center of its nine-point circle. Prove that

$$BE + CE \leq \sqrt{a^2 + R^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S591. Prove that there are infinitely many even positive integers n such that

$$n \mid 2^n - 2 \text{ and } n \nmid 3^n - 3.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

S592. Let ABC be a triangle and let E, F be the foot of the altitude from B, C, respectively. Denote by X the center of nine-point circle of $\triangle ABC$ and assume that the symmedian from A intersects EF in X. Find $\angle BAC$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

S593. Let ABC be a triangle and let N be its Nagel point. Let D, E, F be the orthogonal projections of N onto BC, CA, AB, respectively. Prove that

$$ND + NE + NF \leq r\left(\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c}\right)$$

Proposed by Marian Ursărescu, National College Roman-Vodă, Roman, Romania

S594. Let a, b, c be positive real numbers. Prove that

$$\frac{(4a+b+c)^2}{2a^2+(b+c)^2} + \frac{(4b+c+a)^2}{2b^2+(c+a)^2} + \frac{(4c+a+b)^2}{2c^2+(a+b)^2} \leq \frac{52}{3} + \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}.$$

Proposed by Marius Stănean, Zalău, Romania

Undergraduate Problems

U589. Let $a > 4$ be a positive integer. Prove that there are composite and relatively prime positive integers x_1 and x_2 such that the sequence $\{x_n\}_{n\geq 1}$ defined by $x_{n+1} = ax_n + x_{n-1}$, $n \geq 2$, consists of composite numbers only.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U590. Prove that for all positive real numbers x, y,

$$x^x + y^y \geq 2\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}}$$

Proposed by Toyesh Prakash Sharma, Agra College, India

U591. Prove that

$$\int_0^{\sqrt{\sqrt{7}-1}} (x^3 + x)e^{-x^2} dx \leq \ln 2$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

U592. Evaluate

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{(n+1)(n+2)},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the n^{th} harmonic number.

Proposed by Ovidiu Furdui and Alina Sîntămărian, Cluj-Napoca, Romania

U593. Let ABC be an acute scalene triangle with cirumcenter O and centroid G. Let W be point on line BC such that $GW \perp BC$. Given that $b^2 + c^2 = 3a^2$ show that the line OW is tangent to *Jerabek* hyperbola of triangle ABC.

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

U594. Let n be a positive integer. Evaluate

$$\int_1^n \lfloor\sqrt{x}\rfloor \mathrm{d}x,$$

where $\lfloor a \rfloor$ denotes the integer part of a.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Olympiad Problems

O589. Let x, y, z be positive real numbers. Find the minimum of

$$\frac{xy^2}{z(x^2+xz+z^2)}+\frac{yz^2}{x(y^2+yx+x^2)}+\frac{zx^2}{y(z^2+zy+y^2)}+2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right).$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

O590. Let ABC be a scalene triangle with centroid G and symmedian point K. Prove that if $\angle BAG = \angle ABK$ then GK is parallel to BC.

Proposed by Todor Zaharinov, Sofia, Bulgaria

O591. Real numbers $a_1, a_2, \ldots, a_n$ satisfy

$$a_1+a_2+\cdots+a_n = a_1^2+a_2^2+\cdots+a_n^2 = n-1.$$

Prove that

$$n-1 \le a_1^3+a_2^3+\cdots+a_n^3 < n+1.$$

Proposed by Josef Tkadlec, Czech Republic

O592. Let M be an interior point of a triangle ABC. Let D, E, F be the intersections of lines AM, BM, CM with BC, CA, AB, respectively and P, Q, R be the intersections of lines AM, BM, CM with EF, DF, DE, respectively. Prove that

$$\frac{MA}{MD}+\frac{MB}{ME}+\frac{MC}{MF} \ge \frac{MD}{MP}+\frac{ME}{MQ}+\frac{MF}{MR}.$$

Proposed by Marius Stănean, Zalău, Romania

O593. Let a, b, c, d be non-zero complex numbers such that

$$2|a-b| \le |b|,\ 2|b-c| \le |c|,\ 2|c-d| \le |d|,\ 2|d-a| \le |a|.$$

Prove that

$$\max\{|\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}|,|\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}|\} > 2\sqrt{3}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O594. Find all positive integers a and b such that

$$2-3^{a+1}+3^{3a} = pq^b,$$

for some prime numbers p and q.

Proposed by Mircea Becheanu, Canada