# Solving Functional Equations in $\mathbb{R}^+$ Using Inequalities and Sequences

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Functional equations appear frequently in Mathematical Olympiads. In this article, we focus on a special type of functional equations involving functions defined over $\mathbb{R}^+$ and consider different approaches, mainly using inequalities and sequences for solving them.

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## §1 Finding the value of $f(a)$ with two flipped inequalities

A general idea, which will appear frequently, is to use two inequalities for finding the value of $f$ at a certain point. That is we will somehow show $f(a) \leq c$, $f(a) \geq c$, to obtain $f(a) = c$. In the following problem, we will find $f(1)$ by this method. Then, we will plug some variables to finish the solution.

**Example 1**

Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers $x, y$ such that $x > y$, we have:

$$f(x-y) = f(x) - f(x)f(\frac{1}{x})y.$$

*Solution.* We start the proof with the following claim.

**Claim 1.1** — For all positive real numbers $x$, we have $x \geq f(x)$.

*Proof.* Since $f(x-y) > 0$ for all positive real numbers $x, y$ such that $x > y$, we get that $f(x)\left(1 - yf\left(\frac{1}{x}\right)\right) > 0$, which shows $\frac{1}{y} > f\left(\frac{1}{x}\right)$ for any $x > y$. Hence, choosing $y$ close enough to $x$, leads to the inequality $\frac{1}{x} \geq f\left(\frac{1}{x}\right)$. Plugging $\frac{1}{x}$ instead of $x$ finishes the proof. □

**Claim 1.2** — We have $f(1) = 1$.

*Proof.* Claim 1.1 yields $f(1) \leq 1$. So, it suffices to prove $f(1) \geq 1$. Now, assume that $y$ is an arbitrary positive real number such that $y < 1$. Plugging $x = 1$, implies that $f(1-y) = f(1) - f(1)^2 y$. Using the above inequality, we reach $1 - y \geq f(1) - f(1)^2 y$. Hence,

$$(f(1) - 1)\left(y(1 + f(1)) - 1\right) \geq 0.$$

Therefore, plugging an arbitrary positive real number $y$ that satisfies $1 > y > \frac{1}{(1+f(1))}$, yields $f(1) \geq 1$. □

**Claim 1.3** — For all positive real numbers $x'$ that satisfies $x' < 1$, we have $f(x') = x'$.

*Proof.* Setting $x = 1, y = 1 - x'$, yields the desired result. □

Now, by considering positive real numbers $y < x < 1$, we have:

$$x - y = f(x-y) = f(x) - f(x)f(\frac{1}{x})y = x - xyf(\frac{1}{x}).$$

It follows that $y = xyf(\frac{1}{x})$. Thus, $f(\frac{1}{x}) = \frac{1}{x}$ for all positive real numbers $x < 1$.
Hence, for any positive real number $x > 1$, we have $f(x) = x$. Therefore, for all positive real numbers $x$, there is $f(x) = x$. Which is indeed a solution. □

## §2 "Too Deli"

By *Too Deli* [1] we mean trying to plug variables such that the two expressions wrapped by $f$ in different sides of the equation obtain the same value. In other words, if given a functional equation of the form $(...f(sth)...) = (...f(sthelse)...)$, we would like $sth$ and $sthelse$ to have the same value, so that we can cancel out the two terms and get a simpler equation.
The general procedure here will be the following:
In the case we are able to put $sth = sthelse$, we will reach a useful equation, or more frequently a contradiction!
In the frequent contradiction case, we will deduce that we couldn't put $sth = sthelse$, and this will usually result in an inequality.
we can observe this more visibly through the following examples.

**Example 2** (India, 2008; MEMO, 2012)
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers $x, y$, we have:

$$f(x + f(y)) = yf(xy + 1).$$

*Solution.* First, we apply *Too Deli* by trying to force $f(x+f(y))$ and $f(xy+1)$ obtain the same value. This will be concluded by equalizing $x + f(y), xy + 1$. By solving the equation for $x$, we will get $x = \frac{f(y)-1}{y-1}$. This leads to the following claim.

**Claim 2.1** — For all positive real numbers $y$ satisfying $y \neq 1$, we have $\frac{yf(y)-1}{y-1} \leq 0$.

*Proof.* If there is a positive real number $y_0 \neq 1$ such that $\frac{y_0 f(y_0)-1}{y_0-1} > 0$, plug $x = \frac{f(y_0)-1}{y_0-1}$, to obtain:

$$f(\frac{yf(y_0) - 1}{y_0 - 1}) = y_0 f(\frac{yf(y_0) - 1}{y_0 - 1}).$$

Hence, $y_0 = 1$, which is a contradiction. □

Now, plug $x = 1 - \frac{1}{y}$ for some real positive number $y > 1$, to reach the equation:

$$f(1 - \frac{1}{y} + f(y)) = yf(y).$$

We will use this equation in the following claim.

**Claim 2.2** — For all positive real numbers $y$ satisfying $y > 1$, we have $f(y) = \frac{1}{y}$.

*Proof.* First, we prove that $f(y) \geq \frac{1}{y}$. If $f(t) < \frac{1}{t}$ for some positive real number $t > 1$, then let $y_1 = 1 + f(t) - \frac{1}{t} < 1$. Using the above equation, since $tf(t) < 1$, it follows that $f(y_1) < 1$ while $y_1 < 1$. Which is in contradiction with claim 2.1.
Now, we prove that $f(y) \leq \frac{1}{y}$ for all $y > 1$. If $f(r) > \frac{1}{r}$ for some positive real number $r > 1$, then define $y_2 = 1 + f(r) - \frac{1}{r}$. This time, we have $y_2 > 1$ and $f(y_2) > 1$. Again, this is in contradiction with claim 2.1. □

[1] Too Deli is a Persian phrase, where *Deli* means heart, and *Too* means inside! So, *Too Deli* can be translated as *Heart's Inside*.

Now, plugging $x = 1$, leads to $f(1 + f(y)) = yf(y + 1)$. Since $1 + f(y), y + 1 > 1$, we have:

$$\frac{1}{1 + f(y)} = \frac{y}{(y + 1)}.$$

Hence, for all positive real numbers $x$, there is $f(x) = \frac{1}{x}$. □

The following problem, at first sight, may seem a bit strange. But there is nothing more than the previous methods.

**Example 3**

Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any positive real numbers $x, y$ such that $y \in (0, x^2)$, we have:

$$f(x) = f(\frac{x^2 - y}{f(x)}) + f(\frac{f(y)}{x}).$$

*Solution.* We will prove two claims which will solve the problem immediately.

**Claim 3.1** — For all positive real numbers $x$, we have $f(x) \leq x$.

*Proof.* Assume that there is a positive real number $x_0$ such that $f(x_0) > x_0$. Since $x_0 < \left(\sqrt{f(x_0)}\right)^2 = f(x_0)$, plug $x = \sqrt{f(x_0)}, y = x_0$, to obtain:

$$f(\sqrt{f(x_0)}) = f\left(\frac{f(x_0) - x_0}{f(\sqrt{f(x_0)})}\right) + f(\sqrt{f(x_0)}).$$

So, it follows that $f\left(\frac{f(x_0) - x_0}{f(\sqrt{f(x_0)})}\right) = 0$. Which is impossible.
Hence, for all positive real numbers, $f(x) \leq x$, as desired. □

**Claim 3.2** — For all positive real numbers $x$, we have $f(x) \geq x$.

*Proof.* If there exists some positive real number $y_0$ such that $f(y_0) < y_0$, note that $0 < y_0^2 - y_0 f(y_0) < y_0^2$. Therefore, setting $x = y_0, y = y_0^2 - y_0 f(y_0)$, leads to:

$$f(y_0) = f(y_0) + f\left(\frac{f(y_0^2 - y_0 f(y_0))}{y_0}\right).$$

So, it follows that $f\left(\frac{f(y_0^2 - y_0 f(y_0))}{y_0}\right) = 0$. Which is a contradiction, and thus we proved the claim. □

Comparing claims 3.1, 3.2 for all positive real numbers $x$, we have $f(x) = x$. Which is indeed a solution. □

## §3 Monotonicity

We have so far seen the strength of inequalities in solving functional equations involving functions defined over $\mathbb{R}^+$. Since monotonicity, in essence, is based on an inequality, it is not strange that we use it for solving functional equations involving inequalities. In other words, sometimes we should try to prove that $f$ is either increasing or decreasing.
We demonstrate how this will help by the following example.

**Example 4**
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that for all positive real numbers $x, y$, we have the following inequalities:

$$f(x) + f(y) \leq \frac{f(x+y)}{4}$$
$$\frac{f(x)}{y} + \frac{f(y)}{x} \geq (\frac{1}{x} + \frac{1}{y}) \cdot \frac{f(x+y)}{8}.$$

*Solution.* Combining our inequalities to obtain:

$$8\frac{xf(x) + yf(y)}{x+y} \geq f(x+y) \geq 4f(x) + 4f(y).$$

**Claim 4.1** — The function $f$ is non-decreasing.

*Proof.* Comparing the left-hand side and the right-hand side of the above inequality yields:

$$8\frac{xf(x) + yf(y)}{x+y} \geq 4f(x) + 4f(y).$$

This can be written as:

$$2(xf(x) + yf(y)) \geq (x+y)\left(f(x) + f(y)\right).$$

So,

$$xf(x) + yf(y) - yf(x) - xf(y) = (x-y)(f(x) - f(y)) \geq 0.$$

Note that the last inequality is equivalent to the desired result. □

**Claim 4.2** — For all positive real numbers $x$, we have $f(x) \geq 0$.

*Proof.* Setting $y = x$, we reach that:

$$8\frac{2xf(x)}{2x} = 8f(x) \geq f(2x) \geq 8f(x).$$

It follows that $f(2x) = 8f(x)$. Now, since $f(2x) \geq f(x)$, we find that $8f(x) \geq f(x)$. And hence for all positive real numbers $x$, there is $f(x) \geq 0$. □

Hence, $f(x+y) \geq 4f(x) + 4f(y) \geq 4f(x)$. From here, an easy induction gives:

$$f(x+ny) \geq 4^n f(x).$$

And so plugging $y = \frac{1}{n}$, yields $f(x+1) \geq 4^n f(x), n \in \mathbb{N}$ for all positive real numbers $x$.
Now, consider any fixed positive real number $t$. If $f(t) \neq 0$, then note that $\lim\limits_{n\to\infty} 4^n f(t) = +\infty$, while we showed it's bounded by $f(t+1)$.
This is a contradiction which yields $f(x) = 0$ for all positive real numbers $x$, which is indeed a solution. □

# §4 Defining sequences, using limits, and some overkill techniques!

We will mostly use sequences to reach an inequality for $f$. For example, it will often be the case that $f(x) < f(x_i)$ for some sequence $x_i$, then we obtain $f(x) \leq \lim_{n\to\infty} f(x_n)$.[2]
In some cases, the right-hand side equals to zero, which will give us a contradiction, and perhaps we can show some property of $f$ using it.
Another theme, that appears frequently, is to choose a sequence that satisfies $f(x_i) = f(x_{i+1})$, which implies $f(x_i) = f(x_1) = constant$. And then look for inequalities which compare $x$ to $f(x)$.
Let's start with some examples.

**Example 5**
Prove that for any positive real number $\alpha$ , there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers $x, y$, we have:

$$f(\alpha + xf(y)) = yf(x + y + \alpha).$$

*Solution.* We start the solution with the following claim.

**Claim 5.1 —** For all positive real numbers $x \neq 1$, we have $f(x) \leq 1$.

*Proof.* If there exists a positive real number $y_0$ such that $f(y_0) > 1$, then plugging $x = \frac{y_0}{f(y_0)-1}$, gives:

$$f\left(\alpha + \frac{y_0 f(y_0)}{f(y_0)-1}\right) = y_0 f\left(\alpha + \frac{y_0 f(y_0)}{f(y_0)-1}\right).$$

It follows that $y_0 = 1$, which is a contradiction. □

Therefore, $yf(x + y + \alpha) = f(\alpha + xf(y)) \leq 1$, and so for all positive real numbers $x, y$, we have:

$$f(x + y + \alpha) \leq \frac{1}{y}.$$

We will use this inequality to show the following claim.

**Claim 5.2 —** For all positive real numbers $x$, we have $f(x + \alpha) \leq \frac{1}{x}$.

*Proof.* Fix an arbitrary positive real number $x_0$. Set $x = x_0 - y$, for some positive real number $y$ satisfying $y < x_0$. Now, by rewriting the inequality, we obtain:

$$f(x_0 + \alpha) \leq \frac{1}{y}$$

for any $y < x_0$. Now, note that by considering $y$ close enough to $x_0$, we have:

$$f(x_0 + \alpha) \leq \lim_{y\to x_0^-} \frac{1}{y} = \frac{1}{x_0}.$$

So, there is $f(x_0 + \alpha) \leq \frac{1}{x_0}$, as desired. □

[2] See appendix(The Limit Inequality Theorem)

**Claim 5.3** — We have $f(1) > 1$.

*Proof.* For more convenience, let $f(1) = a$. If $a \leq 1$, define a sequence $\{x_i\}_{i=1}^{\infty}$, as following:

$$x_1 = x \text{ and } x_{n+1} = \frac{x_n}{a} + 1.$$

Since $\frac{1}{a} > 1$, we have the inequality, $x_{n+1} > x_n + 1$. It follows that, $x_n > x + (n-1)$. Thus,

$$\lim_{n\to\infty} x_n = \infty.$$

Plugging $x = \frac{x_n}{a}, y = 1$, implies $f(x_n + \alpha) = f(x_{n+1} + \alpha)$. Indeed, $f(x + \alpha) = f(x_n + \alpha)$. But note that using claim 5.3, we obtain:

$$f(x + \alpha) = f(x_n + \alpha) \leq \frac{1}{x_n}.$$

So,

$$f(x + \alpha) \leq \lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Thus, $0 < f(x + \alpha) \leq 0$, which is a contradiction. Hence, $a > 1$. □

**Claim 5.4** — We have $f(1) \leq 1$.

*Proof.* If $a > 1$, choose an arbitrary positive real number $x$ such that $x > \frac{a}{a-1} > 1$. Define a sequence $\{y_i\}_{i=1}^{\infty}$, as following:

$$y_1 = x \text{ and } y_{n+1} = a(y_n - 1).$$

Now, we show that the sequence is increasing. Moreover, $\lim\limits_{n\to\infty} y_n = +\infty$.
Let $d_n = y_{n+1} - y_n$ note that the definition of $y_n$, implies $d_n = (a-1)y_n - a$.
Hence, if $d_n > 0$, we would also have $y_{n+1} > y_n$. To have this, it suffices to prove $y_n > \frac{a}{a-1}$.
Now, by induction we can easily show that $d_n > 0$ for every $n$. Note that $d_1 = x(a-1) - a$ implies $d_1 > 0$. Proceeding by induction, if $d_n > 0$, we deduce that $y_n > \frac{a-1}{a}$. Hence,

$$y_{n+1} = y_n + d_n > y_n > \frac{a-1}{a}.$$

Therefore, $d_{n+1} > 0$.
So, we conclude that $y_n$ is increasing. Since $d_n = (a-1)y_n - a$, we get $d_n$ is also increasing, which means $d_n > d_1$.
It's easy to see that $y_{n+1} \geq x + d_1 n$. Hence,

$$\lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x + d_1 n = +\infty,$$

as desired.

The last step is straight forward. Setting $x = y_n - 1, y = 1$, implies $f(y_n + \alpha) = f(y_{n+1} + \alpha)$. Thus, $f(x + \alpha) = f(y_n + \alpha) \leq \frac{1}{y_n}$. Hence,

$$f(x + \alpha) \leq \lim_{n\to\infty} \frac{1}{y_n} = 0.$$

□

Therefore, $0 < f(x + \alpha) = 0$, which is a contradiction! And so, no such function exists. □

The following example is a part of IMO Shortlist 2020, A8's official solution.

**Example 6** (Adapted from IMO Shortlist 2020, A8)
If for $f : \mathbb{R}^+ \to \mathbb{R}^+$ and all positive real numbers $x, y$, we have:

$$f(x + f(xy)) + y = f(x)f(y) + 1,$$

prove that: $f(x) \geq x + 1$.

*Solution.* We will solve the problem using the following claims.
Let $M = \inf_{x \in \mathbb{R}^+} f(x) > 0$.[3]

**Claim 6.1** — For all positive real numbers $y$, we have $f(y) \geq \frac{y-1}{M} + 1$.

*Proof.* From the main equation, notice that $f(x)f(y) \geq y - 1 + M$, and hence:

$$f(y) \geq \frac{y-1}{M} + 1.$$

□

**Claim 6.2** — We have $M \leq 1$.

*Proof.* Suppose that $M > 1$. For more convenience, let $f(1) = a$. Substituting $y = \frac{1}{x}$, for any positive real number $x > 1$, in the main equation, we get:

$$f(x + a) = f(x)f\left(\frac{1}{x}\right) + 1 - \frac{1}{x} \geq Mf(x).$$

By a straightforward induction on $\lceil \frac{(x-1)}{a} \rceil$, this yields:

$$f(x) \geq M^{\frac{(x-1)}{a}}.$$

Now, choose an arbitrary positive real number $x_0$, and define a sequence $\{x_i\}_{i=1}^{\infty}$, as following:

$$x_{n+1} = x_n + f(x_n) \geq x_n + M.$$

Notice that the sequence is unbounded. On the other hand, by plugging $y = 1$, in the main equation, we get $f(x + f(x)) = af(x)$. Combining with the previous equation yields:

$$ax_{n+1} > af(x_n) = f(x_{n+1}) \geq M^{\frac{(x_{n+1}-1)}{a}}.$$

Which cannot hold when $x_{n+1}$ is large enough. □

Now, let's prove that for all positive real numbers $y > 0$, we have $f(y) \geq y + 1$, by contradiction. Assume there is a positive real number $t$ such that $f(t) < t + 1$, and choose $\alpha$ such that $f(t) < \alpha < t + 1$. Define a sequence $\{y_i\}_{i=1}^{\infty}$, by choosing a large $y_0 \geq 1$, and the following property:

$$y_{n+1} = y_n + f(y_n t) \geq y_n + M.$$

Note that this sequence is also unbounded. If $y_0$ is large enough, then claim 6.1 implies that $(\alpha - f(t))f(y_n) \geq 1 - t$ for all $n \in \mathbb{N}$. Therefore,

$$f(y_{n+1}) = f(t)f(y_n) + 1 - t \leq \alpha f(y_n).$$

[3]You can study the infimum and supremum here.

On the other hand, since $M \leq 1$, claim 6.1 implies that for all positive real numbers $z \geq 1$, there is $f(z) \geq z$. Hence, if $y_0$ is large enough, we have $y_{n+1} \geq y_n(1+t)$ for all $n$. Therefore,

$$y_0(1+t)^n \leq y_n \leq f\left(y_n\right) \leq \alpha^n f\left(y_0\right),$$

which cannot hold when $n$ is large enough. $\square$

> **Remark.** The main problem asks to find all the functions defined over $\mathbb{R}^+$, satisfying $f(x+f(xy))+y=f(x)f(y)+1$. The readers may try to solve it.

## §5 A final problem to conclude everything

The last example has been selected from the 2021 Korea winter program practice test. This problem seems to be inspired from IMO shortlist 2005, A2.
Example 7 includes literally all the techniques we went through:

**Example 7** (Korea Winter Program Practice Test, 2021)
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers $x, y$, we have:

$$4f(x+yf(x)) = f(x)f(2y).$$

*Solution.* We will solve the problem using the following claims.

**Claim 7.1** — For all positive real numbers $x$, we have $f(x) \geq 2$.

*Proof.* If there is a positive real number $r$ such that $f(r) < 2$, plugging $x = r, y = \frac{r}{2-f(r)}$, yields:

$$4f(r + \frac{r}{2-f(r)} \times f(r)) = f(r) \times f(\frac{2r}{2-f(r)}).$$

But note that the expression wrapped inside left-hand side is $\frac{2r}{2-f(r)}$. Hence, we obtain:

$$4f(\frac{2r}{2-f(r)}) = f(r) \times f(\frac{2r}{2-f(r)}).$$

It follows that $f(r) = 4$. Obviously $f(r) = 4$ is a contradiction to $f(r) < 2$. Hence, $f(x) \geq 2$ for any choice of a positive real number $x$. $\square$

**Claim 7.2** — For all positive real numbers $x$, we have $f(x) \geq 4$.

*Proof.* If there exists a positive real number $s$ such that $f(s) \in [2, 4)$, define a sequence $\{x_i\}_{i=1}^{\infty}$, as following:

$$x_1 > 0 \text{ and } x_{n+1} = x_n + \frac{s}{2}f(x_2).$$

Now, setting $x = x_n, y = \frac{s}{2}$, leads to the following equation:

$$4f(x_{n+1}) = 4f(x_n + \frac{s}{2}f(x_n)) = f(x_n) \times f(s).$$

So,

$$f(x_{n+1}) = f(x_n) \times \frac{f(s)}{4}.$$

Therefore, a simple induction on $n$ implies $f(x_{n+1}) = \left(\frac{f(s)}{4}\right)^n \times f(x_1)$.
Since $\frac{f(s)}{4} \in [\frac{1}{2}, 1)$, we have:

$$2 \leq \lim_{n\to\infty} f(x_{n+1}) = f(x_1) \times \left(\lim_{n\to\infty} \left(\frac{f(s)}{4}\right)^n\right) = 0.$$

This is a contradiction, which yields $f(x) \geq 4$ for all positive real numbers $x$. □

**Claim 7.3** — There exists a positive real number $t$, such that $f(t) = 4$.

*Proof.* Assume there is no such $t$. Therefore, using the above claim, we deduce that $f(x) > 4$ for any positive real number $x$.
Hence, since $f(2y) > 4$, we obtain $f(x + yf(x)) > f(x)$. Set $y = \frac{z}{f(x)}$, to get $f(x + z) > f(x)$ for all positive real numbers $x, z$, which means that $f(x)$ is strictly increasing and therefore it is injective.
Now, note that comparing the equation for $(x, y) = (2x, y)$ and $(x, y) = (2y, x)$ implies:

$$\frac{f(2x)f(2y)}{4} = f(2x + yf(2x)) = f(2y + xf(2y)).$$

So,

$$2x + yf(2x) = 2y + xf(2y).$$

Considering a fixed $y$ (for instance $y = 1$), implies $f(x) = \alpha x + \beta$ for some fixed real numbers $\alpha, \beta$. It's not hard to see that the only such function that can work is $f(x) = 4$, which clearly doesn't satisfy $f(x) > 4$.
With this contradiction, we conclude that there exists a positive real number $t$, such that $f(t) = 4$, as desired. □

Plugging $y = \frac{z}{f(x)}$, in the main equation, we will reach:

$$4f(x + z) = f(x)f(\frac{2z}{f(x)}).$$

By using claim 7.2, now we know that $f(\frac{2z}{f(x)}) \geq 4$. So,

$$4f(x + z) = f(x)f(\frac{2z}{f(x)}) \geq 4f(x).$$

Therefore, $f(x + z) \geq f(x)$, which means $f$ is non-decreasing.
Now, consider the real positive number $t$ which satisfies $f(t) = 4$ (given by claim 7.3). Note that since $f$ is non-decreasing, for any positive real number $x$ satisfying $x \leq t$, we have $f(x) \leq f(t) = 4$.
Also, note that we already know $4 \leq f(x)$ based on claim 7.2. Therefore, for any positive real number $x$ such that $x \leq t$, we have $f(x) = 4$ (i.e. for all positive real numbers $x \in (0, t]$, we have $f(x) = 4$).
As the final step, set $x = y = \frac{t}{2}$, to get:

$$4f(\frac{t}{2} + \frac{t}{2} \times 4) = 4f(\frac{5}{2}t) = f(\frac{t}{2})f(t) = 16.$$

Therefore, $f(\frac{5}{2}t) = 4$. So, for all positive real numbers $x \in (0, \frac{5}{2}t]$, we have $f(x) = 4$.
By repeating the same argument, we can reach that for any $k \in \mathbb{N}$, the following result holds:

$$f(x) = 4 \text{ for any positive real number } x \in \left(0, \left(\frac{5}{2}\right)^k t\right].$$

Note that $\left(\frac{5}{2}\right)^k t$ is obviously boundless when $k$ goes to infinity, therefore any positive real number $x$ will be included in the interval $\left(0, \left(\frac{5}{2}\right)^k t\right]$ eventually. Hence, for all positive real numbers $x$, we have $f(x) = 4$.
And $f \equiv 4$ is indeed a solution. $\square$

> **Remark.** The generalized problem $2\alpha f(x + yf(x)) = f(x)f(\alpha y)$, can be solved by the same approach.

> **Remark.** Here is the IMO shortlist problem we mentioned at the beginning of this section. This problem is also a particular case of Problem 9.3 in [1]. We suggest the readers to solve this problem as an exercise:
>
> **Exercise** (IMO Shortlist 2005, A2)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers $x, y$, we have:
> $$f(x)f(y) = 2f(x + yf(x)).$$

# §6 Appendix

In this part, we will provide a theorem and a useful lemma in solving functional equations defined over $\mathbb{R}^+$. Although these techniques are a bit far from our topic about inequalities and sequences, we recommend the readers to study this part.

**Theorem** (The Limit Inequality Theorem)
If $f(x) \le g(x)$ for all real numbers $x$, and $\lim_{x \to a} f(x) = F$ and $\lim_{x \to a} g(x) = G$, then $F \le G$.

> **Remark.** We regularly use a simpler version, where one of the functions is constant.

As an exercise, we will go through the following well-known lemma:

**Lemma**
Let $f$ be a monotonic and $g$ be a continuous function defined over $\mathbb{R}$ that satisfies $f(r) = g(r)$ for all $r \in \mathbb{Q}$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

*Proof.* For convenience, lets assume that $f$ is non-decreasing (the other case will use the same argument, with all inequalities getting flipped).
Since we already know that $f(x) = g(x)$ for all rational numbers $x$, it suffices to show $f(x) = g(x)$ for all irrational numbers $x$.
Consider any arbitrary irrational number $x_0$, consider two sequences $\{u_i\}_{i=1}^{\infty}, \{v_i\}_{i=1}^{\infty}$ in rational numbers such that:
$$u_i < x_0 < v_i.$$
And,
$$\lim_{n \to +\infty} u_n = \lim_{n \to +\infty} v_n = x_0.$$
We can construct the two sequences since rational numbers are dense in $\mathbb{R}$. If that's not clear, we may also define the two sequences implicitly as follows:
$$u_n = \frac{\lfloor nx_0 \rfloor}{n}, v_n = \frac{\lceil nx_0 \rceil}{n}.$$

Now, note that since $x_0 < v_i$, we have the inequality:

$$f(x_0) < f(v_i) = g(v_i).$$

Now, we can apply The Limit Inequality Theorem to obtain:

$$f(x_0) \leq \lim_{i\to\infty} g(v_i).$$

Continuity of $g$ implies $\lim\limits_{i\to\infty} g(v_i) = g(x_0)$. Therefore, $f(x_0) \leq g(x_0)$.
We can also show $g(x_0) \leq f(x_0)$ by using $u_i$'s and repeating the same argument. Combining the two inequalities, we obtain:

$$g(x_0) \leq f(x_0) \leq g(x_0) \implies f(x_0) = g(x_0).$$

Hence, $f$ and $g$ are also equal over irrational numbers. Thus, they are always equal, as desired. □

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