Junior problems

J589. Let $a, b, c \in [0, 1]$ such that $a + b + c = 2$. Prove that

$$a^3 + b^3 + c^3 + 2abc \le 2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Polyahedra, Polk State College, USA

We have $0 \le (1-a)(1-b)(1-c) = -1 + ab + bc + ca - abc$, so $ab + bc + ca \ge 1 + abc$. Therefore,

$$\begin{aligned} a^3 + b^3 + c^3 + 2abc &= (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 5abc \\ &\le 8 - 6(1+abc) + 5abc = 2 - abc \le 2. \end{aligned}$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Arkady Alt, San Jose, CA, USA; Brittany Turner, SUNY Brockport, NY, USA; Jiang Lianjun, Quanzhou second Middle School, GuiLin, China; Ryan DiPaola, SUNY Brockport, NY, USA; Shannon Forman, SUNY Brockport, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Le Hoang Bao, Tien Giang, Vietnam; A S Arun Sriniavaas, Mumbai, India; Daniel Văcaru, Pitești, Romania.

J590. Let p be a positive integer. Evaluate

$$S_p = \sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}.$$

Proposed by Florică Anastase, Lehliu-Gară, Romania

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Letting $n-k=r$ the sum is

$$\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{k^2+(n-k)^2} = \sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{r=0}^{n-1}\frac{(n-r)^2}{(n-r)^2+r^2}$$

that is

$$\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n-1}\frac{k^2}{k^2+(n-k)^2} + \sum_{m=1}^{p}\sum_{n=1}^{m}1 =$$
$$= \sum_{m=1}^{p}\sum_{n=1}^{m}1 + \sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{r=1}^{n-1}\frac{(n-r)^2}{(n-r)^2+r^2}$$

thus

$$\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n-1}\frac{k^2}{k^2+(n-k)^2} = \frac{1}{2}\left[\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n-1}\frac{k^2}{k^2+(n-k)^2} + \sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n-1}\frac{(n-k)^2}{k+(n-k)^2}\right] =$$
$$= \frac{1}{2}\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n-1}\cdot 1$$

and then

$$\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{k^2}{k^2+(n-k)^2} = \frac{1}{2}\sum_{m=1}^{p}\sum_{n=1}^{m}\sum_{k=1}^{n-1}\cdot 1 + \sum_{m=1}^{p}\sum_{n=1}^{m}\cdot 1 = \frac{p(p^2-1)}{12} + \frac{p(p+1)}{2} =$$
$$= \frac{p(p+5)(p+1)}{12}.$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Polyahedra, Polk State College, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Henry Ricardo, Westchester Area Math Circle, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J591. Let D_A, D_B, D_C be disks in the plane with centers O_A, O_B, O_C, respectively. Consider points $A \in D_A$, $B \in D_B$, $C \in D_C$ such that the area of triangle ABC is maximal. Prove that lines AO_A, BO_B, CO_C are concurrent.

Proposed by Josef Tkadlec, Czech Republic

Solution by Joel Schlosberg, Bayside, NY, USA

Let ℓ be the line through A parallel to BC. Unless ℓ is tangent to D_A, we can take a point A' in D_A on the opposite side of ℓ from BC, so that A' is farther from line BC than A and thus $\triangle A'BC$ has greater area than $\triangle ABC$. Thus, if $\triangle ABC$ has maximal area, $AO_A \perp \ell \parallel BC$ and so is an altitude of $\triangle ABC$. By the same reasoning, BO_B and CO_C are altitudes of $\triangle ABC$ and so concur with AO_A at the orthocenter of $\triangle ABC$.

Also solved by Polyahedra, Polk State College, USA; Theo Koupelis, Cape Coral, FL, USA.

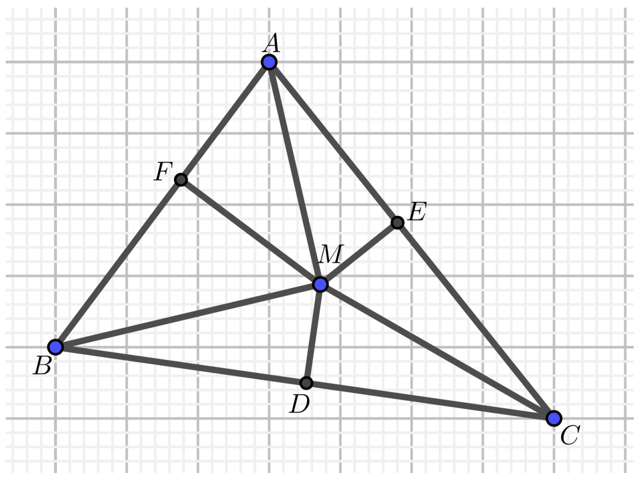
J592. Let M be a point inside triangle ABC. Let D, E, F be the orthogonal projections of M onto sides BC, CA, AB, respectively. Prove that

$$MA \sin\frac{A}{2} + MB \sin\frac{B}{2} + MC \sin\frac{C}{2} \geq MD + ME + MF.$$

When does equality hold?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author



We have

$$ME = MA \cdot \sin \angle MAE, \; MF = MA \cdot \sin \angle MAF.$$

Therefore

$$\begin{aligned} ME + MF &= MA(\sin \angle MAE + \sin \angle MAF) \\ &= 2MA \cdot \sin\frac{A}{2} \cos\frac{\angle MAE - \angle MAF}{2} \\ &\leq 2MA \cdot \sin\frac{A}{2}. \end{aligned}$$

Similarly

$$MF + MD \leq 2MB \cdot \sin\frac{B}{2},$$

$$MD + ME \leq 2MC \cdot \sin\frac{C}{2}.$$

Summing up these three inequalities we obtain the desired result. The equality happens if and only if M is the incenter of triangle ABC. From the above result, when M is the centroid of triangle ABC we get

$$m_a \sin\frac{A}{2} + m_b \sin\frac{B}{2} + m_c \sin\frac{C}{2} \geq \frac{h_a + h_b + h_c}{2}.$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Polyahedra, Polk State College, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; A S Arun Sriniavaas, Mumbai, India; Joel Schlosberg, Bayside, NY, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J593. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{(1+2a)^3} + \frac{1}{(1+2b)^3} + \frac{1}{(1+2c)^3} \geq \frac{1}{3(1+2abc)}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Polyahedra, Polk State College, USA

By Holder's inequality,

$$(1+abc+abc)\left(1+\frac{a}{b}+\frac{a}{c}\right)\left(1+\frac{a}{c}+\frac{a}{b}\right) \geq (1+a+a)^3,$$

thus

$$\frac{1}{(1+2a)^3} \geq \frac{1}{(1+2abc)} \cdot \frac{(bc)^2}{(ab+bc+ca)^2}.$$

The proof is complete by summing this with the other two analogous inequalities, and applying $(bc)^2+(ca)^2+(ab)^2 \geq (ab+bc+ca)^2/3$.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J594. Let a be a positive real number other than 1 and let c, d be real numbers such that

$$a^c + a^d = (a+1)a^{\frac{c+d-1}{2}}.$$

Prove that for all positive real numbers $b \neq 1$,

$$b^c + b^d = (b+1)b^{\frac{c+d-1}{2}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Notice that for $x > 0$,

$$f(x) = (x+1)x^{\frac{c+d-1}{2}} - x^c - x^d = x^{\frac{c+d+1}{2}}\left(1 - x^{\frac{c-d-1}{2}}\right)\left(1 - x^{\frac{d-c-1}{2}}\right).$$

If $f(a) = 0$ for some positive $a \neq 1$, then $c = d+1$ or $d = c+1$. Therefore, $f(b) = 0$ for all positive $b \neq 1$.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Sundaresh H R, Shivamogga, India.

Senior problems

S589. Let a, b, c be real numbers such that

$$\cos(a-b) + 2\cos(b-c) \geq 3\cos(c-a)$$

Prove that

$$|3\cos a - 2\cos b + 6\cos c| \leq 7$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Indeed,

$$\begin{aligned}(3\cos a - 2\cos b + 6\cos c)^2 &\leq (3\cos a - 2\cos b + 6\cos c)^2 + (3\sin a - 2\sin b + 6\sin c)^2 \\ &= 49 + 12(3\cos(c-a) - \cos(a-b) - 2\cos(b-c)) \leq 49.\end{aligned}$$

Also solved by Sundaresh H R, Shivamogga, India; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania.

S590. Let ABC be an acute triangle and let E be the center of its nine-point circle. Prove that

$$BE + CE \leq \sqrt{a^2 + R^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Văcaru, Pitești, Romania

We know that E is the midpoint of OH. We obtain

$$BE^2 = \frac{2\left(BO^2 + BH^2\right) - OH^2}{4} = \frac{2\left(R^2 + 4R^2\cos^2 B\right) - R^2 + 8R^2\cos A\cos B\cos C}{4} =$$

$$= \frac{R^2\left(1 + 8\cos B\left(\cos B + \cos A\cos C\right)\right)}{4} =$$

$$= \frac{R^2}{4}\left(1 + 8\cos B\left(\cos\left(\pi - (A + C)\right) + \cos A\cos C\right)\right) =$$

$$\frac{R^2}{4}\left(1 + 8\cos B\sin A\sin C\right) \Rightarrow BE = \frac{R}{2}\sqrt{1 + 8\cos B\sin A\sin C}$$

In the same manner, we obtain

$$CE = \frac{R}{2}\sqrt{1 + 8\cos C\sin A\sin B.}$$

It follows, using Cauchy-Buniakowski-Schwartz, that

$$BE + CE = \frac{R}{2}\left(\sqrt{1 + 8\cos B\sin A\sin C} + \sqrt{1 + 8\cos C\sin A\sin B}\right) \leq$$

$$\leq \frac{R\sqrt{2}}{2}\left(\sqrt{\left(1 + 8\cos B\sin A\sin C\right) + \left(1 + 8\cos C\sin A\sin B\right)}\right) =$$

$$= R\sqrt{1 + 4\cos B\sin A\sin C + 4\cos C\sin A\sin B} = R\sqrt{1 + 4\sin A\left(\sin B\cos C + \cos B\sin C\right)} =$$

$$R\sqrt{1 + 4\sin A\sin\left(B + C\right)} = R\sqrt{1 + 4\sin^2 A} = \sqrt{R^2 + 4R^2\sin^2 A} = \sqrt{R^2 + a^2}.$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; A S Arun Sriniavaas, Mumbai, India; Marian Ursărescu, Roman, Romania; Telemachus Baltsavias, Kerameies Junior High School Kefalonia, Greece; Corneliu Mănescu-Avram, Ploieşti, Romania; Nandan Sai Dasireddy, Hyderabad, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

S591. Prove that there are infinitely many even positive integers n such that

$$n \mid 2^n - 2, \;\; n \nmid \; 3^n - 3.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Lemma: There are infinitely many even n such that 73 divides n and

$$2^n \equiv 2 \pmod{n}.$$

We know that $m = 2 \cdot 73 \cdot 1103$ satisfies the above congruence. Now, assume that $m = 2r$ satisfies the congruence

$$2^m \equiv 2 \ (\mathrm{mod}\ m).$$

Then,

$$2^{2r-1} \equiv 1 \ (\mathrm{mod}\ r).$$

Take a prime number p dividing $2^{2r-1} - 1$ such that $\mathrm{ord}_p^2 = 2r - 1$ it follows that $2r - 1$ divides $p - 1$. That is,

$$p = (2r - 1)s + 1 > r.$$

We are going to prove that

$$2^{2rp} \equiv 2 \ (\mathrm{mod}\ 2rp).$$

It suffices to prove that

$$2^{2rp-1} \equiv 1 \ (\mathrm{mod}\ p),$$

and

$$2^{2rp-1} \equiv 1 \ (\mathrm{mod}\ r).$$

For the first congruence, it suffices to prove that $\mathrm{ord}_p^2 = 2r - 1$ divides $2rp - 1$. Indeed,

$$2rp - 1 \equiv p - 1 \equiv 0 \ (\mathrm{mod}\ 2r - 1).$$

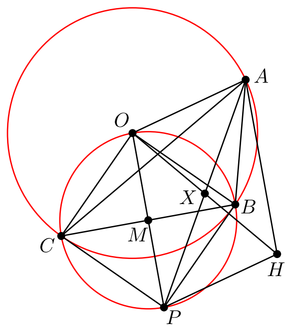
For the second congruence, since r divides $2^{2r-1} - 1$ it suffices to prove that $2^{2r-1} - 1$ divides $2^{2rp-1} - 1$. That is, because $2r - 1$ divides $2rp - 1$, we are done. So, setting $m = 2 \cdot 73 \cdot 1103$. Then, the operation $m \to mp$ preserves the divisibility by 73. This completes our proof.

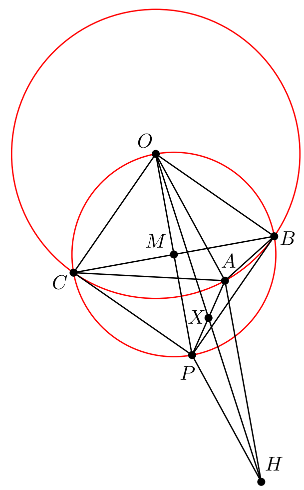
Back to our problem, take an even n such that $\mathcal{N} \mid 2^n - 2$ and 73 divides n. We are going to prove that $n \nmid 3^n - 3$. Otherwise, 73 must divide $3^{n-1} - 1$. Since $\mathrm{ord}_{73}^3 = 12$, it follows that 12 divides $n - 1$. But, n is even. This proves our problem.

S592. Let ABC be a triangle and let E, F be the foot of the altitude from B, C, respectively. Denote by X the center of nine-point circle of $\triangle ABC$ and *assume that the symmedian from* A intersects EF in X. Find $\angle BAC$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Li Zhou, Polk State College, USA





There is no need to mention E, F. We only need to assume that the symmedian from A passes through X. Let O and H be the circumcenter and orthocenter of $\triangle ABC$, respectively. Suppose that the tangents to the circumcircle of $\triangle ABC$ at B and C intersect at P. Then AP is the symmedian, thus passes through X. Also, OP is the perpendicular bisector of BC, thus is parallel to AH. Since X is the midpoint of OH, $AOPH$ is a parallelogram. Let M be the midpoint of BC, then $OP = AH = 2OM$, so $OCPB$ is a square. Therefore, $\angle BAC = 45°$ or $135°$.

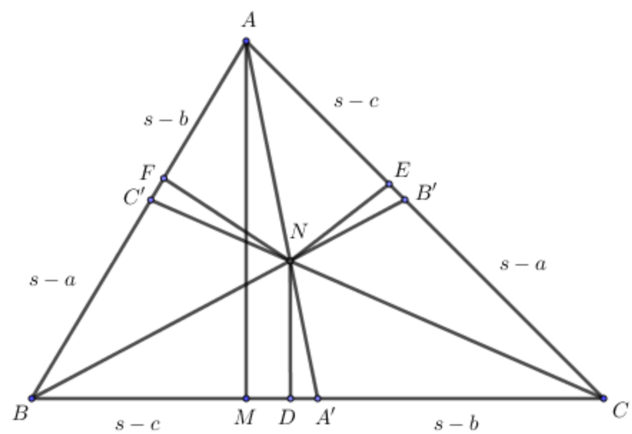
Also solved by Theo Koupelis, Cape Coral, FL, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Telemachus Baltsavias, Kerameies Junior High School Kefalonia, Greece; Nandan Sai Dasireddy, Hyderabad, India.

S593. Let ABC be a triangle and let N be its Nagel point. Let D, E, F be the orthogonal projections of N onto BC, CA, AB, respectively. Prove that

$$ND + NE + NF \le r\left(\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c}\right)$$

Proposed by Marian Ursărescu, National College Roman-Vodă, Roman, Romania

Solution by the author



$\triangle NDA' \sim \triangle AMA'$ then $\dfrac{ND}{AM} = \dfrac{NA'}{AA'}$ hence

$$ND = h_a \cdot \frac{NA'}{AA'} \tag{1}$$

From Van Aubel's theorem, it follows that:

$$\frac{AN}{NA'} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$$

$$\frac{NA'}{AN} = \frac{s-a}{a} \tag{2}$$

From (1) and (2) we get:

$$ND = h_a \cdot \frac{s-a}{s} = \frac{2F}{a}\frac{s-a}{s} = \frac{2r(s-a)}{a} \text{ and analogs}$$

$$ND + NE + NF = 2r\sum_{cyc}\frac{s-a}{a} = 2r\sum_{cyc}\frac{s(s-a)}{as} \tag{3}$$

However, from (3) and (4), it follows that:

$$ND + NE + NF \le r\sum_{cyc}\frac{s(s-a)}{w_a r_a} \tag{5}$$

But

$$s(s-a) \le m_a w_a \tag{6}$$

From (5) and (6) we get:

$$ND + NE + NF \le r\sum_{cyc}\frac{m_a}{r_a}$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Nandan Sai Dasireddy, Hyderabad, India.

S594. Let a, b, c be positive real numbers. Prove that

$$\frac{(4a+b+c)^2}{2a^2+(b+c)^2}+\frac{(4b+c+a)^2}{2b^2+(c+a)^2}+\frac{(4c+a+b)^2}{2c^2+(a+b)^2}\le\frac{52}{3}+\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

The inequality is homogeneous in a, b, c, so we can start by assuming $a+b+c=3$. Hence, the inequality can be rewitten as

$$\frac{(a+1)^2}{a^2-2a+3}+\frac{(b+1)^2}{b^2-2b+3}+\frac{(c+1)^2}{c^2-2c+3}\le\frac{17}{3}+\frac{1}{a^2+b^2+c^2}.$$

or,

$$\sum_{cyc}\frac{2a-1}{a^2-2a+3}\le\frac{4}{3}+\frac{1}{2(a^2+b^2+c^2)}.$$

Let $ab+bc+ca=3(1-t^2)$, $0\le t\le 1$ and $abc=r$, then $a^2+b^2+c^2=3(1+2t^2)$. After expanding and rearranging terms in left hand side, the proposed inequality is equivalent to

$$\frac{-6r(t^2+2)+24-6t^2-9t^4}{r^2+2(3t^2-1)r+27t^4+18t^2+9}\le\frac{4}{3}+\frac{1}{6(1+2t^2)},$$

or after clearing denominators,

$$(16t^2+9)r^2+2(84t^4+101t^2+27)r+9(4t^2-1)((15t^2+7)(t^2+1)\ge 0.$$

We have 2 cases:

Case 1: $t\ge\frac{1}{2}$ then the inequality is clearly true. The equality holds when $r=0$ and $t=\frac{1}{2}$ which means $a=b=\frac{3}{2}$, $c=0$ and its cyclic permutations.

Case 2: $t<\frac{1}{2}$ then $r\ge(1-2t)(1+t)^2$. Hence, it remains to prove

$$(16t^2+9)(1-2t)^2(1+t)^4+2(84t^4+101t^2+27)(1-2t)(1+t)^2+9(4t^2-1)(15t^2+7)(t^2+1)\ge 0$$

that is

$$4t^2(2t-1)(4t-7)(t^2+2)(2t^2+1)\ge 0$$

clearly true. The equality holds when $t=0$ which means $a=b=c$.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania.

U589. Let $a \geq 4$ be a positive integer. Prove that there are two relatively prime composite positive integers x_1, x_2 such that for all $n, \mid x_n \mid$ is composite. Where

$$x_{n+1} = ax_n - x_{n-1}$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

First, prove following lemma:

Lemma: Let $\mid a \mid \geq 4, y_1 = 1, y_2 = a, y_{n+1} = ay_n - y_{n-1}$. Then there are five distinct prime numbers $p_1, \ldots, p_5$ such that

$$p_1 \mid y_2, p_2 \mid y_3, p_3 \mid y_4, p_4 \mid y_6, p_5 \mid y_{12}$$

Let p_1 be any divisor of $y_2 = a$ and let $p_2 \neq 2$ be any divisor of $y_3 = a^2 - 1 = (a = 1)(a+1)$. Indeed such p_2 exists because $\mid a \mid \geq 4$. Clearly $p_2 \neq p_1$. Since $a^2 - 2 \equiv 2, 3 (\bmod 4)$ it is not divisible by 4 . So $a^2 - 2$ must have an odd prime divisor p_3. Clearly $p_3 \neq p_1, p_2$. We select p_3 as a prime dividing y_4. Further, 9 doesn't divide $a^2 - 3$. Hence, there is a prime $p_4 \neq 3$ that divides $a^2 - 3$ and it is clear that $p_4 \neq p_1, p_2, p_3$. Select p_4 as a divisor of y_6. Finally, we shall show that $a^4 - 4a^2 + 1$ has a prime divisor p_5 different from p_1, p_2, p_3, p_4. Note that

$$a^4 - 4a^2 + 1 = (a^2 - 1)(a^2 - 3) - 1$$

And

$$a^4 - 4a^2 + 1 = (a^2 - 2)^2 - 3$$

Finally, note that

$$y_3 = a^2 - 1, y_4 = a(a^2 - 2), y_6 = a(a^2 - 1)(a^2 - 3), y_{12} = a(a^2 - 1)(a^2 - 2)(a^2 - 3)(a^4 - 4a^2 + 1)$$

Thus, the lemma is proved.

Next , note that sets $0 (\bmod 2), 0 (\bmod 3), 1 (\bmod 4), 5 (\bmod 6), 7 \pmod{12}$ cover the positive integers. Consider now the primes p_1, p_2, p_3, p_4, p_5 and following systems:

$$\begin{aligned} s &\equiv y_2 (\bmod p_1), \\ s &\equiv y_3 (\bmod p_2), \\ s &\equiv y_3 (\bmod p_3), \\ s &\equiv y_1 (\bmod p_4), \\ s &\equiv y_5 (\bmod p_5). \end{aligned}$$

And

$$\begin{aligned} r &\equiv y_3 (\bmod p_1), \\ r &\equiv y_4 (\bmod p_2), \\ r &\equiv y_4 (\bmod p_3), \\ r &\equiv y_2 (\bmod p_4), \\ r &\equiv y_6 (\bmod p_5). \end{aligned}$$

If p divides y_n, y_{n-1} then p divides y_{n-2}. Hence it divides y_1, y_2, which is impossible. Let $P = p_1 p_2 \dots p_5$ Then for every $x_1 \equiv s(\text{mod} P)$ and $x_2 \equiv r(\text{mod} P)$ we have $x_3 \equiv y_4(\text{mod} p_1), y_5(\text{mod} p_2), \dots, y_7(\text{mod} p_5)$. By the same argument:

$$x_{n+1} \equiv y_{n+2}(\text{mod} p_1), y_{n+3}(\text{mod} p_2), \dots, y_{n+5}(\text{mod} p_5)$$

Since each $n \geq 0$ belongs to one of $0(\text{mod}2), 0(\text{mod}3), 1(\text{mod}4), 5(\text{mod}6), 7(\text{mod}12)$. Letting for example $n = 7 + 12k$ for some $k \geq 0$. Then p_5 divides $y_{12(k+1)}$. Hence,

$$x_{n+1} \equiv y_{n+5} \equiv y_{12k+12} = y_{12(k+1)} \equiv 0(\text{mod} p_5)$$

It remains to choose two composite and relatively prime numbers $x_1 \equiv s(\text{mod} P)$ and $x_2 \equiv r(\text{mod} P)$ such that $| x_n |> \max(p_1, \dots, p_5)$ for each n. Choose $x_1 > \max(p_1, \dots, p_5)$ and $x_1 \equiv s(\text{mod} P)$. We are done.

Also solved by Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Le Hoang Bao, Tien Giang, Vietnam.

U590. Prove that for all positive real numbers x, y,

$$x^x + y^y \geq 2\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}}.$$

Proposed by Toyesh Prakash Sharma, Agra College, India

Solution by the author

Let a function $f(x) = x^x$ then $\ln f(x) = x \ln x$,

Differentiate both sides with respect to x.

$$\frac{f'(x)}{f(x)} = 1 + \ln x \Rightarrow f'(x) = f(x) + f(x)\ln x$$

Again, differentiate it w.r.t. x. as a result we obtained

$$f''(x) = f'(x) + f'(x)\ln x + \frac{f(x)}{x} > 0$$

So, now we can claim that $f(x)$ is convex then from Jensen's Inequality we can say

$$f(x) + f(y) \geq 2 \cdot f\left(\frac{x+y}{2}\right)$$

$$\Rightarrow x^x + y^y \geq 2 \cdot \left(\frac{x+y}{2}\right)^{\left(\frac{x+y}{2}\right)}$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Anmol Kumar, IISc Bangalore, India; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Kumar Satyadarshi, Bihar, India; Matthew Too, Brockport, NY, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Michail Prousalidis, Evangeliki Model High School of Smyrna, Athens, Greece; Sundaresh H R, Shivamogga, India; Arkady Alt, San Jose, CA, USA.

U591. Prove that

$$\int_0^{\sqrt{\sqrt{7}-1}} (x^3+x)\, e^{-x^2}\, dx \le \ln 2.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by the author

Let I be the definite integral in the left-hand side. We have

$$e^x = 1 + x + \frac{x^2}{2} + ...,$$

for all real numbers x so $e^{x^2} > 1 + x^2 + \dfrac{x^4}{2}$. Then

$$(x^3+x)e^{(-x^2)} < \frac{x^3+x}{1+x^2+\dfrac{x^4}{2}} = \frac{(x^4+2x^2+2)'}{2(x^4+2x^2+2)},$$

implying

$$I < \frac{1}{2}(ln((x^2+1)^2+1)\Big|_0^{\sqrt{\sqrt{7}-1}} = \frac{1}{2}(\ln 8 - \ln 2) = \ln 2.$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Le Hoang Bao, Tien Giang, Vietnam; G. C. Greubel, Newport News, VA, USA; Anmol Kumar, IISc Bangalore, India; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Kumar Satyadarshi, Bihar, India; Matthew Too, Brockport, NY, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sundaresh H R, Shivamogga, India; Arkady Alt, San Jose, CA, USA.

U592. Evaluate

$$\sum_{n=1}^{\infty} \frac{H_n\, H_{n+1}}{(n+1)(n+2)},$$

where H_n denotes the n^{th} harmonic number.

Proposed by Ovidiu Furdui and Alina Sîntămărian, Cluj-Napoca, Romania

Solution by the author

The series equals $\zeta(2) + \zeta(3)$. The series telescopes. We have

$$\begin{aligned}
\frac{H_n H_{n+1}}{(n+1)(n+2)} &= \frac{H_n H_{n+1}}{n+1} - \frac{H_n H_{n+1}}{n+2} \\
&= \frac{H_n\left(H_n + \frac{1}{n+1}\right)}{n+1} - \frac{\left(H_{n+1} - \frac{1}{n+1}\right) H_{n+1}}{n+2} \\
&= \frac{H_n^2}{n+1} - \frac{H_{n+1}^2}{n+2} + \frac{H_n}{(n+1)^2} + \frac{H_{n+1}}{(n+1)(n+2)} \\
&= \frac{H_n^2}{n+1} - \frac{H_{n+1}^2}{n+2} + \frac{H_{n+1} - \frac{1}{n+1}}{(n+1)^2} + \frac{H_{n+1}}{n+1} - \frac{H_{n+1}}{n+2} \\
&= \frac{H_n^2}{n+1} - \frac{H_{n+1}^2}{n+2} + \frac{H_{n+1}}{(n+1)^2} - \frac{1}{(n+1)^3} + \frac{H_{n+1}}{n+1} - \frac{H_{n+2}}{n+2} + \frac{1}{(n+2)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{(n+1)(n+2)} &= \sum_{n=1}^{\infty}\left(\frac{H_n^2}{n+1} - \frac{H_{n+1}^2}{n+2}\right) + \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{H_{n+1}}{n+1} - \frac{H_{n+2}}{n+2}\right) + \sum_{n=1}^{\infty} \frac{1}{(n+2)^2} \\
&= \frac{H_1^2}{2} + \sum_{i=2}^{\infty} \frac{H_i}{i^2} - \sum_{i=2}^{\infty} \frac{1}{i^3} + \frac{H_2}{2} + \zeta(2) - 1 - \frac{1}{4} \\
&= \frac{1}{2} + \sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{1}{i^3} + \frac{3}{4} + \zeta(2) - \frac{5}{4} \\
&= 2\zeta(3) - \zeta(3) + \zeta(2) \\
&= \zeta(2) + \zeta(3),
\end{aligned}$$

since $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$.

For the sake of completeness we give the proof of the above Euler series below.

One can check that $\sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{H_n}{n}$. It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n^2} &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{nk(n+k)} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{nk} \int_0^1 x^{n+k-1} \mathrm{d}x \\
&= \int_0^1 \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{k=1}^{\infty} \frac{x^{k-1}}{k} \mathrm{d}x \\
&= \int_0^1 \frac{\ln^2(1-x)}{x} \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{\ln^2 y}{1-y}\mathrm{d}y \\
&= \int_0^1 \ln^2 y \sum_{n=0}^{\infty} y^n \mathrm{d}y \\
&= \sum_{n=0}^{\infty} \int_0^1 y^n \ln^2 y \,\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{2}{(n+1)^3} \\
&= 2\zeta(3).
\end{aligned}
$$

The problem is solved.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Matthew Too, Brockport, NY, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Yunyong Zhang.

U593. Let ABC an acute scalene triangle with circumcenter O and barycenter G. Let W be point on line BC such that $GW \perp BC$. Given that $b^2 + c^2 = 3a^2$ show that the line OW is tangent to Jerabek hyperbola of the triangle ABC.

Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece

Solution by the author

Let H be the orthocenter of ABC and $BH \cap AC = E, CH \cap AB = F$.

Claim: $G \in EF$.

Proof: Let M be the midpoint of BC and $EF \cap BC = T, AH \cap BC = D$. We have $(T, D; B, C) = -1$ thus

$$\frac{TB}{TC} = \frac{DB}{DA} = \frac{AB}{AC}\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{c\cos B}{b\cos C} \Leftrightarrow \frac{TB}{TB + a} = \frac{c\cos B}{bcosC} \Leftrightarrow TB = \frac{ac\cos B}{b\cos C - c\cos B}.$$

Using Menelaus's theorem it suffice to show that

$$\frac{GA}{GM}\cdot\frac{TM}{TB}\cdot\frac{FB}{FA} = 1 \Leftrightarrow 2\cdot\frac{2ac\cos B + a(b\cos C - c\cos B)}{2ac\cos B}\cdot\frac{a\cos B}{b\cos A} = 1 \Leftrightarrow a(b\cos C + c\cos B) = bc\cos A.$$

Observe that $b\cos C + c\cos B = \dfrac{a^2 + b^2 - c^2}{2a} + \dfrac{a^2 + c^2 - b^2}{2a} = a$ sowe need to show that $bc\cos A = a^2 \Leftrightarrow b^2 + c^2 - a^2 = 2a^2 \Leftrightarrow b^2 + c^2 = 3a^2$ which is true.

Now let L be the Lemoine point of ABC. Set $BL \cap AC = B_1, CL \cap AB = C_1$ and $AL \cap BC = A_1$.

Claim: $A_1 \equiv W$.

Proof: Let R be the $A-$ *humpty* point of ABC. Then T, H, R are collinear and $MA\cdot MR = MC^2 = MB^2$. Observe that $MA^2 = \dfrac{2(b^2 + c^2) - a^2}{4} = \dfrac{5a^2}{4}$ and thus $MA^2\cdot\dfrac{MR}{MA} = \dfrac{a^2}{4} \Leftrightarrow MA = 5MR$. Since $GA = 2GM$ it's easy to show that $(M, G; R, A) = 1$. Since $TH \perp AM$ if $X = TH \cap GW$ then X is the ortho-center of TGM. Let $Z = MX \cap EF$. Since $(M, G; R, A) = -1$ we have that the points W, Z, A are collinear. Now observe that $\angle BAW = \angle BAH + \angle GWA = \angle OAC + \angle GMZ = \angle OAC + 90^\circ - \angle ZGM = \angle OAC + \angle GAO = \angle MAC$ and we are done.

Now we'll prove the following lemma.

Lemma: Let Q an arbitrary point on line P_1P_2. Denote by P^*, Q^* the isogonal conjugates points of P, Q with respect to ABC. If $BQ^* \cap AC = Q_1, CQ^* \cap AB = Q_2$ then $P^* \in Q_1Q_2$.

Proof: We will use moving points. Move the point Q on line P_1P_2. Then $Q \to CQ \to CQ^* \to Q_1$. Also $Q_1 \equiv A \Leftrightarrow Q_2 \equiv A$(this happens only when $Q = P_1P_2 \cap BC$). Thus the line Q_1Q_2 must pass through a fixed point. Checking some simple cases for Q we see that this point is P^*.

Now, using lemma for $P \equiv H, Q \equiv G$ we have that $O \in B_1C_1$. Let $B_1C_1 \cap AW = Y$. Note that from the complete quadrilateral AC_1LB_1BC we have that $(A, L; Y, W) = -1$. If $B_1C_1 \cap BC = J$ then $(J, W; B, C) = -1$. and since Jerabek hyperbola passes through A, B, C, L it follows that B_1C_1 is the polar of W and we are done.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania.

U594. Let n be a positive integer. Evaluate

$$\int_1^n \lfloor x \rfloor dx$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Let p be the unique integer such that $p^2 \le n < (p+1)^2$. The integral is

$$\int_1^n \lfloor x \rfloor dx = \sum_{i=1}^{p-1} \int_{i^2}^{(i+1)^2} i dx + \int_{p^2}^n p dx = \sum_{i=1}^{p-1} i(2i+1) + p(n-p^2)$$
$$= \frac{p(4p+1)(p-1)}{6} + p(n-p^2)$$

Second solution by the author

$$\begin{aligned}\int_1^n \lfloor \sqrt{x} \rfloor \mathrm{d}x &= \sum_{k=1}^{\lfloor \sqrt{n}-1 \rfloor} \int_{k^2}^{(k+1)^2} \lfloor \sqrt{x} \rfloor \mathrm{d}x + \int_{(\lfloor \sqrt{n}-1 \rfloor+1)^2}^n \lfloor \sqrt{x} \rfloor \mathrm{d}x \\ &= \sum_{k=1}^{\lfloor \sqrt{n}-1 \rfloor} \int_{k^2}^{(k+1)^2} k \mathrm{d}x + \int_{(\lfloor \sqrt{n}-1 \rfloor+1)^2}^n (\lfloor \sqrt{n}-1 \rfloor+1) \mathrm{d}x \\ &= \sum_{k=1}^{\lfloor \sqrt{n}-1 \rfloor} k(2k+1) + (\lfloor \sqrt{n}-1 \rfloor+1)\left(n-(\lfloor \sqrt{n}-1 \rfloor+1)^2\right) \\ &= \frac{m(m+1)(4m+5)}{6} + (m+1)(n-(m+1)^2) \qquad (\text{where } m = \lfloor \sqrt{n}-1 \rfloor).\end{aligned}$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Rohan Dalal and Tommy Goebeler, The Episcopal Academy, Newtown Square, PA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Yunyong Zhang; Henry Ricardo, Westchester Area Math Circle, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Anmol Kumar, IISc Bangalore, India; Matthew Too, Brockport, NY, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Sundaresh H R, Shivamogga, India; Arkady Alt, San Jose, CA, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania.

O589. Let x, y, z be positive real numbers. Find the minimum of

$$\frac{xy^2}{z(x^2+xz+z^2)}+\frac{yz^2}{x(y^2+yx+x^2)}+\frac{zx^2}{y(z^2+zy+y^2)}+2\left(\frac{x}{y}+\frac{y}{z}+\frac{x}{z}\right)$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by the author

By Cauchy-Schwarz's inequality we have

$$\frac{(xy)^2}{xz(x^2+xz+z^2)}+\frac{(yz)^2}{xy(y^2+yx+x^2)}+\frac{(zx)^2}{zy(z^2+zy+y^2)}\geq\frac{(xy+yz+zx)^2}{xz(x^2+xz+z^2)+xy(y^2+yx+x^2)+zy(z^2+zy+y^2)}\quad (*)$$

We have, $(x^2+y^2)(x-y)^2+(y^2+z^2)(y-z)^2+(z^2+x^2)(z-x)^2\geq 0; \forall x,y,z>0$

$$\Leftrightarrow (x^2+y^2)^2-2xy(x^2+y^2)+(y^2+z^2)^2-2yz(y^2+z^2)+(z^2+x^2)^2-2zx(z^2+x^2)\geq 0$$

$$\Leftrightarrow x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2)\geq xy(x^2+xy+y^2)+yz(y^2+yz+z^2)+zx(z^2+zx+x^2)$$

$$xy(x^2+xy+y^2)+yz(y^2+yz+z^2)+zx(z^2+zx+x^2)\leq(x^2+y^2+z^2)^2$$

Hence, (*):$\Rightarrow \dfrac{xy^2}{z(x^2+xz+z^2)}+\dfrac{yz^2}{x(y^2+yx+x^2)}+\dfrac{zx^2}{y(z^2+zy+y^2)}\geq\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Then $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}=\dfrac{x^2}{xy}+\dfrac{y^2}{yz}+\dfrac{z^2}{zx}\geq\dfrac{(x+y+z)^2}{xy+yz+zx}=\dfrac{x^2+y^2+z^2}{xy+yz+zx}+2$; let $t=\dfrac{x^2+y^2+z^2}{xy+yz+zx}>0$ and by Cauchy's inequality we have $P\geq 2(t+2)+\dfrac{1}{t^2}=t+t+\dfrac{1}{t^2}+4\geq 3\sqrt[3]{t\cdot t\cdot\dfrac{1}{t^2}}+4=3+4=7$

Hence, the minimum value of expression is 7 when $x=y=z>0$.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Arkady Alt, San Jose, CA, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.

O590. Let ABC be a scalene triangle with centroid G and symmedian point K. Knowing that $\angle BAG = \angle ABK$, prove that GK is parallel to BC.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution by Li Zhou, Polk State College, USA

Let M be the midpoint of BC. Since $\angle BAG = \angle ABK = \angle GBC$, $\triangle ABM \sim \triangle BGM$. Thus,

$$\frac{a^2}{4} = BM^2 = AM \cdot GM = \frac{AM^2}{3} = \frac{2b^2 + 2c^2 - a^2}{12},$$

so $2a^2 = b^2 + c^2$. Suppose AK intersects BC at D. Since $[ABK]/[BCK] = c^2/a^2$ and $[CAK]/[BCK] = b^2/a^2$,

$$\frac{AK}{KD} = \frac{[ABK] + [CAK]}{[BCK]} = \frac{c^2 + b^2}{a^2} = \frac{1}{2} = \frac{AG}{GM},$$

completing the proof.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Kousik Sett, India; Nandan Sai Dasireddy, Hyderabad, India; Telemachus Baltsavias, Kerameies Junior High School Kefalonia, Greece.

O591. Real numbers $a_1, \ldots, a_n$ satisfy

$$a_i + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = n - 1.$$

Prove that

$$n - 1 \le a_1^3 + a_2^3 + \cdots + a_n^3 < n + 1.$$

Proposed by Josef Tkadlec, Czech Republic

First solution by Li Zhou, Polk State College, USA

By the given conditions, $(a_1 - 1)^2 + \cdots + (a_n - 1)^2 = (n-1) - 2(n-1) + n = 1$, so $|a_i - 1| \le 1$ for $1 \le i \le n$. Consequently, $a_i \ge 0$ for $1 \le i \le n$. By the Cauchy-Schwarz inequality,

$$(n-1)(a_1^3 + \cdots + a_n^3) = (a_1 + \cdots + a_n)(a_1^3 + \cdots + a_n^3) \ge (a_1^2 + \cdots + a_n^2)^2 = (n-1)^2,$$

thus $a_1^3 + \cdots + a_n^3 \ge n-1$. Also, for each i, $(a_i - 1)^3 \le (a_i - 1)^2$, with equality if and only if $a_i \in \{1, 2\}$. Therefore,

$$a_1^3 + \cdots + a_n^3 = (a_1 - 1)^3 + \cdots + (a_n - 1)^3 + 3(a_1^2 + \cdots + a_n^2) - 3(a_1 + \cdots + a_n) + n \le 1 + n.$$

Since it is impossible to have $a_i \in \{1, 2\}$ for all $i \in \{1, \ldots, n\}$, equality cannot be achieved.

Second solution by the author

The lower bound follows immediately from Cauchy-Schwarz inequality in the form $(\sum_i a_i^3)(\sum_i a_i) \geq (\sum_i a_i^2)^2$.

For the upper bound, let $b_i = a_i - 1$. Then

$$\sum_{i=1}^{n} b_i = (n-1) - n = -1 \quad \text{a} \quad \sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{n}(a_i^2 - 2a_i + 1) = (n-1) - 2(n-1) + n = 1.$$

At the same time

$$\sum_{i=1}^{n} b_i^3 = \left(\sum_{i=1}^{n} a_i^3\right) - 3(n-1) + 3(n-1) - n = \left(\sum_{i=1}^{n} a_i^3\right) - n,$$

hence we need to show $\sum_{i=1}^{n} b_i^3 \in [-1, 1)$. Note that since $\sum_{i=1}^{n} b_i^2 = 1$, we have $b_i \in [-1, 1]$.

For $x \in [-1, 1]$ we have $x^3 \leq x^2$ (with equality if and only if $x \in \{0, 1\}$), hence

$$\sum_{i=1}^{n} b_i^3 \leq \sum_{i=1}^{n} b_i^2 \leq 1.$$

To show that the inequality is in fact sharp, suppose $b_i \in \{0, 1\}$. Then we have $a_i = b_i + 1 \geq 1$, hence $a_1 + a_2 + \cdots + a_n \geq n > n - 1$, a contradiction.

Alternative proof of the lower bound.

Once we restate the problem in terms of b_i, the lower bound can be done similarly to the upper bound: For $x \in [-1, 1]$ we have $(x+1) \cdot x^2 \geq 0$ (with equality when $x \in \{-1, 0\}$), hence

$$\sum_{i=1}^{n} b_i^3 \geq -\sum_{i=1}^{n} b_i^2 \geq -1.$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Anmol Kumar, IISc Bangalore, India; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania.

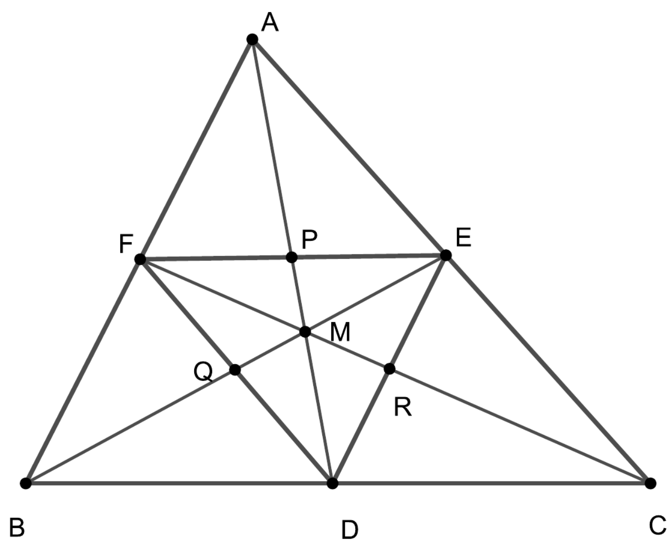
O592. Let M be an interior point of a triangle ABC. Let D, E, F be the intersection points of the lines AM, BM, CM with BC, CA, respectively AB and P, Q, R be the intersection points of the lines AM, BM, CM with EF, DF, respectively DE. Prove that

$$\frac{MA}{MD}+\frac{MB}{ME}+\frac{MC}{MF}\geq\frac{MD}{MP}+\frac{ME}{MQ}+\frac{MF}{MR}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

We have



$$(A,P,M,D)\text{ is harmonic }\Longrightarrow\frac{AD}{AP}=\frac{MD}{MP}.$$

It follows that

$$\frac{MD}{MP}>1\text{ and }\frac{MA}{MD}=\frac{2}{\frac{MD}{MP}-1},$$

$$\frac{MB}{ME}=\frac{2}{\frac{ME}{MQ}-1},$$

$$\frac{MC}{MF}=\frac{2}{\frac{MF}{MR}-1},$$

Now, if we denote with S_1,S_2,S_3 the area of triangle MEF,MDF, respectively MDE, we deduce that

$$\frac{MD}{MP}=\frac{S_2+S_3}{S_1},\quad\frac{ME}{MQ}=\frac{S_3+S_1}{S_2},\quad\frac{MF}{MR}=\frac{S_1+S_2}{S_3}.$$

Since $S_2 + S_3 > S_1$ let $x = S_2 + S_3 - S_1 > 0$. Similar, $y = S_3 + S_1 - S_2 > 0$ and $z = S_1 + S_2 - S_3 > 0$. Then

$$\frac{MD}{MP} = \frac{2x}{y+z} + 1, \quad \frac{MA}{MD} = \frac{y+z}{x},$$

$$\frac{ME}{MQ} = \frac{2y}{z+x} + 1, \quad \frac{MA}{MD} = \frac{z+x}{y},$$

$$\frac{MF}{MR} = \frac{2z}{x+y} + 1, \quad \frac{MA}{MD} = \frac{x+y}{z}.$$

Our inequality becomes

$$\sum_{cyc} \left(\frac{x}{y} + \frac{y}{x} \right) \geq 3 + 2 \sum_{cyc} \frac{x}{y+z},$$

that is

$$\sum_{cyc} \frac{(x-y)^2}{xy} \geq \sum_{cyc} \frac{(x-y)^2}{(x+z)(y+z)},$$

clearly true.

Also solved by Theo Koupelis, Cape Coral, FL, USA.

O593. Let a, b, c, d be four non-zero complex numbers such that

$$2|a-b| \leq |b|, 2|b-c| \leq |c|, 2|c-d| \leq |d|, 2|d-a| \leq |a|.$$

Prove that

$$\max\{|\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}|, |\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}|\} > 2\sqrt{3}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

We can rewrite the assumptions of our problem as follows

$$|\frac{a}{b}-1| \leq \frac{1}{2}, |\frac{b}{c}-1| \leq \frac{1}{2}, |\frac{c}{d}-1| \leq \frac{1}{2}, |\frac{d}{a}-1| \leq \frac{1}{2}.$$

Let$(\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}) = (x, y, z, t)$ it follows that $|x-1| \leq \frac{1}{2}, |y-1| \leq \frac{1}{2}, |z-1| \leq \frac{1}{2}, |t-1| \leq \frac{1}{2}$. We are then going to prove that

$$|x+y+z+t||\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}| > 12.$$

This will also prove the statement of our problem. Letting $x = u + iv$, for some real numbers u, v such that $u^2 + v^2 \leq 2u - \frac{3}{4}$. It follows that $u \geq \frac{3}{8}$. Since

$$\frac{1}{x} = \frac{u - iv}{u^2 + v^2},$$

We would obtain

$$|\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}| = |\sum \frac{u-iv}{u^2+v^2}| \geq |\mathfrak{Re}(\sum \frac{u-iv}{u^2+v^2})| = [\sum \frac{u}{u^2+v^2}].$$

On the other hand,

$$|x+y+z+t| \geq |\mathfrak{Re}(x+y+z+t)| = |\sum u|.$$

That is,

$$|x+y+z+t||\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}| \geq (\sum u)(\sum \frac{u}{u^2+v^2}).$$

Since $u > 0$ we can use *Cauchy-Schwarz* inequality to obtain;

$$(\sum u)(\sum \frac{u}{u^2+v^2}) \geq (\sum \sqrt{\frac{u^2}{u^2+v^2}})^2.$$

Furthermore,

$$\frac{u^2}{u^2+v^2} \geq \frac{u^2}{2u-\frac{3}{4}} = \frac{4u^2}{8u-3} = \frac{1}{16}(\frac{64u^2}{8u-3}) = \frac{1}{16}(\frac{64u^2-9+9}{8u-3}) = \frac{1}{16}(8u+3+\frac{9}{8u-3}).$$

Yielding to;

$$8u+3+\frac{9}{8u-3} = 8u-3+\frac{9}{8u-3}+6 \geq 2\sqrt{(8u-3)\frac{9}{8u-3}}+6 \geq 12.$$

Hence,

$$\frac{u^2}{u^2+v^2} \geq \frac{12}{16} = \frac{3}{4}.$$

The equality case occurs whenever, $u = \frac{3}{4}$. Thus, $\sqrt{\frac{u^2}{u^2+v^2}} \geq \frac{\sqrt{3}}{2}$. Hence,

$$(\sum \sqrt{\frac{u^2}{u^2+v^2}})^2 \geq (4 \cdot \frac{\sqrt{3}}{2})^2 = 12.$$

The equality holds whenever $v = 0$ and $u = \frac{3}{4}$. That is, $x = y = z = t = \frac{3}{4}$. But, this doesn't happen because $xyzt = 1$. Hence, the inequality is strict.

Also solved by Anmol Kumar, IISc Bangalore, India.

O594. Find all positive integers a and b such that

$$2 - 3^{a+1} + 3^{3a} = pq^b,$$

for some prime numbers p and q.

Proposed by Mircea Becheanu, Canada

Solution by the author

From the decomposition $t^3 - 3t + 2 = (t+2)(t-1)^2$ we have the following form of the equation:

$$(3^a + 2)(3^a - 1)^2 = pq^b.$$

For $a = 1$ we have $5 \times 2^2 = pq^b$, so we may consider $p = 5$, $q = 2$ and $b = 2$.
For $a = 2$ we have $11 \times 2^6 = pq^b$, so we may consider $p = 11$, $q = 2$ and $b = 6$.
From now one we assume $a > 2$. Observe that $\gcd(3^a + 2, 3^a - 1) = 1$. If $p | 3^a - 1$ we have $p^2 | pq^b$, which is impossible. Hence

$$p | 3^a + 2 \Rightarrow p = 3^a + 2 \text{ and } q^b = (3^a - 1)^2 \Rightarrow q = 2 \text{ and } b = 2c,\ c \geq 1.$$

For $c = 1$ one obtains $q^2 = 2^2 = (3^a - 1)^2$, giving $a = 1$ and this case was removed. Therefore, $c \geq 2$ and we have $2^c = 3^a - 1$. Taking this equality modulo 4 we have $(-1)^a \equiv 1 \bmod 4$, giving that a is even. Put $a = 2d$. From

$$3^{2d} - 1 = 2^c \Rightarrow (3^d + 1)(3^d - 1) = 2^c$$

it follows that $3^d - 1 = 2^u$ and $3^d + 1 = 2^v$ with $u < v$. By subtracting these equalities we have $2^u(2^{v-u} - 1) = 2$, giving that $u = 1$, $v = 2$, $c = 3$, $d = 1$ $a = 2$ and $b = 6$. This case was already discussed.

Therefore, the solutions are only: $a = 1,\ b = 2,\ p = 5,\ q = 2$ and $a = 2,\ b = 6,\ p = 11,\ q = 2$.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Anmol Kumar, IISc Bangalore, India.