Local Monotony and Continuity

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December 5, 2021

Abstract

This paper intends to provide some answers regarding the necessity of the concept of local monotony in the study of continuous functions.

1 Introduction

The behavior of the functions is the main goal when studying real-valued functions. It is known that continuity does not imply monotony, as well as monotony, does not imply continuity. Anyway, if a continuous function changes from negative to positive values, an increasing behavior of the function might be suspected. This article provides a few answers to this concern and establishes the existence of an interval of monotony in certain plain conditions of differentiability. A new concept of "local" monotony is justified to be introduced, and relationships between the local monotony and monotony of continuous functions on a compact interval are being established. From a didactic perspective, the narrative intends to support the AP Calculus students with a better understanding of the continuity and differentiability of the functions.

2 Preliminaries

The introductory section presents the theme of the study and the terminology used throughout the article.

If nothing else is specified, real-valued functions defined on an non-empty interval of real numbers will be considered in the following sections.

The following standard mathematical notations will be used throughout the article:
⋄ *iff* meaning *if and only if*
⋄ $(\forall)$ denotes *for all*
⋄ $(\exists)$ stands for *there exists*
⋄ $\Longrightarrow$ denotes *implies*
⋄ $\Longleftrightarrow$ denotes *equivalent*
⋄ *s.t.* is an abbreviation for *such that*
⋄ *i.e.* is an abbreviation for *idem est* (latin) meaning *the same as previously given or mentioned*
⋄ $a \in A$ stands for the element *a belongs* to the set A
⋄ $A \subseteq B$ stands for the set A *included* into the set B

Definition 2.1. *Let f be a real-valued function defined on the interval (a, b).*

(1) The function f changes from negative to positive values on the interval (a, b) iff there is a point $c \in (a, b)$ s.t.

(i) $f(x) < 0$ for $x \in (a, c)$

(ii) $f(c) = 0$

(iii) $f(x) > 0$ for any $x \in (c, b)$ *(Figure 1)*

(2) The function f changes from positive to negative values on the interval (a, b) iff there is a point $c \in (a, b)$ s.t.

(i) $f(x) > 0$ for $x \in (a, c)$

(ii) $f(c) = 0$

(iii) $f(x) < 0$ for any $x \in (c, b)$

(3) The function f changes the sign at c on the interval (a, b) iff the function changes from negative to positive values on the interval (a, b) or from positive to negative values on the interval (a, b).

Example:

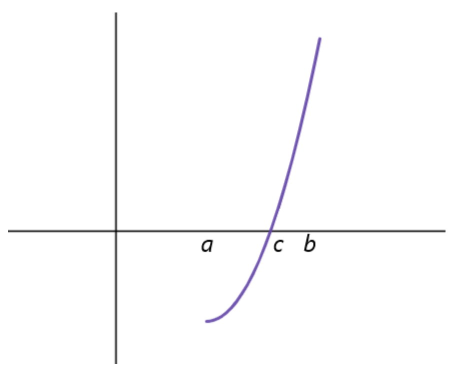

Figure 1: Example of function that changes from negative to positive values on the interval (a, b)

Definition 2.2. *Let c be a real number. A neighborhood of the point c is an open interval containing the point c.*

Example: The interval (-2, 1) is a neighborhood of 0. (Figure 2)
*Note: It can be proved that an open interval is a neighborhood of any of its points.

Definition 2.3. *Let f be a real-valued function defined on the interval I. The function f is called*

Figure 2: The interval $(-2, 1)$ is neighborhood of any of its points

(i) increasing on the interval I *iff for any* $x, y \in I$, $x < y$, *it results* $f(x) \leq f(y)$

(ii) strictly increasing on the interval I *iff for any* $x, y \in I$, $x < y$, *it results* $f(x) < f(y)$

(iii) decreasing on the interval I *iff for any* $x, y \in I$, $x < y$. *it results* $f(x) \geq f(y)$

(iv) strictly decreasing on the interval I *iff for any* $x, y \in I$, $x < y$, *it results* $f(x) > f(y)$

(v) monotonic on the interval I *iff the function* f *is increasing on the interval* I *or decreasing on the interval* I

(vi) strictly monotonic on the interval I *iff the function* f *is strictly increasing on the interval* I *or strictly decreasing on the interval* I

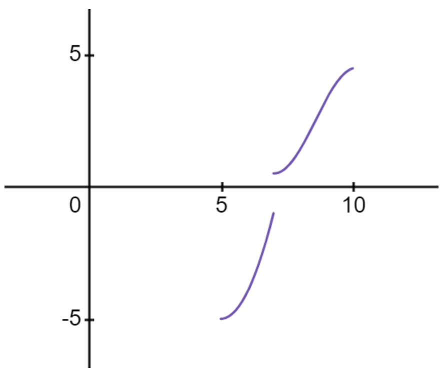

Figure 3: Example of function that is strictly increasing on the interval $(5, 10)$

Definition 2.4. *Let* f *be a real-valued function defined on the open interval* I. *Define the difference quotient of the function* f *on the non-empty interval* $[x, y] \subseteq I$ *as:*

$$R(x, y) = \frac{f(y) - f(x)}{y - x}$$

*Note that $R(x, y)$ represents the slope of the secant line determined by the points $(x, f(x))$ and $(y, f(y))$ of the graph of the function f (Figure 4).

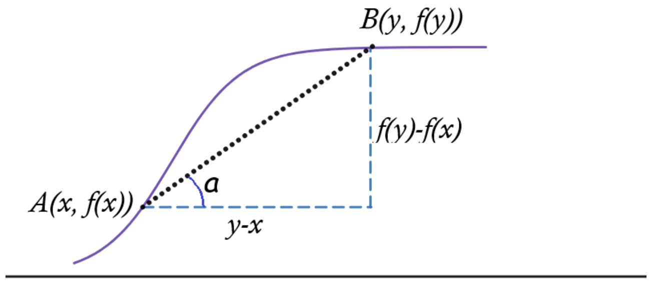

Figure 4: The difference quotient $R(x, y)$ is the slope of the secant line AB to the graph of the function f and $R(x, y) = \tan\alpha$

Lemma 2.1. *Let f be a real-valued function defined on the open interval I. The following holds:*

(i) the function f is increasing on the interval I iff for any $x, y \in I$, $x \neq y$, it results $R(x, y) \geq 0$

(ii) the function f is strictly increasing on the interval I iff for any $x, y \in I$, $x \neq y$. it results $R(x, y) > 0$

(iii) the function f is decreasing on the interval I iff for any $x, y \in I$, $x \neq y$, it results $R(x, y) \leq 0$

(iv) the function f is strictly decreasing on the interval I iff for any $x, y \in I$, $x \neq y$, it results $R(x, y) < 0$

Proof. The proof is immediate from the Definition 1.3. □

Remark 2.1. *In the context of Definition 1.1, the following holds:*

If a function f changes from negative to positive values, then a certain increase tendency in the values of the function can be described by:

$$(\forall)x, y \in (a, b),\ x < c < y \Rightarrow f(x) < 0 < f(y) \quad (1)$$

Proof. (1) is a reformulation of the conditions from Definition 1.1(1) □

*Note: The characterization from Definition 1.1(1) does not describe the increasing behavior of the function f on the interval (a, b) in the sense of Definition 1.3(i)-(ii). In fact, the characterization from Definition 1.1(1) is a weaker condition than conditions from Definition 1.3(i)-(ii) because it establishes the inequality $f(x) < f(y)$ for any $a < x < c < y < b$ and not for any $x < y$ of the interval (a, b) as in Definition 1.3 (i)-(ii).

The following definition will enable a specific terminology for the behavior described in (1).

Definition 2.5. *Let f be a real-valued function defined on the open interval I and c a point of this interval. The function f is called:*

(i) locally increasing at the point c iff there is a neighborhood (a, b) of c, $(a, b) \subseteq I$ s.t.

$$(\forall)x, y \in (a, b),\ x < c < y \implies f(x) \leq f(c) \leq f(y)$$

(ii) locally strictly increasing at the point c iff there is a neighborhood (a, b) of c, $(a, b) \subseteq I$ s.t.

$$(\forall)x, y \in (a, b),\ x < c < y \implies f(x) < f(c) < f(y)$$

(iii) locally decreasing at the point c iff there is a neighborhood (a, b) of c, $(a, b) \subseteq I$ s.t.

$$(\forall)x, y \in (a, b),\ x < c < y \implies f(x) \geq f(c) \geq f(y)$$

(iv) locally strictly decreasing at the point c iff there is a neighborhood (a, b) of c, $(a, b) \subseteq I$ s.t.

$$(\forall)x, y \in (a, b),\ x < c < y \implies f(x) > f(c) > f(y)$$

*Note 1. f locally increasing or f locally strictly increasing at a point c are defined only for the points c that are interior of the domain of definition of f, that is an open interval I.
The similar holds when defining f locally decreasing or f strictly locally decreasing at the point c.
2. If the function f changes from negative to positive values at the point c on the interval (a, b), then f is locally strictly increasing at the point c, and conversely, if f is locally strictly increasing at the point c then the function $k(x) = f(x) - f(c)$ changes from negative to positive at the point c on some interval $(g, h) \subseteq (a, b)$.
The similar holds for f when it changes from positive to negative values at the point c on the interval (a, b).

Lemma 2.2. *Let f be a real-valued function defined on the open interval I. The following hold:*

(i) the function f is locally increasing at the point c of the interval I iff

$$(\exists)\ (g, h) \subseteq I,\ g < c < h \text{ s.t. } (\forall)\ x, y \in (g, h),\ x < c < y \implies R(x, c) \geq 0,\ R(y, c) \geq 0$$

(ii) the function f is locally strictly increasing at the point c of the interval I iff

$$(\exists)\ (g, h) \subseteq I,\ g < c < h \text{ s.t. } (\forall)\ x, y \in (g, h),\ x < c < y \implies R(x, c) > 0,\ R(y, c) > 0$$

(iii) the function f is locally decreasing at the point c of the interval I iff

$$(\exists)\ (g, h) \subseteq I,\ g < c < h \text{ s.t. } (\forall)\ x, y \in (g, h),\ x < c < y \implies R(x, c) \leq 0,\ R(y, c) \leq 0$$

(iv) the function f is locally strictly decreasing at the point c of the interval I iff

$$(\exists)\ (g, h) \subseteq I,\ g < c < h \text{ s.t. } (\forall)\ x, y \in (g, h),\ x < c < y \implies R(x, c) < 0,\ R(y, c) < 0$$

Proof. (i) According to Definition 1.5, f locally increasing at the point c implies

$$(\exists)\ (g,h) \subseteq I,\ g < c < h \ s.t.\ (\forall)\ x, y \in (g,h),\ x < c < y \implies f(x) \geq f(c) \leq f(y)$$

From the inequalities above, it results $c-x > 0,\ f(c)-f(x) \geq 0$ and also $y-c > 0,\ f(y)-f(c) \geq 0$.

In consequence, the following holds

$$\frac{f(c)-f(x)}{c-x} \geq 0, \quad \frac{f(y)-f(c)}{y-c} \geq 0$$

Finally, it follows $R(x,c) \geq 0$ and $R(y,c) \geq 0$. So the conclusion is proved.

(ii)-(iv) can be proved analogously □

Definition 2.6. *Let f be a real-valued function defined on the interval (a,b).*
The function f is called:

(i) locally increasing on the interval (a,b) iff the function f is locally increasing at every point c of the interval (a,b)

(ii) locally strictly increasing on the interval (a,b) iff the function f is locally strictly increasing at every point c of the interval (a,b)

(iii) locally decreasing on the interval (a,b) iff the function f is locally decreasing at every point c of the interval (a,b)

(iv) locally strictly decreasing on the interval (a,b) iff the function f is locally strictly decreasing at every point c of the interval (a,b)

Definition 2.7. *Let f be a real-valued function defined on the interval (a,b).*
The function f is called:

(i) locally monotonic at the point c of the interval (a,b) iff the function f is locally increasing at the point c or the function f is locally decreasing at the point c

(ii) locally strictly monotonic at the point c of the interval (a,b) iff the function f is locally strictly increasing at the point c or the function f is locally strictly decreasing at the point c

(iii) locally monotonic on the interval (a,b) iff the function f is locally increasing on the interval (a,b) or the function f is locally decreasing on the interval (a,b)

(iv) locally strictly monotonic on the interval (a,b) iff the function f is locally strictly increasing on the interval (a,b) or the function f is locally strictly decreasing on the interval (a,b)

Proposition 2.1. *Let f be a function defined on the interval $(a,b.)$ The following hold:*

(i) if f is increasing on the interval (a,b), then f is locally increasing on the interval (a,b)

(ii) if f is strictly increasing on the interval (a,b), then f is locally strictly increasing on the interval (a,b)

(iii) if f is decreasing on the interval (a,b), then f is locally decreasing on the interval (a,b)

(iv) if f is strictly decreasing on the interval (a,b), then f is locally strictly increasing on the interval (a,b)

Proof. (i) Fix c a point of the interval (a,b). The intent is to prove that f is locally increasing at c. From f increasing on the interval (a,b), it follows that for any $x, y \in (a,b)$ s.t. $x < c < y$, it results $f(x) \leq f(c) \leq f(y)$. Thus, it results f locally increasing at c (according to Definition 1.5).
(ii)-(iv) it can be proved analogously with (i)

□

*Note that the converses statements from Lemma 1.1 are not true (Figure 5)

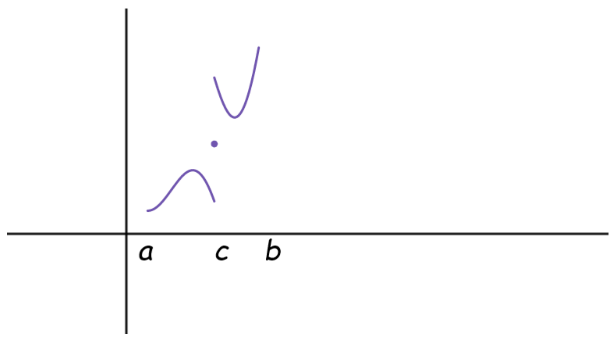

Figure 5: Example of function that is locally strictly increasing at $x = c$ and not increasing on the interval (a,b)

It seems natural to question an eventual potential of increasing in the behavior of the function f in the hypothesis of Definition 1.3 (1) i.e. if the function f changes from negative to positive values on the interval (a,b) at the point c or, in other terms said, if the function f is locally strictly increasing at the point c on some neighborhood $(g,h) \subseteq (a,b)$:

$(*)$ Is there a neighborhood of the point c on which the function f is increasing?

$(**)$ Is there an open interval, not necessarily a neighborhood of the point c, on which the function f is increasing?

where c is the zero of the function f as in the Definition 1.3 (1).

Finally, what will be the answers to the questions above, if specific conditions of continuity or differentiability are imposed to the function f?

3 Exploratory Examples

If a function changes from negative to positive values on the interval (a, b) at the point c, the answers to the questions $(*)$ and $(**)$ are both negative, as provided by the next example.

Example 3.1. *Consider f the piecewise function defined by:*

$$f(x) = \begin{cases} -x-1 & 0 < x < 2 \\ 0 & x = 2 \\ -x+4 & 2 < x < 4 \end{cases}$$

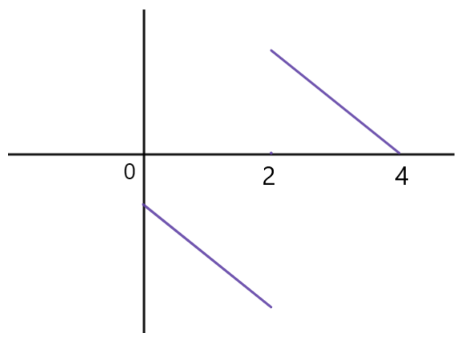

Figure 6: f changes from negatives to positives values at $x = 2$ and f is not increasing on any interval that contains $x = 2$

Indeed, the function f has the following properties:

→ f is strictly decreasing on each of the intervals $(0, 2)$ and $(2, 4)$

→ f changes from negatives to positives values at the point $x = 2$ on the interval $(0, 4)$ i.e.

$$f(x) < 0, \quad 0 < x < 2$$

$$f(2) = 0$$

$$f(x) > 0, \quad 2 < x < 4$$

→ f is not increasing on the interval $(0, 4)$

→ f is not increasing on any interval included in its domain of definition

→ f is locally strictly increasing at $x = 2$

→ f is locally strictly decreasing at each point of the intervals $(0, 2)$ and $(2, 4)$

The properties of the function from Example 2.1 justify why the answers to the questions:

(∗) Is there a neighborhood of the point c on which the function f is increasing?

(∗∗) Is there an open interval, not necessarily a neighborhood of the point c, on which the function f is increasing?

are both negative.

Conclusion 3.1.1. *A function f that changes from negative to positive values on the interval (a, b) does not necessarily have an interval on which the function is increasing. (Example 2.1, Figure 6) or, in other terms said,*
a function defined on the interval (a, b) that is locally increasing at the point c, it is not necessarily increasing on an interval included in (a, b). Furthermore, a function f defined on an open interval can have only one single point at which the function f is locally strictly increasing and at any other point of its domain the function f is locally strictly decreasing.

Examples 1.1 and 2.1 intuitively support the idea that a function changing from negative to positive values might involve an eventual potential of increasing behavior on a neighborhood of the point where it changes the sign only if its "jumping" behavior is canceled.
More specifically, any removable or non-removable point of discontinuity is not welcomed in the search of an interval of increasing due to the fact that an increasing behavior of the function can be easily troubled at a point of discontinuity by designing a backward jump that lowers the increasing tendency of the function. The condition of continuity seems to be essential for the existence of an increasing interval for functions changing from negative to positive values i.e. functions locally increasing at a point.

Now, back to the intuitive image of a *continuous* function that changes from negative to positive values as in the Figures 1 and 7, it seems obvious at the first glance that the answers to the questions:

(∗) Is there a neighborhood of the point c on which the function f is increasing?

(∗∗) Is there an interval in the neighborhood of c, not necessarily containing the point c, on which the function f is increasing?

are YES (Figure 1) and respectively YES (Figure 7).

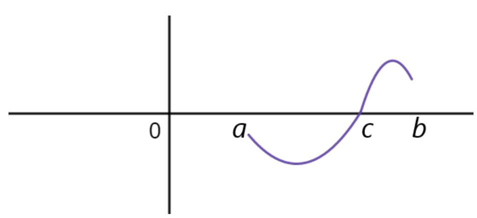

Figure 7: f continuous changes from positives to negatives and f is increasing on an interval that contains the point c

Motto: What seems obvious can be misleading.

Example 3.2. *Consider the piecewise function f defined by:*

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(Figure 8)

This well-known example enables the following:

Conclusion 3.2.1. *The following are true:*

(i) the points where the function f changes from negative to positive values are $x = \pm\frac{1}{\pi}, \pm\frac{1}{3\pi}, \pm\frac{1}{5\pi}, ...$

(ii) the function f is continuous on $(-\infty, 0) \cup (0, \infty)$

(iii) for any integer k, there is an interval that contains the point $x = (2k+1)\pi$ on which the function f is strictly increasing. More precisely, f is strictly increasing on each of the intervals $\left(\frac{2}{(4k+3)\pi}, \frac{2}{(4k+1)\pi}\right)$

Similarly,

(iv) the points where the function f changes from positive to negative values are $x = \pm\frac{1}{2\pi}, \pm\frac{1}{4\pi}, \pm\frac{1}{6\pi}, ...$

(v) for any integer $k, k \neq 0$, there is an interval that contain the point $x = 2k\pi$ on which the function f is strictly decreasing. More precisely, f is strictly decreasing on each of the intervals $\left(\frac{2}{(4k+1)\pi}, \frac{2}{(4k-1)\pi}\right)$

In consequence, the answers to the questions $(*)$ and $(**)$ for the points $x = \pm\frac{1}{\pi}, \pm\frac{1}{3\pi}, \pm\frac{1}{5\pi}, ...$ are both positive (Figure 8).

Proof. see [2] pag.182 □

*Note: What about the answers to the questions $(*)$ and $(**)$ when $c = 0$?

The most interesting part of the investigation it's just about to start ...

Also, Figure 8 suggests that the pattern from Conclusion 2.2.1 continues to appear for any integer k. One might say that for $k \longrightarrow \infty$, the claim referring to the existence of a neighborhood on which the function is increasing in the sense of the questions $(*)$ and $(**)$ seems to remain valid at $c = 0$, i. e. there is a neighborhood of $c = 0$ on that the function f is increasing.

Motto: Where technology fails to reveal

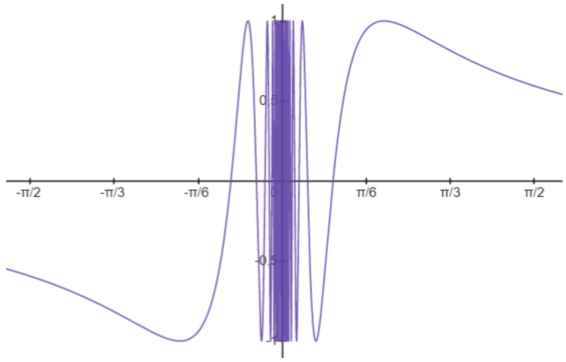

Figure 8: f discontinuous and does NOT change from positives to negatives or from negatives to positives at 0, in the sense of Definition 1.1

It is true that when $k \longrightarrow \infty$, it results $\frac{1}{2k\pi} \longrightarrow 0$ and this implies $\sin(2k\pi) \longrightarrow 0$. On the other hand, the corresponding neighborhood of increasing $\left(\frac{2}{(4k+1)\pi}, \frac{2}{(4k-1)\pi}\right) \longrightarrow [0,0] = 0$ is not an open interval, so the expected neighborhood of $c = 0$ on which the function would be increasing basically does not exist.
Finally, the answers at the questions $(*)$ and $(**)$ at the point $c = 0$ are given by:

Conclusion 3.2.2. *At $x = 0$, the following are true:*

(i) in any neighborhood of $x = 0$, the function f does NOT change from negatives to positives values or from positive to negative values at $x = 0$ as it changes at $x = \pm\frac{1}{\pi}, \pm\frac{1}{3\pi}, \pm\frac{1}{5\pi}, ...$ or at $x = \pm\frac{1}{2\pi}, \pm\frac{1}{4\pi}, \pm\frac{1}{6\pi}, ...$

(ii) the function f is NOT continuous at $x = 0$

(iii) there is no open interval that contains $x = 0$ on which the function f is increasing

(iv) any interval that contains $x = 0$ has an subinterval on which the function f is increasing as well a subinterval on which the function f is decreasing (see Example 2.2, Figure 8)

*Note: Due to Conclusion 2.2.2.(i), the case $c = 0$ does not correspond to the initial declared investigation, so the question $(*)$ does not make sense at $c = 0$ because in any neighborhood of 0, the function f does not change the sign at $x = 0$ in the sense of Definition 1.1. A detailed proof for (iii) and (iv) can be found in see [2] pag.182.

The question that arises after Conclusion 2.2.2 is referring to continuity: if the function f from Example 2.2 would be continuous overall its domain, then Conclusion 2.2.1.(iii) or (v) can be extended at the point $x = 0$? The answer suggested by the limits-considerations above seems to be negative. Indeed ...

The next example will confirm the fact that even if the function is continuous overall its domain, then it does not necessarily exist a neighborhood of $x = 0$ where the function changes from negative to positive values or from positive to negative values.

Example 3.3. *Consider the piecewise function f defined by:*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(Figure 9)

Intuitive, some monotonic patterns in the behavior of the function f can be noticed in any open interval centered at $x = 0$.

Indeed, it is not difficult to observe that the function f is increasing on any interval $\left(\frac{2}{(4k+3)\pi}, \frac{2}{(4k+1)\pi}\right)$ and decreasing on any interval $\left(\frac{2}{(4k+1)\pi}, \frac{2}{(4k-1)\pi}\right)$, for any integer k.

For proof see [3] pag.106 and [2] pag.187.

Example 2.3 provides reason to formulate the following:

Conclusion 3.3.1. *If a non-constant continuous function defined on an open interval has a zero at c and in any neighborhood of the point c the function takes positive and negatives values, then the function does NOT necessarily changes the sign from negatives to positives values or from positive to negatives values at c in the sense of Definition 1.1 (see Example 2.3, Figure 9)*

Moral: What we see (Figure 1 or similar) is not all what it is (Figure 9 or similar). Conclusion 2.3.1 justify more clearly now the reason of introducing Definition 1.1.

* Note

In other terms said, if the Figure 1 were representative for the graph of any continuous function that changes from negative to positive values in an appropriate neighborhood of its zero, what sense could one find in the introduction of the Definition 1.1? Obviously, there would be no sense in introducing Definition 1.1 if the scenarios described in Definition 1.1 were the only option for a continuous function that would change the sign from negative to positive values or from positive to negative values in an appropriate neighborhood of its zero. Examples 2.3 shows that Definition 1.1 makes sense for continuous functions as well because continuous functions may have zeros not necessarily described by a neighborhood where the function changes from negative to positive values or from positive to negative values as in the scenarios described by Definition 1.1.

Examples 2.3 and 2.2 show that if a continuous or discontinuous function has a zero, then the function does not necessarily change from negative to positive values or from positive to negative values on a neighborhood of its zero. This unpredictable behavior surprises the mind because a first insights that one gets when starting the learning of continuous functions is the intuitive representation of a continuous function over an interval: a function whose graph can be drawn without lifting the pen from the paper. In this context, it seems obvious that the graph of a continuous function that changes from negative to positive values can be drawn in a sufficiently small neighborhood of 0 without lifting the pen from the paper, in an approximate manner as in Figure 1. The intuitive drawing pen model might trouble seriously the perception of continuity if the learning of the continuity isn't backed up by accurate mathematical definitions and judicious examples and counterexamples similar to those discussed here.

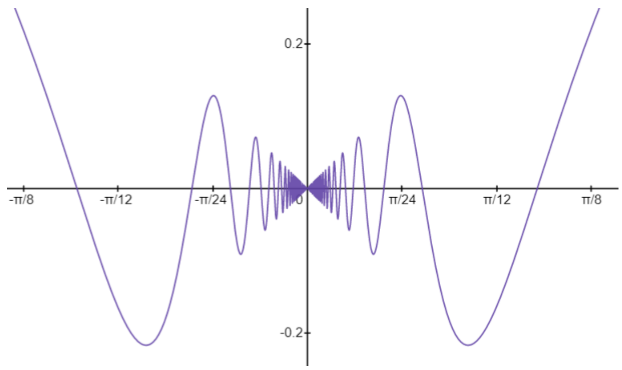

Figure 9: f continuous and f does NOT change from positives to negatives or from negatives to positives at the point $x = 0$ (in the sense of Definition 1.1)

Mathematical analysis taught us that beyond intuition, an unexplored universe might be hidden. For instance, the next conclusion. Before that, a fascinating animation provided by the magnification of the Koch curve: Koch Snow Flake (link) reminds us of the self-similarity(link) of the fractals and helps us in understanding the behavior of the function from Example 2.3 when "zooming on" in the neighborhood of the point 0.
If the Weierstrass Function is considered the very first beginning of fractals, then a function as one from Figure 9 might be considered the ancestor of the Weierstrass function.
In consequence, the following conclusion can be enabled:

Conclusion 3.3.2. *Example 2.3 shows that even though the function is continuous, its behavior in the neighborhood of one of its zeros could be:*

◇ *unpredictable (referring to the graph from Figure 9 in the neighborhood of $x = 0$ versus the 'foreseeing' graphs, Figure 1 in the neighborhood of $x = c$)*

◇ *erratic (referring to the intervals of monotony that appears in any neighborhood of $x = 0$, Figure 9)*

◇ *fascinating (referring to the 'compressed self-similarity' of the graph in the neighborhood of $x = 0$; imagine the perfect 'zoom in' animation technology of the function represented in Figure 9)*

Consider a non-continuous function that has zeros, not necessarily changing the sign in a neighborhood of its zeros. The weirdness of the behavior of this function from the perspective of the existence of an interval of monotony in the neighborhood of those zeros, it does not end with the above examples and the erratic behavior as it is described in Conclusion 2.3.2.

There are continuous functions that have such an erratic behavior in the neighborhoods of its zeros (and not only there) that they will not admit not even a single interval of monotony in these neighborhoods. Here it comes the famous Weierstrass function, an intricate real-valued function that disrupted the terms in that the continuity of functions was seen in the 19th century. [1]. It is the very first example of fractal curve and it has marked a new milestone in the further development of the mathematical analysis.

Example 3.4. *(The Weierstrass Function)*
Let f be the function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right)$$

for $0 < a < 1$ and b an odd positive integer greater than 1 s.t. $ab > 1 + \frac{3\pi}{2}$.

This function is everywhere continuous but nowhere differentiable.

See the graph here: the Weierstrass Function on the interval $[-2, 2]$.

*Note A complete proof can be found in [6], pag. 22-25

Consider c a fixed point on the real axis. The core of the proof in [5] is based on the construction of two sequences $(y_m)_{m\geq 1}$ and $(z_m)_{m\geq 1}$ with the properties

$y_m < c < z_m$ and $y_m \longrightarrow c$, $z_m \longrightarrow c$ as $m \longrightarrow \infty$

that generate the difference quotients $R(y_m, c)$ and $R(z_m, c)$ defined by

$$R(y_m, c) = (-1)^{\alpha_m} (ab)^m \eta_1 \left(\frac{2}{3} + \epsilon_1 \frac{\pi}{ab-1}\right), \quad R(z_m, c) = -(-1)^{\alpha_m} (ab)^m \eta_2 \left(\frac{2}{3} + \epsilon_2 \frac{\pi}{ab-1}\right) \tag{2}$$

where $\alpha_m \in \mathbb{Z}$, $\eta_1, \eta_2 > 1$ and $\epsilon_1, \epsilon_2 \in [-1, 1]$.

The next proposition will enable the difference quotients $R(y_m, c)$ and $R(z_m, c)$ as essential in the study of the monotony or the local monotony of the Weierstrass function on any interval of the real axis and respectively, at any point of the real axis.

Proposition 3.1. *Let be f the Weierstrass function defined by*

$$f(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right)$$

for $0 < a < 1$ and b an odd positive integer greater than 1 s.t. $ab > 1 + \frac{3\pi}{2}$.

[1] "deplorable evil", "lamentable scourge", "pathological function"

The difference quotients $R(y_m, c)$ and $R(z_m, c)$ defined in (2) have the following properties

(i) $R(y_m, c)$ and $R(z_m, c)$ have oscillating opposite signs for any m

(ii) $R(y_m, c)$ and $R(z_m, c)$ do not approach 0 as $m \longrightarrow \infty$

Proof. (i) Consider c a fixed point on the real axis and the two sequences $(y_m)_{m\geq 1}$ and $(z_m)_{m\geq 1}$ that generate the difference quotients $R(y_m, c)$ and $R(z_m, c)$ described in (2).
According to the hypothesis it follows $ab - 1 > \dfrac{3\pi}{2}$ or, equivalently $\dfrac{2}{3} > \dfrac{\pi}{ab-1}$.

Because $\eta_1 > 1$ and $\epsilon_1 \in [-1, 1]$, the following inequalities hold successively

$$\frac{2}{3} > \frac{\pi}{ab-1} \geq \left|\epsilon_1 \frac{\pi}{ab-1}\right| \geq \pm\epsilon_1 \frac{\pi}{ab-1}$$

It follows that $\dfrac{2}{3} > -\epsilon_1 \dfrac{\pi}{ab-1}$ or, equivalently

$$\frac{2}{3} + \epsilon_1 \frac{\pi}{ab-1} > 0$$

Analogously, it can be proved that

$$\frac{2}{3} + \epsilon_2 \frac{\pi}{ab-1} > 0$$

The relationships above allow to notice that the difference quotients $R(y_m, c)$ and $R(z_m, c)$ have opposite signs for any $m \in \mathbb{Z}$.
(ii) According to the hypothesis, $ab > 1 + \dfrac{3\pi}{2}$. This implies

$$\lim_{m\to\infty} (ab)^m = \infty$$

From (2), it results that there exists the limits of the absolute values of the difference coefficients $R(y_m, c)$ and $R(z_m, c)$ when $m \to \infty$ i.e.

$$\lim_{m\to\infty} \left|R(y_m, c)\right| = \infty, \quad \lim_{m\to\infty} \left|R(z_m, c)\right| = \infty$$

In consequence, (ii) is proved.

□

From the perspective of the interest of this paper, at a closer look, Proposition 2.1 seem to support the idea that Weierstrass function is nowhere *locally monotonic* in the sense of Definition 1.5 introduced in section 1. Indeed...

Proposition 3.2. *The Weierstrass function*

$$f(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right)$$

for $0 < a < 1$ *and* b *an odd positive integer greater than 1 s.t.* $ab > 1 + \frac{3\pi}{2}$,

has the following properties:

(i) f *is not monotonic on any interval*

(ii) f *is not locally monotonic at any point*

Proof. (i) Assume by contradiction that there is an interval (d, e) on that f is monotonic.
To fix the ideas, consider a point c in the interval (d, e). Notice that the interval (d, e) is a neighborhood of the point c in the sense of Definition 1.2.
According to Proposition 2.1, there are two sequences $y_m < c < z_m$ with $y_m \longrightarrow c$, $z_m \longrightarrow c$ as $m \longrightarrow \infty$ s. t. the difference quotients $R(y_m, c)$ and $R(z_m, c)$ have oscillating opposite signs for any m i.e., for a fixed integer m, the following hold

$$(j): \quad R(y_m, c) < 0,\, R(z_m, c) > 0$$

or

$$(jj): \quad R(y_m, c) > 0,\, R(z_m, c) < 0$$

Let's prove case (j), case (jj) being similar.
From $y_m \longrightarrow c$ when $m \longrightarrow \infty$, it follows that exists a rank m_1 from where y_m is in the interval (d, e) for any $m \geq m_1$.
Analogously, from $z_m \longrightarrow c$ when $m \longrightarrow \infty$, it follows that exists a rank m_2 from where z_m is in the interval (d, e) for any $m \geq m_2$.
Defining $m_0 = \max\{m_1, m_2\}$ it follows:

$$(\forall)\, m \geq m_0 \implies y_m,\ z_m \in (d, e),\ y_m < c < z_m$$

Fixing $n > m_0$, it results $y_n,\ z_n \in (d, e),\ y_n < c < z_n$.
From case $(j): \quad R(y_m, c) < 0,\, R(z_m, c) > 0$, so $f(y_m) < f(c)$ and $f(z_m) > f(c)$ for all $m \geq m_0$.
In particular, $n > m_0$ implies $f(y_n) < f(c)$ and $f(z_n) > f(c)$.
Finally,

$$y_n,\ z_n \in (d, e),\ y_n < c < z_n\ ,\ f(y_n) < f(c) > f(z_n)$$

The last relationships prove that f is neither increasing nor decreasing on the interval (d, e).
This is a contradiction with the assumption that f is monotonic on the interval (d, e).
Analogously, it can be proved that case (jj) leads to contradiction.
In conclusion, f is not monotonic on any interval (d, e).
(ii) the proof is similar to the proof given at (i) in the context of Definition 1.5.
Indeed, assume by contradiction that there is a point c at that f is locally monotonic. That means f is locally increasing at c or f is locally decreasing at c. In consequence, there is a neighborhood (d, e) of the point c s.t.

$$(k)\ (\forall)\ x, y \in (d, e), x < c < y \Longrightarrow f(x) \leq f(c) \leq f(y)$$

or

$$(kk)\ (\forall)\ x, y \in (d, e), x < c < y \Longrightarrow f(x) \geq f(c) \geq f(y)$$

According to Proposition 2.1, there are two sequences $y_m < c < z_m$ with $y_m \longrightarrow c$, $z_m \longrightarrow c$ as $m \longrightarrow \infty$ s. t. the difference quotients $R(y_m, c)$ and $R(z_m, c)$ have oscillating opposite signs for any m i.e.

$$(l): \quad R(y_m, c) < 0,\, R(z_m, c) > 0$$

or

$$(ll): \quad R(y_m, c) > 0,\, R(z_m, c) < 0$$

Let's consider case (l), case (ll) being similar.
From $y_m \longrightarrow c$ when $m \longrightarrow \infty$, it follows that exists a rank m_1 from where y_m is in the interval (d, e) for any $m \geq m_1$. Analogously, from $z_m \longrightarrow c$ when $m \longrightarrow \infty$, it follows that exists a rank m_2 from where z_m is in the interval (d, e) for any $m \geq m_2$.
Define $m_0 = \max\, \{m_1, m_2\}$, so

$$(\forall)\, m \geq m_0 \implies y_m,\ z_m \in (d, e),\ y_m < c < z_m$$

Fixing $n > m_0$ it results $y_n,\ z_n \in (d, e),\ y_n < c < z_n$.
From case $(l): \quad R(y_m, c) < 0,\, R(z_m, c) > 0$, so $f(y_m) > f(c)$ and $f(z_m) > f(c)$ for all $m \geq m_0$.
In particular, $n > m_0$ implies $f(y_n) > f(c)$ and $f(z_n) > f(c)$.
Finally,

$$y_n,\ z_n \in (d, e),\ y_n < c < z_n\ ,\ f(y_n) > f(c) < f(z_n)$$

The last relationships prove that f is neither locally increasing nor locally decreasing at the point c (see relationships (k) and (kk)).
This is a contradiction with the assumption that f is locally monotonic at the point c.
Analogously, case (ll) leads to contradiction.
In conclusion, f is not locally monotonic at the point c of the interval (a, b).

□

Conclusion 3.4.1. *Example 2.4 and Proposition 2.2 show that even though a function is continuous, its behavior can NOT be monotonic on any interval nor locally monotonic at any point. In particular, from the interest of questions $(*)$ and $(**)$, Example 2.4 and Proposition 2.2 show that even though a function is continuous, the function can NOT be monotonic in any neighborhood of its zero nor locally monotonic at any of its zeros.*

4 On the Sign Change of a Continuous Function

In spite of the unpredictable, erratic, or 'fickle' behavior of a continuous function as it was revealed in the previous section, if the function changes from negative to positive values and it is differentiable with the derivative continuous, then a certain degree of predictability in its increasing behavior in a neighborhood of its zero exists. This is the goal of the present section.

The following definitions and lemmas are needed.

Definition 4.1. *($\varepsilon - \delta$ definition)*

Let f be a real-valued function defined on the open interval (a, b) and α a point of the interval (a, b) .

The function f is continuous at the point α iff for any $\varepsilon > 0$, there exists $\delta > 0$ s.t.

for any x of the interval (a, b) with $\alpha - \delta < x < \alpha + \delta$, it results $f(\alpha) - \varepsilon < f(x) < f(\alpha) + \varepsilon$.

*Note that the $\varepsilon - \delta$ definition of continuity at a point α can be rewritten equivalently:

The function f is continuous at the point α iff for any $\varepsilon > 0$, there exists $\delta > 0$ s.t.

for any x of the interval (a, b) with $|x - \alpha| < \delta$, it results $|f(x) - f(\alpha)| < \varepsilon$

or

f is continuous at the point α iff

$$(\forall)\, \varepsilon > 0, (\exists)\, \delta > 0 \ s.t. (\forall)\, x \in (a, b), |x - \alpha| < \delta \ \Rightarrow |f(x) - f(\alpha)| < \varepsilon \quad (3)$$

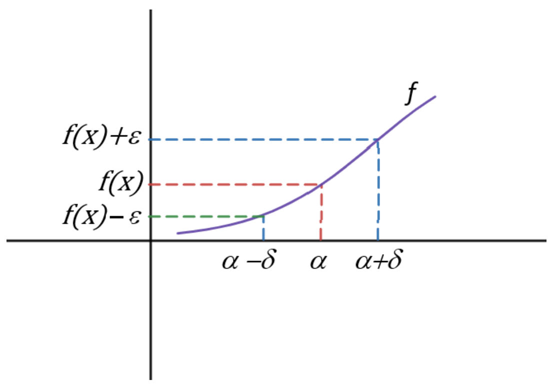

Figure 10: $\varepsilon - \delta$ definition of continuity at the point $x = \alpha$

One of the most powerful piece of information about the values of a continuous function that $\varepsilon - \delta$ definition provides is that if the function is continuous and strictly positive at a point, then the function is strictly positive on some neighborhood of that point. In other words, for a function defined on an open interval, the characteristic of the function to be strictly positive at a point of continuity is not isolated, it is induced on a whole neighborhood of that point.

The following lemma will prove the result mentioned above.

Lemma 4.1. *(The inertia property of a continuous function)*

Let f be a real-valued function defined on the open interval (a, b) and α a point of the interval (a, b). The following hold:

(i) if the function f is continuous and strictly positive at the point α, then there is a neighborhood of the point α included in (a, b) on which function f is strictly positive.

(ii) if the function f is continuous and strictly negative at the point α, then there is a neighborhood of the point α included in (a,b) on which function f is strictly negative.

Shortly,

(i) if f is continuous at α and $f(\alpha)>0$, then

$$(\exists)\,(c,d)\subseteq(a,b)\,s.t.\,(\forall)\,x\in(c,d)\Rightarrow f(x)>0 \quad (4)$$

(ii) if f is continuous at α and $f(\alpha)<0$, then

$$(\exists)\,(c,d)\subseteq(a,b)\,s.t.\,(\forall)\,x\in(c,d)\Rightarrow f(x)<0 \quad (5)$$

Proof. (i) It is given that the function f is continuous at $x=\alpha$ and $f(\alpha)>0$

The $\varepsilon-\delta$ definition offers the opportunity to choose $\varepsilon>0$ on purpose. Because $f(\alpha)>0$ and the intention is to get a neighborhood of $f(\alpha)$ on which the function f remains positive, the ε-value will be chosen s.t. $\varepsilon<f(\alpha)$. In this way, the desired interval $(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$ will contain only positive values.

To fix the ideas, choose for instance $\varepsilon=\dfrac{f(\alpha)}{2}$.

According to the $\varepsilon-\delta$ definition, there exists $\delta>0$ s.t. for any x in the interval $(\alpha-\delta,\alpha+\delta)\cap(a,b)$ it results $f(x)$ in the interval $(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$, so $f(x)$ is positive. (6)

The intersection $(\alpha-\delta,\alpha+\delta)\cap(a,b)$ is also an open neighborhood of α. Indeed, considering $c=max\,(\alpha-\delta,a)\,,d=min\,(\alpha+\delta,b)$, it follows $\alpha\in(c,d)$ and $(\alpha-\delta,\alpha+\delta)\cap(a,b)=(c,d)$. (7)

From (6) and (7) it follows that for any x in the interval $(c,d)\subseteq(a,b)$, it results $f(x)$ in the interval $(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$, so $f(x)$ is positive. In consequence, (4) is proved.
(ii) can be proved analogously. □

Lemma 4.2. *If the function f defined on the interval (a,b) is differentiable, then the following implications hold:*

(i) if f is increasing on (a,b) then $f'(x)\geq 0,(\forall)\,x\in(a,b)$

(ii) if f is strictly increasing on (a,b) then $f'(x)\geq 0,(\forall)\,x\in(a,b)$

(iii) if f is decreasing on (a,b) then $f'(x)\leq 0,(\forall)\,x\in(a,b)$

(iv) if f is strictly decreasing on (a,b) then $f'(x)\leq 0,(\forall)\,x\in(a,b)$.

Proof. see [3] pag. 108

□

Lemma 4.3. *If the function f defined on the interval (a, b) is differentiable, then the following implications hold:*

(i) if $f'(x) \geq 0, (\forall)\, x \in (a, b)$, then f is increasing on (a, b)

(ii) if $f'(x) > 0, (\forall)\, x \in (a, b)$, then f is strictly increasing on (a, b)

(iii) if $f'(x) \leq 0, (\forall)\, x \in (a, b)$, then f is decreasing on (a, b)

(iv) if $f'(x) < 0, (\forall)\, x \in (a, b)$, then f is strictly decreasing on (a, b)

Proof. see [4], pag. 201

□

Example 2.4 and Conclusion 2.4.1 revealed that even though a function is continuous, its behavior in any neighborhood of its zeros can NOT be locally monotonic.

Even though the previous statement holds, back to Definition 1.1 and the terms of the initial investigative questions $(*)$ and $(**)$, a certain expected behavior in terms of monotonicity can be proved in some conditions of differentiability.

More specifically, the following will be proved:

Theorem 1. *Let f be a function defined on the interval (a, b) that changes from negative to positive values on the interval (a, b).*

If the function f is differentiable with the derivative continuous, then there is an subinterval (g, h) of (a, b) on which the function f is strictly increasing i.e.

$$(\exists)\,(g, h) \subseteq (a, b)\ s.t.\ f\, is\, strictly\, increasing\, on\, the\, interval\, (g, h)$$

Proof. Notice that the derivative f' cannot be negative on the interval (a, b). (8)
Indeed, if $f'(x) \leq 0$ for any x from (a, b) then f will be decreasing on (a, b) (Lemma 3.3) and the difference quotient $R(x, y) \leq 0$ for any x, y, $x \neq y$ from (a, b) (Lemma 1.1). (9)

From f changes from negative to positive values on (a, b), it results that there is a point c of (a, b) s.t.

(i) $f(x) < 0$ for $x \in (a, c)$

(ii) $f(c) = 0$

(ii) $f(x) > 0$ for any $x \in (c, b)$.

Choosing x, y from (a, b) s.t. $x < c < y$, it results $f(x) < f(c) = 0 < f(y)$ and thus it follows immediately that $\dfrac{f(y) - f(x)}{y - x} > 0$ or equivalently $R(x, y) > 0$.
The last inequality is in contradiction with the inequality (9).

Therefore, f' cannot be negative on (a, b), so statement (8) is proved.

It follows that there exists a point α in the interval (a, b) where the derivative f' is strictly positive i.e. $f'(\alpha) > 0$.
The derivative f' is continuous and, therefore, the property of inertia holds for the continuous function f', according to Lemma 3.1 (i):
there exists a neighborhood (g, h) of α, $(g, h) \subseteq (a, b)$ s.t. $f'(x) > 0$ for all $x \in (g, h)$.
According to Lemma 3.3 (ii), it results now that the function f is strictly increasing on the interval (g, h). □

Conclusion 2.2.2 as well as Theorem 1 and its proof allow to formulate the following:

Conclusion 4.0.1. *If the function f defined on the interval (a, b) is differentiable with the derivative continuous and it changes from negative to positive values at the point c of the interval (a, b), then the answers to the initial questions:*

$(*)$ *Is there a neighborhood of the point c on which the function f is strictly increasing?*

$(**)$ *Is there an open interval, not necessarily a neighborhood of the point c, on which the function f is increasing?*

are NO (Conclusion 2.2.2 (iii)) and respectively, YES (Theorem 3.3 and its proof).

**Note Similar results as those provided by Theorem 1 and Conclusion 3.0.1 for functions that change from negative to positive values can be proved for functions that change from positive to negative values.*

5 Locally Monotonic Functions

Proposition 5.1. *Let f be a function defined on the interval (a, b) and c a point of this interval. The following holds:*

(i) if the function f is strictly increasing on each of the both intervals (a, c), (c, b) and locally strictly increasing at the point c, then the function f is strictly increasing on the interval (a, b)

(ii) if the function f is increasing on each of the both intervals (a, c), (c, b) and locally increasing at the point c, then the function f is increasing on the interval (a, b)

(iii) if the function f is strictly decreasing on each of the both intervals (a, c), (c, b) and locally strictly decreasing at the point c, then the function f is strictly decreasing on the interval (a, b).

(iv) if the function f is decreasing on each of the both intervals (a, c), (c, b) and locally decreasing at the point c, then the function f is decreasing on the interval (a, b).

Proof. (i) It is given f strictly increasing on each of the intervals (a, c), (c, b), and locally strictly increasing at c. To prove the conclusion, it is needed to prove the following:

$$(\forall)\, x, y \in (a, b),\ x < y \implies f(x) < f(y) \quad (10)$$

Let be $x, y \in (a,b), x < y$. The following cases are considered:

(j) if $a < x < y < c$, from f strictly increasing on the interval (a,c), it results immediately $f(x) < f(y)$

(jj) if $c < x < y < b$, from f strictly increasing on the interval (a,c), it results immediately $f(x) < f(y)$

(jjj) if $a < x < c < y < b$, from f locally increasing at c it results $f(x) < f(c) < f(y)$, so $f(x) < f(y)$.

In consequence, (10) is proved.
(ii)-(iv) can be proved analogously. □

To avoid the boredness of the reader, the following theorems will be stated only in the case when f is locally increasing. The immediate consequence is that the theorems can be stated analogously in the case when f is locally decreasing.

Theorem 2. *Let be f a function defined on the interval (a,b) and c a point of the interval (a,b). If the function f is:*

(i) differentiable with the derivative continuous

(ii) locally strictly increasing at the point c

then $(\exists)\,(g,h) \subseteq (a,b)$ s. t. f is strictly increasing on the interval (g,h).

Proof. It results from applying Theorem 1. to the function $k(x) = f(x) - f(c)$.
Indeed, from f locally strictly increasing at c it follows from Definition 1.5 that there is an interval $(p,q) \subseteq (a,b)$ s.t.

$$(\forall)\ x, y \in (p,q),\ x < c < y \Longrightarrow f(x) < f(c) < f(y)$$

or equivalently,

$$(\forall)\ x, y \in (p,q),\ x < c < y \Longrightarrow f(x) - f(c) < 0,\ f(y) - f(c) > 0$$

or equivalently,

$$(\forall)\ x, y \in (p,q),\ x < c < y \Longrightarrow k(x) < 0,\ k(c) = 0,\ k(y) > 0$$

In consequence, the function k changes from negative to positive values on the interval (p,q) at the point c (Definition 1.1). Because the function f is differentiable with the derivative continuous, it follows immediately that the function k is differentiable with the derivative continuous. Finally, according to Theorem 1, it results that

$$(\exists)\,(g,h) \subseteq (a,b) \text{ s. t. } k \text{ is strictly increasing on the interval } (g,h)$$

i.e.

$$(\exists)\,(g,h) \subseteq (a,b)\ s.t.\ (\forall)\ x, y \in (g,h),\ x < y \implies k(x) < k(y)$$

or equivalently,

$$(\exists)\,(g,h) \subseteq (a,b)\ s.t.\ (\forall)\ x,y \in (g,h),\ x < y \implies f(x) - f(c) < f(y) - f(c)$$

or, equivalently,

$$(\exists)\,(g,h) \subseteq (a,b)\ s.t.\ (\forall)\ x,y \in (g,h),\ x < y \implies f(x) < f(y)$$

or, equivalently,

$(\exists)\,(g,h) \subseteq (a,b)$ s.t. f is strictly increasing on the interval (g,h).

In consequence, Theorem 2 is proved. ☐

Theorem 3. *If the function f defined on the interval (a,b) is differentiable and locally increasing on the interval (a,b), then the function f is increasing on the interval (a,b).*

Proof. From f locally increasing on the interval (a,b) (Definition 1.6) it follows that f is locally increasing at each point c of the interval (a,b). For the next part of the proof, fix the point c on the interval (a,b). From f locally increasing at the point c, it follows that there exists an open neighborhood (g,h) of c, $(g,h) \subseteq (a,b)$ s.t. for any $x,y \in (g,h)$ with $x < c < y$, it results the difference quotients $R(x,c) \geq 0$ and $R(c,y) \geq 0$ (Lemma 1.2).
Because f is differentiable at the point c, both limits of $R(x,c)$ and $R(c,y)$ exists when $x \longrightarrow c$ and $y \longrightarrow c$, and they should be equal.
Furthermore, it follows $f'(c) \geq 0$. (11)
Due to the fact that the relationship (11) holds for every c from (a,b), it results $f'(x) \geq 0$ for any $a < x < b$. From Lemma 3.3 (i), it follows that f is increasing on the interval (a,b). ☐

Corollary 3.1. *If the function f defined on the interval (a,b) is differentiable and locally strictly increasing at every point c from (a,b), then the function f is increasing on the interval (a,b).*

Proof. Although f is locally strictly increasing at the point c and not only locally increasing as in the Theorem 4, the conclusion remains the same due to the fact that even if $R(x,c) > 0$ for any x, c from some interval $(g,h) \subseteq (a,b)$ with $x \neq c$, the limit of $R(x,c)$ when $x \longrightarrow c$ will be greater or equal than 0 and not strictly greater than 0.

☐

Motto: There is always a way to greatness ...

The following theorem enables the converse of Theorem 3 under the additional hypothesis of continuity.

Theorem 4. *Let f be a continuous function defined on the interval $[a,b]$.
If f is locally strictly increasing on the interval (a,b), then f is strictly increasing on the interval $[a,b]$.*

Proof. The proof will be performed in four steps:

Step 1: proof that f is monotonic on the open interval (a,b)
Step 2: proof that f is increasing on the open interval (a,b)

Step 3: proof that f is increasing on the closed interval $[a, b]$
Step 4: proof that f is strictly increasing on the closed interval $[a, b]$

Step 1: proof that f is monotonic on the open interval (a, b)
Assume by contradiction that f is not monotonic on the interval (a, b) i. e. f is not increasing nor decreasing on the interval (a, b). In consequence, the following holds:

$$(\exists)\ c, d, e \in (a, b),\ c < d < e\ s.t.\ f(c) < f(d) > f(e) \quad (12) \quad or \quad f(c) > f(d) < f(e) \quad (13)$$

Let's prove the case (12), the proof for the other case (13) being similar.
According to the hypothesis, it follows that f is continuous on the interval $[c, e]$. The Extreme Value Theorem implies that f has a maximum on the interval $[c, e]$, i.e.

$$(\exists)\ m \in [c, e]\ s.t.\ (\forall)\ c \le x \le e \implies f(m) \ge f(x) \quad (14)$$

From (12) it results that $m \ne c,\ m \ne e,\ f(m) \ge f(d)$. In consequence, the following holds:

$$c < m < e,\ f(c) < f(m) > f(e) \quad (15)$$

Due to the fact that $a < c < m < e < b$, it follows, according to the hypothesis that f is locally increasing at m i.e.

$$(\exists (p, q) \subseteq (a, b),\ p < m < q\ s.t.\ (\forall)\ x, y,\ p < x < m < y < q \Longrightarrow f(x) < f(m) < f(y) \quad (16)$$

Note that both q and e are greater than m. Choose $r = min\{e, q\}$.
From (14), it results:

$$(\forall)\ z,\ m < z < r \implies f(m) \ge f(z)$$

From (16), it results:

$$(\forall)\ z,\ m < z < r \implies f(m) < f(z)$$

The last two statements prove the contradiction. In consequence, (12) leads to contradiction.
In the case (13), it will be used the fact that the Extreme Value Theorem implies the existence of the minimum of the function f on the interval $[c, e]$. Next, the proof is similar with the proof given in the case (12). Analogously, it will result that case (13) leads to contradiction.
Finally, because both cases, (12) and (13), lead to contradiction, it results that f is monotonic on the interval (a, b).

Step 2: proof that f is increasing on the open interval (a, b)

Indeed, f monotonic on the interval (a, b) implies f increasing on the interval (a, b) or f decreasing on the interval (a, b).
According to the hypothesis, f is locally strictly increasing at any point w of the interval (a, b). In consequence,

$$(\exists)\ (k, l) \subseteq (a, b),\ k < w < l\ s.t.\ (\forall)\ x, y \in (k, l),\ x < w < y \implies f(x) < f(w) < f(y)$$

Therefore, it results that the only option for f to be monotonic on (a, b) is f increasing on (a, b).

Step 3: proof that f is increasing on the closed interval $[a,b]$
Due to the fact that f is continuous on $[a,b]$, it can be proved that f is increasing on the closed interval $[a,b]$.
Indeed, fix a point $\alpha \in (a,b)$. Obviously, there exists a point $x \in (a,\alpha)$, so $a < x < \alpha < b$.
From f continuous at the point a, it follows that exists the left limit of the function f at a, i.e.

$$\lim_{x \to\ a^+} f(x) = f(a) \quad (17)$$

On the other hand, from $a < x < \alpha < b$ and f increasing on (a,b) it follows $f(x) \leq f(\alpha)$.
Again, f continuous at the point a implies the existence of the left limit of the function f at a, so the limit when $x \to a^+$ can be considered in the previous inequality. Therefore,

$$\lim_{x \to\ a^+} f(x) \leq \lim_{x \to\ a^+} f(\alpha) \quad (18)$$

that is equivalent to

$$\lim_{x \to\ a^+} f(x) \leq f(\alpha) \quad (19)$$

From (17) and (19) it results $f(a) \leq f(\alpha)$. Analogously, it can be proved $f(\alpha) \leq f(b)$.
The last two inequalities lead to $f(a) \leq f(\alpha) \leq f(b)$ and are valid for any point α of the interval (a,b). This, with f increasing on the open interval (a,b) implies f increasing on the closed interval $[a,b]$.

Step 4: proof that f is strictly increasing on the closed interval $[a,b]$
Assume by contradiction that f is not strictly increasing on the interval $[a,b]$.
This means that exists m,n on the interval $[a,b]$ s.t. $m < n$ and $f(m) = f(n)$.
Firstly, the intent is to prove that $m \neq a$ or $n \neq b$.
Indeed, if $m = a$ and $n = b$ then $f(m) = f(n)$ implies $f(a) = f(b)$. At Step 3 it was proved that f is increasing on the interval (a,b). Thus, from f increasing on the interval $[a,b]$ and $f(a) = f(b)$, it results f constant on the interval $[a,b]$.
On the other hand, f locally strictly increasing at each point c of the interval (a,b) means f non-constant in some neighborhood of each point c, so basically, f is not constant.
The last two statements imply the contradiction. So, $m \neq a$ or $m \neq b$.

To fix the ideas for the next part of the proof, consider $m \neq a$ (the other case can be proved similarly).
From $m < n, f(m) = f(n)$ and f increasing on $[a,b]$ it follows:

$$(\forall)\ x,\ m < x < n \Longrightarrow f(m) = f(x) = f(n) \quad (20)$$

i.e. f is constant on the interval $[m,n]$.
From $a < m < n \leq b$, it follows $a < m < b$.
Because $a < m < b$, it results according to the hypothesis that f is locally strictly increasing at m i.e.

$$(\exists)\ (g,h) \subseteq (a,b),\ g < m < h\ s.t.\ (\forall)\ i,j \in (g,h)\ i < m < j \Longrightarrow f(i) < f(m) < f(j) \quad (21)$$

Choosing $s = min\{n,h\}$, for any z s.t. $m < t < s$, the following holds:

- from (20) it results $f(m) = f(t)$
- from (21) it results $f(m) < f(t)$

The last two relationships imply the contradiction.
In consequence, f is strictly increasing on the interval $[a, b]$.

□

Corollary 4.1. *Let be f a continuous function defined on the interval $[a, b]$. The following holds:*

(i) f is strictly increasing on the interval (a, b) iff f is locally strictly increasing on the interval (a, b).

(ii) f is strictly decreasing on the interval (a, b) iff f is locally strictly decreasing on the interval (a, b).

Proof. The proof results immediately from Proposition 1.1 and Theorem 4.

□

Motto: ... and there is no upper limit where the research is not bringing joy

Dropping the condition of continuity, the following question arises:

Let f be a function defined on the interval (a, b).
If f is locally strictly increasing on the interval (a, b), is then f strictly increasing on the interval (a, b) ?

6 Instead of Epilogue

There is no magic more captivating than mathematics: where one sees a path, one can find soon, or later a way. Definition 1.1 framed the intent: if a continuous function has a zero while changing from negative to positive values, it must definitely show a form of increase in its values. Then, the questions rose naturally:
($*$) If the function is continuous, could an inertia property of increase hold in a sufficiently small neighborhood of the zero?
($**$) If the function is continuous, could an eventual inertia property of increase hold in a sufficiently small neighborhood of the zero which eventually does not contain the zero?
($\diamond$) Can a weaker pattern of increase define better the changing from negative to positive behavior?
($\diamond\diamond$) For continuous functions, the concepts of locally strictly increasing/decreasing and strictly increasing/decreasing on an open interval are equivalent?
Finally, a question emerges for further developments:
For any function, not necessarily continuous, are the concepts of locally increasing/decreasing and increasing/decreasing on an open interval equivalent?

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5 December 2021

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