Formulas for Diagonals of any Quadrilateral

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Abstract

In this paper, novel formulas for the diagonals of any quadrilateral are given and proven. The novelty of this work is in the ability to represent each diagonal of any quadrilateral separately. We provide formulas for representing a diagonal of any cyclic quadrilateral, as well as a formula of representing a diagonal of any quadrilateral.

Introduction

As of now, there are no published results that represent each diagonal of any quadrilateral in terms of its sides and the other diagonal. In the literature, there are some formulas for finding the diagonals of cyclic quadrilaterals. Nevertheless, the advent of dealing with any convex quadrilaterals is a common one. And the following formulas are often extremely useful for these situations. In this paper, we will prove two theorems. Theorem 1 provides short and beautiful formulas for finding the diagonals of any cyclic quadrilateral using its side lengths. Moreover, it also serves as a novel and straightforward proof of Ptolemy's theorem (and Ptolemy's inequality). Theorem 2 gives a way of representing a diagonal of any convex quadrilateral in terms of its sides and other diagonal.

Theorem 1 (Diagonal of a cyclic quadrilateral in terms of its sides)**.** *Let a, b, c, d be the side lengths of any cyclic quadrilateral and e, f be its diagonals (a, b, e form a triangle), then*

$$e^2 = \frac{(ac+bd)(ad+bc)}{ab+cd},$$

and

$$f^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}.$$

Theorem 2 [Sedrakyan-Mozayeni] (Diagonal of any quadrilateral in terms of its sides and the other diagonal)**.** *Let a, b, c, d be the side lengths of a convex quadrilateral and e, f be its diagonals (a, b, e form a triangle), then*

$$2f^2 = a^2+b^2+c^2+d^2-e^2+\frac{(a^2-b^2)(c^2-d^2)}{e^2}+\frac{\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{e^2}.$$

Assumptions and Results

Formulas for Diagonals of Quadrilaterals and Their Proofs

Theorem 1 (Diagonal of a cyclic quadrilateral in terms of its sides)**.** *Let a, b, c, d be the side lengths of any cyclic quadrilateral and e be one of its diagonals (a, b, e form a triangle), then*

$$e^2 = \frac{(ac+bd)(ad+bc)}{ab+cd},$$

and

$$f^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}.$$

Proof. Let us denote $\angle ABC$ by α. Then, note that $\angle ADC = 180 - \alpha$.
Hence, by the law of cosines from triangles ABC and ADC, we obtain that

$$\cos\alpha = \frac{a^2+b^2-e^2}{2ab}, \cos\alpha = \frac{e^2-c^2-d^2}{2cd}.$$

Combining the last two equations, we get that

$$\frac{a^2+b^2-e^2}{2ab} = \frac{e^2-c^2-d^2}{2cd}.$$

Next we cross multiply to find that,

$$2abe^2 - 2abc^2 - 2abd^2 = 2a^2cd + 2b^2cd - 2cde^2.$$

Now, we divide everything by 2 and reorder to find that

$$(ab+cd)e^2 = abc^2 + abd^2 + a^2cd + b^2cd.$$

Thus, it follows that

$$e^2 = \frac{abc^2 + abd^2 + a^2cd + b^2cd}{ab+cd}.$$

This can be factored into

$$e^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

This ends the proof of the formula. The formula for f can be proven similarly. □

Remark (**Application of Theorem 1**)**.** *Note that Ptolemy's theorem (and Ptolemy's inequality) are direct consequences of theorem 1, as*

$$e^2f^2 = \frac{(ac+bd)(ad+bc)}{ab+cd} \cdot \frac{(ac+bd)(ab+cd)}{ad+bc} = (ac+bd)^2.$$

Therefore

$$ef = ac+bd.$$

Theorem 2 [Sedrakyan-Mozayeni] (Diagonal of any quadrilateral in terms of its sides and the other diagonal). *Let a, b, c, d be the side lengths of a convex quadrilateral and e, f be its diagonals (a, b, e form a triangle), then*

$$2f^2 = a^2+b^2+c^2+d^2-e^2+\frac{(a^2-b^2)(c^2-d^2)}{e^2}+\frac{\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{e^2}.$$

Proof. We will prove the above formula by finding two ways to represent the area of a quadrilateral.
Note that the area of quadrilateral $ABCD$ is equal to the sum of the areas of triangles ABC and ADC. Now, let us apply Heron's formula in triangles ABC and ADC to obtain that the area of quadrilateral $ABCD$ can be represented in the following way

$$\sqrt{\frac{(a+b+e)}{2}\cdot\frac{(a-b+e)}{2}\cdot\frac{(a+b-e)}{2}\cdot\frac{(b+e-a)}{2}}+$$

$$+\sqrt{\frac{(c+d+e)}{2}\cdot\frac{(c-d+e)}{2}\cdot\frac{(c+d-e)}{2}\cdot\frac{(d+e-c)}{2}}.$$

This can be further simplified into the following expression

$$\frac{1}{4}\Big(\sqrt{(e^2-(a-b)^2)((a+b)^2-e^2)}+\sqrt{(e^2-(c-d)^2)((c+d)^2-e^2)}\Big).$$

Now, we can open the parenthesis and factor again to find that,

$$\frac{1}{4}\Big(\sqrt{(2ab-a^2-b^2+e^2)(2ab+a^2+b^2-e^2)}+$$

$$\sqrt{(2cd-c^2-d^2+e^2)(2cd+c^2+d^2-e^2)}\Big),$$

and

$$\frac{1}{4}\Big(\sqrt{4a^2b^2-(a^2+b^2-e^2)^2}+\sqrt{4c^2d^2-(c^2+d^2-e^2)^2}\Big).$$

Next, we will represent the area of quadrilateral $ABCD$ using a variation of Bretshneider's formula [2]. It can be represented as,

$$\frac{1}{4}\sqrt{4e^2f^2-(a^2-b^2+c^2-d^2)^2}.$$

Now, we will solve the following equation.

$$\frac{1}{4}\Big(\sqrt{4a^2b^2-(a^2+b^2-e^2)^2}+\sqrt{4c^2d^2-(c^2+d^2-e^2)^2}\Big)=$$

$$\frac{1}{4}\sqrt{4e^2f^2-(a^2-b^2+c^2-d^2)^2}$$

$$\sqrt{4a^2b^2-(a^2+b^2-e^2)^2}+\sqrt{4c^2d^2-(c^2+d^2-e^2)^2}=$$

$$\sqrt{4e^2f^2-(a^2-b^2+c^2-d^2)^2}.$$

Now we will square each side. Here, for the sake of simplicity, we will denote,

$$\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}=M.$$

Then, we have that

$$4a^2b^2-(a^2+b^2-e^2)^2+4c^2d^2-(c^2+d^2-e^2)^2+2M=4e^2f^2-(a^2-b^2+c^2-d^2)^2.$$

Let us open the parenthesis.

$$4a^2b^2-a^4-b^4-e^4-2a^2b^2+2a^2e^2+2b^2e^2+4c^2d^2-c^4-d^4-e^4-$$
$$2c^2d^2+2c^2e^2+2d^2e^2+2M=4e^2f^2-a^4-b^4-c^4-d^4+2a^2b^2-$$
$$-2a^2c^2+2a^2d^2+2b^2c^2-2b^2d^2+2c^2d^2.$$

Now we will cancel terms and simplify.

$$-2e^4+2a^2e^2+2b^2e^2+2c^2e^2+2d^2e^2+2M=4e^2f^2-2a^2c^2+2a^2d^2+2b^2c^2-2b^2d^2.$$

Dividing by 2, we obtain that

$$a^2e^2+b^2e^2+c^2e^2+d^2e^2-e^4+M=2e^2f^2-a^2c^2+a^2d^2+b^2c^2-b^2d^2.$$

This can be rewritten in the following way

$$2f^2=a^2+b^2+c^2+d^2-e^2+\tfrac{(a^2-b^2)(c^2-d^2)}{e^2}+\tfrac{M}{e^2}.$$

Now we replace M to get that

$$2f^2=a^2+b^2+c^2+d^2-e^2+\tfrac{(a^2-b^2)(c^2-d^2)}{e^2}+\tfrac{\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{e^2}.$$

This proves the formula. □

Remark (**Application of Theorem 2. HMMT 2009, Feb., Geometry, Problem 8**). *Triangle ABC has side lengths $AB=231$, $BC=160$, and $AC=281$. Point D is constructed on the opposite side of line AC as point B such that $AD=178$ and $CD=153$. Compute the distance from B to the midpoint of segment AD.*

Solution. Let M be the midpoint of AD. By Apollonius theorem (in $\triangle ACD$) we have $2\cdot 89^2+2\cdot CM^2=281^2+153^2$, so $CM=208$. By theorem 2 (in $ABCM$)

$$2\cdot BM^2=231^2+160^2+208^2+89^2-281^2+\frac{(231^2-160^2)(208^2-89^2)}{281^2}+$$
$$\frac{\sqrt{(4\cdot 231^2\cdot 160^2-(231^2+160^2-281^2)^2)(4\cdot 208^2\cdot 89^2-(208^2+89^2-281^2)^2)}}{281^2}.$$

Hence, we deduce that

$$BM=208.$$

References

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