Junior Problems

J595. Solve the equation

$$\sqrt[3]{(x-1)^2} - \sqrt[3]{2(x-5)^2} + \sqrt[3]{(x-7)^2} = \sqrt[3]{4x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J596. Let x and y be positive real numbers. Prove that

$$\frac{1}{2x+y} + \frac{x}{y+2} + \frac{y}{x+y+1} \geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

J597. Let a, b, c be positive real numbers such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 2.$$

Prove that

$$\frac{5}{3} \leqslant \frac{a+b+c}{\max(a,b,c)} \leqslant 2.$$

Proposed by Marius Stănean, Zalău, România

J598. Solve in integers the equation

$$(x^2 - y^2)^2 - 23y = 8.$$

Proposed by Mihaela Berindeanu, Bucharest, România

J599. Let a, b, c be positive real numbers. Prove that

$$(a^2+b^2+c^2)(a+b+c) \geq 3abc\left(\sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} + \sqrt{\frac{a}{c}}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J600. Let ABC be a triangle with side-lengths a, b, c. Prove that

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 4 - \frac{2r}{R},$$

where r and R are the inradius and circumradius of the triangle, respectively.

Proposed by Mihaly Bencze, Braşov, and Neculai Stanciu, Buzău, România

Senior Problems

S595. Find all triples (x, y, z) of real numbers such that:

$$\sqrt[4]{1-x}+\sqrt[4]{16+y}=\sqrt[4]{1-y}+\sqrt[4]{16+z}=\sqrt[4]{1-z}+\sqrt[4]{16+x}=3.$$

Proposed by Mihaly Bencze, Braşov and Neculai Stanciu, Buzău, România

S596. Let a, b, c be the side-lengths of a triangle. Prove that

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\geq\frac{3(a^2+b^2+c^2)}{ab+bc+ca}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S597. Let $a, b, c, d \geq -1$ be real numbers such that $a^3+b^3+c^3+d^3=0$. Find maximum value of $a+b+c+d$.

Proposed by Marius Stănean, Zalău, România

S598. Let ABC be a triangle and let Δ be its area. Prove that

$$(a^2+b^2+c^2)^6\geq(4\sqrt{3}\Delta)^6+(2a^2-b^2-c^2)^6.$$

Proposed by An Zhenping, Xianyang Normal University, China

S599. Let $ABCD$ be a parallelogram. The tangent at C to the circumcircle of triangle BCD intersects AB in E and AD in F. The tangents at E and F to the circumcircle of triangle AEF intersect at X. Show that the points A, C, X are collinear.

Proposed by Mihaela Berindeanu, Bucharest, România

S600. Let a, b, c be positive real numbers. Prove that

$$\frac{8a}{3b^2+2bc+3c^2}+\frac{8b}{3c^2+2ca+3a^2}+\frac{8c}{3a^2+2ab+3b^2}\geq\frac{9}{a+b+c}$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U595. Find a nonconstant function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

(i) $f(x)f(y+1) = f(x+1)f(y)$, for all $x, y \in \mathbb{R}$,

(ii) f is integrable on every interval $[a, b] \subset \mathbb{R}$.

Proposed by Mircea Becheanu, Canada

U596. Let p be a prime number. We denote by N_p the number of triples (a, b, c) with $a, b, c \in \{0, 1, \ldots, p-1\}$ and such that

$$a^3 + b^3 + c^3 \equiv 3abc \pmod{p}.$$

Find all primes p for which $N_p > p^2 + p$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U597. Evaluate

$$\sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+4} - \cdots\right)^2.$$

Proposed by Ovidiu Furdui, Cluj-Napoca, and Alina Sîntămărian, Cluj-Napoca, România

U598. Let ABC be a triangle with $\angle BAC = 90^\circ$, and let F be its Feuerbach point. Find $\angle ABC$ knowing that $AF = OF$, where O is the circumcenter of the triangle.

Proposed by Corneliu Mănescu-Avram, Ploieşti, România

U599. Evaluate

$$\int_0^{\infty} \frac{\ln x}{1 + x + x^2 + x^3 + x^4 + x^5}\, dx.$$

Proposed by Ankush Kumar Parcha, Indira Gandhi National Open University,India

U600. We say that a positive integer k is *good* if there is a non-constant polynomial $P(x)$ such that

$$P(n^k) = P(n)P(n-1)\ldots P(n-k+1)$$

for all positive integers n. Find all *good* integers k.

Proposed by Kaan Bilge, Ataturk High School of Science, Turkey

Olympiad Problems

O595. Let A be a set of integers greater than 1 such that all positive divisors greater than 1 of $a_1a_2\dots a_n - 1$ belong to A, whenever $a_1, a_2, \dots, a_n$ are distinct elements from A and $n \geq 2$. We also assume that A has at least two elements. Prove that A contains all integers greater than 1.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

O596. Let a, b, c be real numbers such that $a \geq b \geq c \geq 0$ and $a^2 + b^2 + c^2 = 3$. Prove that

$$\sqrt{3abc(a+b+c)} + 2(a-c)^2 \geq 3.$$

Proposed by Marius Stănean, Zalău, România

O597. Let ABC be a triangle and let x, y, z be positive real numbers. Prove that

$$4 + \frac{r}{R} + \frac{x}{y+x}(1+\cos A) + \frac{y}{z+x}(1+\cos B) + \frac{z}{x+y}(1+\cos C) \geq (\sin A + \sin B + \sin C)^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O598. Let $a_1, a_2, \dots, a_n$ be real numbers such that

$$a_1 + a_2 + \dots + a_n = a_1^2 + a_2^2 + \dots + a_n^2 = n - 1.$$

Prove that

$$a_1^3 + a_2^3 + \dots + a_n^3 \leq n + 1 - \frac{6n-4}{n^2}.$$

When does equality hold?

Proposed by Josef Tkadlec, Czech Republic

O599. There are n children in a school. They form groups with each other, of various sizes, in a such a way that no child is left alone. Then, all of these children go to a park, where they have to sit around circular tables, each group around its table. Both the order and sense of the seating arrangements matter. Find in terms of n a closed formula for the number of ways this whole thing can be orchestrated; i.e breaking up into groups together with their seating arrangement around circles.

Proposed by Arpon Basu, AECS-4 School, Mumbai, India

O600. Prove that in any triangle ABC the following inequality holds:

$$\frac{\sin A}{1+\cos^2 B+\cos^2 C} + \frac{\sin B}{1+\cos^2 C+\cos^2 A} + \frac{\sin C}{1+\cos^2 A+\cos^2 B} \leq \sqrt{3}.$$

Proposed by An Zhenping, Xianyang Normal University, China