Junior problems

J595. Solve the equation

$$\sqrt[3]{(x-1)^2} - \sqrt[3]{2(x-5)^2} + \sqrt[3]{(x-7)^2} = \sqrt[3]{4x}.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by author

Let $a = \sqrt[3]{(x-1)^2}, b = -\sqrt[3]{2(x-5)^2}, c = \sqrt[3]{(x-7)^2}$.

We have

$$a^3 + b^3 + c^3 = x^2 - 2x + 1 - 2(x^2 - 10x + 25) + x^2 - 14x + 49 = 4x =$$

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a),$$

implying $a + b = 0$ or $b + c = 0$ or $c + a = 0$, the last of which being impossible.

It follows that $x - 1 = \pm\sqrt{2}$ or $\pm\sqrt{2} = x - 7$, yielding the solutions $x = 9 \pm 4\sqrt{2}$ and $x = 3 \pm 2\sqrt{2}$.

Also solved by Polyahedra, Polk State College, FL, USA; G. C. Greubel, Newport News, VA, USA; Alina Craciun, Miron Costin Theoretical High School, Pascani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Pranjal Jha, Whitefield Global School, India; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Adam John Frederickson, Utah Valley University, UT, USA; Le Hoang Bao, Tien Giang, Vietnam; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Arkady Alt, San Jose, CA, USA; Dao Van Nam, Phúóng Nam High School, Hoáng Mai, Ha Noi, Vietnam.

J596. Let x and y be positive real numbers. Prove that

$$\frac{1}{2x+y}+\frac{x}{y+2}+\frac{y}{x+y+1}\geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
By the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{1}{2x+y}+\frac{x}{y+2}+\frac{y}{x+y+1} &= \frac{1}{2x+y}+\frac{x^2}{xy+2x}+\frac{y^2}{xy+y^2+y}\\ &\geq \frac{(1+x+y)^2}{y^2+2xy+4x+2y}.\end{aligned}$$

Thus, it suffices to show that

$$\frac{(1+x+y)^2}{y^2+2xy+4x+2y}=\frac{1+x^2+y^2+2x+2y+2xy}{y^2+2xy+4x+2y}\geq 1,$$

which is equivalent to $(x-1)^2\geq 0$. This is clearly true. Equality holds when $x=1$.

Also solved by Polyahedra, Polk State College, FL, USA; Adam John Frederickson, Utah Valley University, UT, USA; Le Hoang Bao, Tien Giang, Vietnam; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Alina Craciun, Miron Costin Theoretical High School, Pascani, Romania; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Daniel Văcaru, Pitești, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Hyunbin Yoo, South Korea; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Ryan DiPaola, SUNY Brockport, USA; Soham Dutta, DPS Ruby Park, West Bengal, India; Sundaresh Harige; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Daniel Pascuas, Barcelona, Spain; Prajnanaswaroopa S, Bangalore, India; Arkady Alt, San Jose, CA, USA; Dao Van Nam, Phúóng Nam High School, Hoáng Mai, Ha Noi, Vietnam.

J597. Let a, b, c be positive real numbers such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 2.$$

Prove that

$$\frac{5}{3} \le \frac{a+b+c}{\max(a,b,c)} \le 2.$$

Proposed by Marius Stănean, Zalău, România

Solution by Theo Koupelis, Cape Coral, FL, USA
Without loss of generality let $a = \max(a,b,c) > 0$. Clearing denominators, from the given condition we get

$$a^3 + b^3 + c^3 = a^2(b+c) + b^2(a+c) + c^2(a+b) + abc. \tag{1}$$

If $b + c \ge a$, then the right-hand-side of (1) is clearly greater than the left-hand-side because $a + c > b$ and $a + b > c$. Therefore $b + c < a$ and thus

$$\frac{a+b+c}{a} < 2;$$

if we were to allow $c = \min(a,b,c) = 0$, then equality is achieved when $b = a$. Rewriting (1) as

$$(a-b-c)^2(a+b+c) = bc\left[4(b+c) - a\right],$$

we get $b + c > \frac{a}{4}$, and thus

$$(a-b-c)^2(a+b+c) \le \frac{(b+c)^2}{4} \cdot \left[4(b+c) - a\right].$$

Simplifying the above we get

$$3a(b+c)^2 + 4a^2(b+c) - 4a^3 \ge 0.$$

Solving the above quadratic in $(b+c)$ we get $b + c \ge \frac{2}{3}a$, and thus

$$\frac{a+b+c}{a} \ge \frac{5}{3},$$

with equality when $b = c > 0$. In summary, we have

$$\frac{5}{3} \le \frac{a+b+c}{\max(a,b,c)} < 2.$$

Also solved by Polyahedra, Polk State College, FL, USA; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Prajnanaswaroopa S, Bangalore, India; Arkady Alt, San Jose, CA, USA; Dao Van Nam, Phúóng Nam High School, Hoáng Mai, Ha Noi, Vietnam; Daniel Văcaru, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.

J598. Solve in integers the equation

$$(x^2 - y^2)^2 - 23y = 8$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Blerta Gashi, SUNY Brockport, NY, USA

$$23y + 8 = (x^2 - y^2)^2 \geq 0 \implies 23y \geq -8 \iff y \geq -8/23 \implies y \geq 0$$

If x is a negative solution of the equation then $-x$ is also a solution because x is squared. Thus there are always positive solutions for x. Let (x, y) be a solution with x positive. Then

$$(x - y)^2 (x + y)^2 = 23y + 8 \implies (x + y)^2 | 23y + 8 \implies (x + y)^2 \leq 23y + 8$$

$$\iff x^2 + 2xy + y^2 = 23y + 8 \iff y^2 - 23y - 8 = -x^2 - 2xy \leq 0$$

because both x and y are positive.
Thus,

$$y^2 - 23y - 8 \leq 0 \iff y \in \left[\frac{23 - \sqrt{561}}{2}, \frac{23 + \sqrt{561}}{2}\right] \implies y \in [0, 23]$$

because y is a positive integer.

Therefore

$$y \in \{0, 1, 2, \ldots, 22, 23\}$$

Checking all these values for y we get integer solution only for $y = 7$ when $x = \pm 6$.

Also solved by Polyahedra, Polk State College, FL, USA; Andrew Kim, Cresskill High School, Cresskill, NJ, USA; G. C. Greubel, Newport News, VA, USA; Sundaresh Harige; Theo Koupelis, Cape Coral, FL, USA; Woojin Juhn, Northern Valley High School, Demarest, NJ, USA; Titu Zvonaru, Comănești, Romania.

J599. Let a, b, c be positive real numbers. Prove that

$$(a^2+b^2+c^2)(a+b+c) \geq 3abc\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Pascuas, Barcelona, Spain

Without loss of generality, we may assume that $a+b+c=1$ (just divide both terms of the inequality by $(a+b+c)^3$), and then we have to prove that

$$3abc\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}\right) \leq a^2+b^2+c^2. \tag{$*$}$$

Now we write the left-hand side term of this inequality as

$$3abc\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{a}{c}}\right) = 3\big(c\sqrt{ab^3}+a\sqrt{bc^3}+b\sqrt{a^3c}\big)$$

Then the concavity of the square root function together with Jensen's inequality give that

$$c\sqrt{ab^3}+a\sqrt{bc^3}+b\sqrt{a^3c} \leq \sqrt{ab^3c+abc^3+a^3bc} = \sqrt{abc}\sqrt{a^2+b^2+c^2}.$$

Thus, in order to prove $(*)$, we only have to show that $3\sqrt{abc} \leq \sqrt{a^2+b^2+c^2}$, or equivalently $9abc \leq a^2+b^2+c^2$. And that follows from the identity $a+b+c=1$ and the AM-GM inequality:

$$\begin{aligned} a^2+b^2+c^2 &= (a^2+b^2+c^2)(a+b+c) \\ &= a^3+ab^2+ac^2+a^2b+b^3+bc^2+a^2c+b^2c+c^3 \geq 9abc. \end{aligned}$$

Also solved by Polyahedra, Polk State College, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Alina Craciun, Miron Costin Theoretical High School, Pascani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Dao Van Nam, Phúóng Nam High School, Hoáng Mai, Ha Noi, Vietnam.

J600. Let ABC be a triangle with side-lengths a, b, c. Prove that

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 4 - \frac{2r}{R},$$

where r and R are the inradius and circumradius of the triangle, respectively.

Proposed by Mihaly Bencze, Braşov, and Neculai Stanciu, Buzău, România

Solution by Theo Koupelis, Cape Coral, FL USA

Let s be the triangle's semiperimeter, and E its area. We have $E = rs = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$. Thus,

$$\begin{aligned} 4 - \frac{2r}{R} = 4 - \frac{8E^2}{sabc} &= \frac{1}{abc} \cdot [4abc - (a+b-c)(a-b+c)(a+b-c)] \\ &= \frac{1}{abc} \cdot \left[a^3 + b^3 + c^3 - a^2(b+c) - b^2(a+c) - c^2(a+b) + 6abc\right]. \end{aligned}$$

Also,

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = \frac{1}{abc} \cdot (a^3 + b^3 + c^3).$$

Therefore, the desired inequality is equivalent to

$$a^2(b+c) + b^2(a+c) + c^2(a+b) \geq 6abc,$$

which is obvious by AM-GM. Equality holds for an equilateral triangle.

Also solved by Polyahedra, Polk State College, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Le Hoang Bao, Tien Giang, Vietnam; Daniel Pascuas, Barcelona, Spain; Dao Van Nam, Phúóng Nam High School, Hoáng Mai, Ha Noi, Vietnam; Arkady Alt, San Jose, CA, USA; Telemachus Baltsavias Kerameies Junior High School Kefalonia, Greece; Corneliu Mănescu-Avram, Ploieşti, România; Daniel Văcaru, Pitești, Romania; Emon Suin, West Bengal, India; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Soham Dutta, DPS Ruby Park, West Bengal, India; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

Senior problems

S595. Find all triples (x, y, z) of real numbers such that:

$$\sqrt[4]{1-x}+\sqrt[4]{16+y}=\sqrt[4]{1-y}+\sqrt[4]{16+z}=\sqrt[4]{1-z}+\sqrt[4]{16+x}=3.$$

Proposed by Mihaly Bencze, Braşov, and Neculai Stanciu, Buzău, România

Solution by G. C. Greubel, Newport News, VA, USA
Starting with $\sqrt[4]{1-x}+\sqrt[4]{16+y}=3$ then the square of both sides yields

$$2\sqrt[4]{(1-x)(16+y)}=9-\sqrt{1-x}-\sqrt{16+y}$$

and the fourth power of both sides yields

$$64+x-y=4\sqrt[4]{(1-x)(16+y)}\,(9-2\sqrt[4]{(1-x)(16+y)})+6\sqrt{(1-x)(16+y)}.$$

Multiplying both sides by 2 and using the equation obtained by squaring both sides yields

$$49-\frac{x-y}{2}=9\sqrt{1-x}+9\sqrt{16+y}+\sqrt{(1-x)(16+y)}.$$

This equation has the integer solutions $(x, y) \in \{(-80,-16),(-80,1280),(0,0),(1,65)\}$. From one of the remaining equations, namely,

$$\sqrt[4]{1-y}+\sqrt[4]{16+z}=3$$

with the y values obtained it is found that the only one that gives an integer solution for z is $(y, z)=(0,0)$. This implies that the integer solution set is $(x, y, z)=(0,0,0)$.

Also solved by Alina Craciun, Miron Costin Theoretical High School, Pascani, Romania; Daniel Văcaru, Pitești,Romania; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S596. Let a, b, c be the side-lengths of a triangle. Prove that

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c} \geq \frac{3\left(a^2+b^2+c^2\right)}{ab+bc+ca}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Let's change $a+b-c=z$, $b+c-a=x$, $c+a-b=y$, yielding $a=(y+z)/2$, $b=(a+b)/2$, $c=(x+y)/2$. The inequality becomes equivalent to

$$]\sum_{\text{sym}}(x^4y+4x^3y^2) \geq \sum_{\text{sym}}(3x^3yz+2x^2y^2z)$$

and this is simply AGM.

$$3(x^3y^2+x^3z^2) \geq 6x^3yz \text{ and cyclic}$$

yield

$$\sum_{\text{sym}}(x^4y+x^3y^2) \geq \sum_{\text{sym}} 2x^2y^2z$$

$$x^3y^2+x^3y^2+z^3y^2 \geq 3x^2y^2z \text{ and cyclic} \qquad x^4z+y^4z \geq 2x^2y^2z \text{ and cyclic}$$

complete the proof.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Daniel Văcaru, Pitești,Romania; Taes Padhihary, Disha Delphi Public School, Kota, Rajasthan, India; Arkady Alt, San Jose, CA, USA; Dao Van Nam, Phúóng Nam High School, Hoáng Mai, Ha Noi, Vietnam; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Daniel Pascuas, Barcelona, Spain; Jiang Lianjun, Quanzhou second Middle School, GuiLin, China; Soham Dutta, DPS Ruby Park, West Bengal, India; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Marian Ursărescu, Roman-Vodă National College, Roman, Romania.

S597. Let $a, b, c, d \ge -1$ be real numbers such that $a^3 + b^3 + c^3 + d^3 = 0$. Find maximum value of $a + b + c + d$.

Proposed by Marius Stănean, Zalău, România

Solution by the author

Without loss of generality suppose that $a \le b \le c \le d$ then $-1 \le a \le 0 \le d$. We have the following cases:

1. $-1 \le a \le b \le c \le 0 \le d$, we have

$$a + b + c + d \le a^3 + b^3 + c^3 + d = d - d^3 \overset{(*)}{<} \frac{1}{2}.$$

2. $-1 \le a \le b \le 0 \le c \le d$, we have

$$a + b + c + d \le a^3 + b^3 + c + d = c - c^3 + d - d^3 \overset{(*)}{<} \frac{1}{2} + \frac{1}{2} = 1.$$

For the majorizations denoted by $(*)$ we have used the inequality $t - t^3 < \frac{1}{2}$, $\forall t > 0$. This is obvious if $t \ge 1$, while for $t \in [0, 1)$ it follows from the AM-GM Inequality:

$$t - t^3 = t(1 - t)(1 + t) = \frac{1}{2} \cdot t(2 - 2t)(1 + t) < \frac{1}{2}\left(\frac{t + 2 - 2t + 1 + t}{3}\right)^3 = \frac{1}{2}.$$

3. $-1 \le a \le 0 \le b \le c \le d$.

Since $b^3 + c^3 + d^3 = -a^3 \in [0, 1]$ it follows that $b, c, d \in [0, 1]$. By the Hölder's Inequality, we have

$$b + c + d \le \sqrt[3]{9(b^3 + c^3 + d^3)} = -a\sqrt[3]{9}.$$

Then

$$a + b + c + d \le a - a\sqrt[3]{9} = -a\left(\sqrt[3]{9} - 1\right) \le \sqrt[3]{9} - 1.$$

The equality holds when $b = c = d = \dfrac{1}{\sqrt[3]{3}}$ and $a = -1$. Since $\sqrt[3]{9} - 1 > 1$ it follows that the maximum value of $a + b + c + d$ is $\sqrt[3]{9} - 1$.

Also solved by Adam John Frederickson, Utah Valley University, UT, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania.

S598. Let ABC be a triangle and let Δ be its area. Prove that

$$(a^2+b^2+c^2)^6 \geq (4\sqrt{3}\Delta)^6 + (2a^2-b^2-c^2)^6.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Theo Koupelis, Cape Coral, FL, USA

We have

$$(2a^2-b^2-c^2)^2 = 4(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)-3(b^2-c^2)^2.$$

Let $p = a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \geq 0$ and $x = 3(b^2-c^2)^2 \geq 0$. Also, let $q = a^2b^2+b^2c^2+c^2a^2 > 0$. Then we have $(a^2+b^2+c^2)^2 = p+3q$ and $16\Delta^2 = q-p > 0$ (for a non-degenerate triangle). The given inequality is now equivalent to

$$(p+3q)^3 \geq [3(q-p)]^3 + (4p-x)^3.$$

Expanding and simplifying we get the equivalent inequality

$$36p(q-p)(3q+p) + x\left[(x-6p)^2+12p^2\right] \geq 0,$$

which is obvious because both terms are non-negative. For a non-degenerate triangle, equality occurs when $x=0$ and $p=0$, that is, when the triangle is equilateral.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania.

S599. Let $ABCD$ be a parallelogram. The tangent at C to the circumcircle of triangle BCD intersects AB in E and AD in F. The tangents at E and F to the circumcircle of triangle AEF intersect at X. Show that the points A, C, X are collinear.

Proposed by Mihaela Berindeanu, Bucharest, România

First solution by the author

Notations: Γ_1 = circumcircle of $\triangle BCD$ and Γ_2 = circumcircle of $\triangle AEF$.

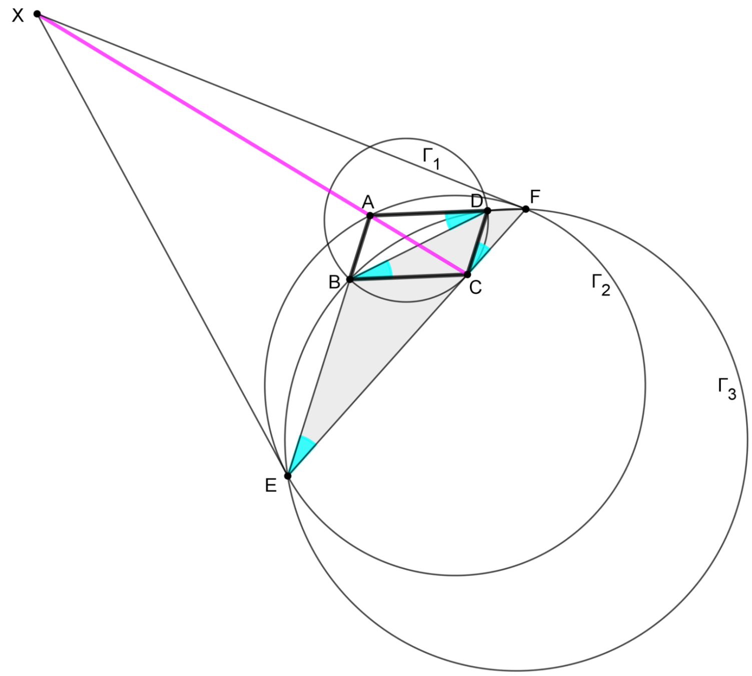


Figura 1:

- Show that AC is symmedian in $\triangle AEF$

  In the parallelogram $ABCD$,

$$\measuredangle BDA \equiv \measuredangle DBC \text{ (alternate interior angles)} \tag{1}$$

EF is tangent in C to $\Gamma_1 \Rightarrow$

$$\measuredangle DCF \equiv \measuredangle DBC = \frac{\overset{\frown}{CD}}{2} \tag{2}$$

And $BE \parallel CD \Rightarrow$

$$\measuredangle BEC \equiv \measuredangle DCF \text{ (corresponding angles)} \tag{3}$$

From (1),(2),(3)$\Rightarrow$ $\measuredangle BEC \equiv \measuredangle BDA \Rightarrow BD$ and EF are antiparallel lines with respect to AE and $AF \Rightarrow EFDB$ is an inscribed quadrilateral in the circle $\mathit{\Gamma}_3$
From the power of the point A with respect to circle $\mathit{\Gamma}_3$:

$$AD \cdot AF = AB \cdot AE \tag{4}$$

$$\begin{cases} CD \parallel AE \Rightarrow \dfrac{AD}{AF} = \dfrac{EC}{EF} \\ \\ BC \parallel AF \Rightarrow \dfrac{AB}{AE} = \dfrac{CF}{EF} \end{cases} \Rightarrow \begin{cases} AD = \dfrac{AF \cdot EC}{EF} \\ \\ AB = \dfrac{AE \cdot CF}{EF} \end{cases} \Rightarrow (4) \text{ becomes } \frac{AF \cdot EC}{EF} \cdot AF = \frac{AE \cdot CF}{EF} \cdot AE \Rightarrow$$

$$\frac{AF^2}{AE^2} = \frac{CF}{EC} \Rightarrow AC \text{ symmedian in } \triangle AEF$$

- Show that $X,\ A,\ C$ are collinear points

$$\left.\begin{array}{l} XE, XF \text{ tangents to } \mathit{\Gamma}_2 \\ \mathit{\Gamma}_2 \text{ circumcircle of } \triangle AEF \\ AC \text{ simmedian of } \triangle AEF \end{array}\right\} \Rightarrow X, A, C \text{ are collinear points}$$

.

Second solution by Theo Koupelis, Cape Coral, FL, USA

Let (O) be the circumcircle of triangle BCD, and (K) the circumcircle of triangle AEF. Also, let $\angle DBC = \theta$, and $\angle BDC = \phi$. Then $\angle DCF = \theta$ because EF is tangent to (O) at C, $\angle AEF = \theta$ because $DC \parallel AE$, and $\angle AFX = \theta$ because FX is tangent to (K) at F. Similarly, $\angle BCE = \angle BEX = \angle CFD = \phi$ because EF is tangent to (O) at C, $FA \parallel BC$, and EX is tangent to (K) at E. Let I, J, C' be the intersection points of FA, EA, XA with XE, XF, EF, respectively. Then from Ceva's theorem in triangle EXF we get

$$\frac{IX}{IE} \cdot \frac{C'E}{C'F} \cdot \frac{JF}{JX} = 1.$$

Using the law of sines in triangles XIF and EIF we get

$$\frac{IX}{IE} = \frac{\sin\theta \cdot \sin(\theta + \phi)}{\sin\phi \cdot \sin\angle EXF}.$$

Similarly, using the law of sines in triangles XJE and FJE we get

$$\frac{JF}{JX} = \frac{\sin\theta \cdot \sin\angle EXF}{\sin\phi \cdot \sin(\theta + \phi)}.$$

Substituting the above two expressions into the equation from Ceva's theorem we get

$$\frac{C'E}{C'F} = \frac{\sin^2\phi}{\sin^2\theta}.$$

However, from the similarity of triangles EBC and EAF we get

$$\frac{CE}{CF} = \frac{BE}{BA} = \frac{BE}{CD},$$

and from the similarity of triangles EBC and BCD we get

$$\frac{BE}{BC} = \frac{BC}{CD}.$$

From the above two equations and using the law of sines in triangle BCD we get

$$\frac{CE}{CF} = \frac{BC^2}{CD^2} = \frac{\sin^2\phi}{\sin^2\theta}.$$

Therefore,

$$\frac{C'E}{C'F} = \frac{CE}{CF},$$

and thus $C \equiv C'$, and the points A, C, X are collinear.

Also solved by Polyahedra, Polk State College, FL, USA; Alina Craciun, Miron Costin Theoretical High School, Pascani, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Titu Zvonaru, Comănești, Romania.

S600. Let a, b, c be positive real numbers. Prove that

$$\frac{8a}{3b^2+2bc+3c^2}+\frac{8b}{3c^2+2ca+3a^2}+\frac{8c}{3a^2+2ab+3b^2}\geq\frac{9}{a+b+c}$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Mircea Becheanu, Canada

Using the inequality Cauchy-Schwartz we have

$$\sum\frac{8a}{3b^2+2bc+3c^2}=8\frac{a^2}{3ab^2+2abc+3ac^2}\geq\frac{8(a+b+c)^2}{3\sum_{sym}a^b+6abc}.$$

Therefore, it is enough to show that

$$8(a+b+c)^3\geq 27(\sum_{sym}a^2b+2abc).$$

But $(a+b+c)^3=\sum a^3+3\sum a^2b+6abc$. So we have to prove

$$8\sum a^3+24\sum a^2b+48abc\geq 27\sum a^2b+54abc.$$

This is equivalent to

$$8\sum a^3-6abc\geq 3\sum a^2b,$$

which can be written

$$3\left(\sum a^3+3abc-\sum a^2b\right)+5\left(\sum a^3-3abc\right)\geq 0.$$

This is clear, since first member is positive by Schur inequality and the second by AM-GM inequality.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Adriana Andreescu, TX, USA; Dao Van Nam, Phúóng Nam High School, Hoáng Mai, Ha Noi, Vietnam; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Daniel Văcaru, Pitești, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Soham Dutta, DPS Ruby Park, West Bengal, India; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA.

U595. Find a nonconstant function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

(i) $f(x)f(y+1) = f(x+1)f(y)$, for all $x, y \in \mathbb{R}$,

(ii) f is integrable on every interval $[a, b] \subset \mathbb{R}$.

Proposed by Mircea Becheanu, Canada

Solution by the author

It is clear that every constant functions $f(x) = c$ satisfy the equation, but not the condition from the statement. It is also clear that the Dirichlet function

$$f(x) = \begin{cases} 1 \text{ for } x \in \mathbb{Q} \\ 0 \text{ for } x \notin \mathbb{Q} \end{cases}$$

satisfy the condition (i) but does not satisfy the condition (ii).

We will study the properties of a non constant function f which satisfy the property (i). First we remark that the propery (i) is symmetric with respect of the pair (x, y). Secondly, we remark that if $f(x)$ is a non constant solution, then $cf(x)$ is also a non constant solution for every number $c \neq 0$.

For the pair $(x, 0)$ we obtain $f(x)f(1) = f(x+1)f(0)$. This shows that $f(0) = 0$ if an only if $f(1) = 0$. Assume that $f(0) = f(1) = 0$. Then, for the pair $(x, 1)$ we have $f(x+1)f(1) = f(x)f(2)$, showing that $f(2) = 0$. For the pair $(x-1, -1)$ we have $f(x)f(-1) = f(x-1)f(0)$, showing that $f(-1) = 0$. By induction up and down we obtain that $f(n) = 0$ for all $n \in \mathbb{Z}$. Now define the function

$$f(x) = \begin{cases} 0 \text{ for } x \in \mathbb{Z} \\ 1 \text{ for } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

It is easy to see that this function is a solution of the problem. Also, we can see that there are infinitely many such functions by replacing 1 with any non zero constant c.

Also solved by Adam John Frederickson, Utah Valley University, UT, USA; Prajnanaswaroopa S, Bangalore, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL, USA.

U596. Let p be a prime number we denote by N_p the number of triples (a,b,c) such that $a,b,c \in \{0,1,\ldots,p-1\}$ such that

$$a^3+b^3+c^3 \equiv 3abc \pmod p.$$

Find all primes for which $N_p > p^2+p$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Note that $a^3+b^3+c^3-3abc = \frac{1}{2}\cdot(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)$. Now, letting $a-b=x, b-c=y$ it follows that $c-a=-x-y$. Then, $\frac{1}{2}((a-b)^2+(b-c)^2+(c-a)^2) = x^2+xy+y^2$. Thus, We use the following facts:

i. If $p \equiv 2 \pmod 3$ such that p divides x^2+xy+y^2 then p divides x,y.

ii. If $p \equiv 1 \pmod 3$ the equation $z^2+z+1 \equiv 0 \pmod p$ has two roots namely r, r^{-1}. Thus, $x^2+xy+y^2 \equiv (x-ry)(x-r^{-1}y) \pmod p$. Thus, if p divides x^2+xy+y^2 then $x \equiv ry \pmod p$ or $x \equiv r^{-1}y \pmod p$.

Now, if $p \equiv 2 \pmod 3$ then either $a+b+c \equiv 0 \pmod p$ and this leads to p^2 triples or p divides x^2+xy+y^2 which according to fact i leads to $a-b \equiv b-c \equiv 0 \pmod p$. Hence, $a \equiv b \equiv c \pmod p$. We only need to remove the duplicate case, that is, when $a \equiv b \equiv c \pmod p$ and $a+b+c \equiv 0 \pmod p$. This leads to the removal of $(0,0,0)$. Hence, we have p^2+p-1 solutions. Thus, $N_p < p^2+p$ for all $p \equiv 2 \pmod 3$. If $p \equiv 1 \pmod 3$ then we have p^2 solutions for $a+b+c \equiv 0 \pmod p$. If p divides x^2+xy+y^2 then, according to the second fact, we either have $a-b \equiv r(b-c) \pmod p$ or $a-b \equiv r^{-1}(b-c) \pmod p$. In the former case, we would obtain $a \equiv (r+1)b-rc \pmod p$ and in the latter case we have $a \equiv (r^{-1}+1)b-r^{-1}c \pmod p$. In each case, we have p^2 cases for (a,b,c). The last two cases have common solutions whenever $a \equiv b \equiv c \pmod p$. Thus, we should delete p solutions. If the first two equations have a common solutions then $(r+2)b-(r-1)c \equiv 0 \pmod p$. Hence, we have p solutions here. Finally, if all three equations have a common solution it follows that $a \equiv b \equiv c \equiv 0 \pmod p$. So, according to the *inclusion-exclusion principle*, we have $3p^2-3p+1$ solutions.

U597. Evaluate

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+2}+\frac{1}{n+4}-\cdots\right)^2.$$

Proposed by Ovidiu Furdui, Cluj-Napoca, and Alina Sîntămărian, Cluj-Napoca, România

Solution by the author

The series equals $-\frac{\pi^2}{32}-\frac{\pi}{8}+\frac{\ln 2}{4}$. A calculation shows that,

$$\frac{1}{n}-\frac{1}{n+2}+\frac{1}{n+4}-\cdots=\int_0^1\left(x^{n-1}-x^{n+1}+x^{n+3}-\cdots\right)\mathrm{d}x=\int_0^1\frac{x^{n-1}}{1+x^2}\mathrm{d}x. \tag{1}$$

We have, based on (1), that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+2}+\frac{1}{n+4}-\cdots\right)^2=\sum_{n=1}^{\infty}(-1)^n\int_0^1\frac{x^{n-1}}{1+x^2}\mathrm{d}x\int_0^1\frac{y^{n-1}}{1+y^2}\mathrm{d}y\\
&=-\int_0^1\int_0^1\frac{1}{(1+x^2)(1+y^2)}\sum_{n=1}^{\infty}(-xy)^{n-1}\mathrm{d}x\mathrm{d}y\\
&=-\int_0^1\int_0^1\frac{1}{(1+x^2)(1+y^2)(1+xy)}\mathrm{d}x\mathrm{d}y\\
&=-\int_0^1\frac{1}{1+x^2}\left(\int_0^1\frac{1}{(1+y^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x\\
&=-\int_0^1\frac{1}{(1+x^2)^2}\left(\int_0^1\left(\frac{1-xy}{1+y^2}+\frac{x^2}{1+xy}\right)\mathrm{d}y\right)\mathrm{d}x\\
&=-\int_0^1\frac{1}{(1+x^2)^2}\left(\int_0^1\frac{\mathrm{d}y}{1+y^2}-x\int_0^1\frac{y}{1+y^2}\mathrm{d}y+x^2\int_0^1\frac{1}{1+xy}\mathrm{d}y\right)\mathrm{d}x\\
&=-\int_0^1\frac{1}{(1+x^2)^2}\left(\frac{\pi}{4}-\frac{x\ln 2}{2}+x\ln(1+x)\right)\mathrm{d}x\\
&=-\frac{\pi}{4}\int_0^1\frac{\mathrm{d}x}{(1+x^2)^2}+\frac{\ln 2}{2}\int_0^1\frac{x}{(1+x^2)^2}\mathrm{d}x-\int_0^1\frac{x\ln(1+x)}{(1+x^2)^2}\mathrm{d}x.
\end{aligned} \tag{2}$$

We calculate the preceding integrals.

We integrate by parts and we have

$$\begin{aligned}
\frac{\pi}{4}=\int_0^1\frac{1}{1+x^2}\mathrm{d}x&=\frac{x}{1+x^2}\bigg|_0^1+\int_0^1\frac{2x^2}{(1+x^2)^2}\mathrm{d}x\\
&=\frac{1}{2}+2\int_0^1\left(\frac{1}{1+x^2}-\frac{1}{(1+x^2)^2}\right)\mathrm{d}x
\end{aligned}$$

and it follows that

$$\int_0^1\frac{1}{(1+x^2)^2}\mathrm{d}x=\frac{\pi}{8}+\frac{1}{4}. \tag{3}$$

An easy calculation shows that

$$\int_0^1\frac{x}{(1+x^2)^2}\mathrm{d}x=-\frac{1}{2(1+x^2)}\bigg|_0^1=\frac{1}{4}. \tag{4}$$

Now we calculate the third integral. We integrate by parts, with $f(x) = \ln(1+x)$, $f'(x) = \frac{1}{1+x}$, $g'(x) = \frac{x}{(1+x^2)^2}$, $g(x) = -\frac{1}{2(1+x^2)}$, and we have that

$$\begin{aligned}
\int_0^1 \frac{x\ln(1+x)}{(1+x^2)^2}\mathrm{d}x &= -\frac{\ln(1+x)}{2(1+x^2)}\bigg|_0^1 + \frac{1}{2}\int_0^1 \frac{1}{(1+x)(1+x^2)}\mathrm{d}x \\
&= -\frac{\ln 2}{4} + \frac{1}{4}\int_0^1 \left(\frac{1}{1+x} + \frac{1-x}{1+x^2}\right)\mathrm{d}x \\
&= -\frac{\ln 2}{4} + \frac{1}{4}\left(\ln 2 + \frac{\pi}{4} - \frac{\ln 2}{2}\right) \\
&= \frac{\pi}{16} - \frac{\ln 2}{8}.
\end{aligned} \tag{5}$$

Combining (2), (3), (4), and (5) we have that

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+4} - \cdots\right)^2 = -\frac{\pi^2}{32} - \frac{\pi}{8} + \frac{\ln 2}{4},$$

and the problem is solved.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Matthew Too, Brockport, New York, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Yunyong Zhang, Chinaunicom.

U598. Let ABC be a triangle with $\angle BAC = 90°$, and let F be its Feuerbach point. Find $\angle ABC$ knowing that $AF = OF$, where O is the circumcenter of the triangle.

Proposed by Corneliu Mănescu-Avram, Ploieşti, România

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

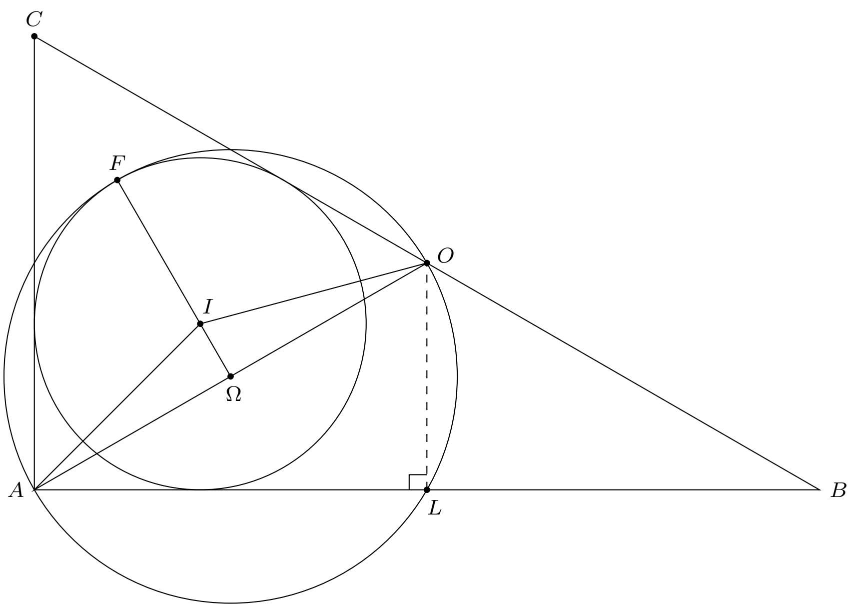
The answer is $\angle ABC = 30°$ or $\angle ABC = 60°$.

Let I denote the incenter of $\triangle ABC$ with sides a, b, c in the usual order and semiperimeter s.

The right angle at A, then, implies the special relation that

$$\text{the inradius of } \triangle ABC = s - a.$$

Let AB be bisected by L. Since $\triangle OAB$ is isosceles with $OA = OB$, $\angle ALO$ is a right angle and the circle on OA as diameter is the nine-point circle of $\triangle ABC$, whose center is denoted by Ω.



Since the line joining the centers of touching circles passes through the point of contact, then, F, I, and Ω are collinear, and since $FA = FO$, F lies on the perpendicular bisector of AO, and therefore

$$F\Omega \perp AO,$$

that is,

$$I\Omega \perp AO.$$

Hence, in $\triangle A\Omega I$, $\angle\Omega$ is a right angle, $\Omega A = \frac{1}{2}AO = \frac{1}{2}\left(\frac{1}{2}BC\right) = \frac{1}{4}a$, $AI =$ (by (1)) $= (s-a)\sqrt{2}$ and $I\Omega = F\Omega - FI = \frac{1}{4}a - (s-a)$. By the Pythagorean theorem,

$$\left(\frac{1}{4}a\right)^2 + \left(\frac{1}{4}a - (s-a)\right)^2 = \left((s-a)\sqrt{2}\right)^2,$$

which is equivalent to

$$a^2 = 4(s-a)(b+c)$$

or

$$(a-2b)(a-2c) = 0$$

whence

$$a = 2b, \text{ which is equivalent to } \angle ABC = 30^\circ$$

or

$$a = 2c, \text{ which is equivalent to } \angle ABC = 60^\circ.$$

Bonus:

Suppose $\angle ABC = 30^\circ$. Then C, F, I, and Ω are collinear and F bisects segment IC.

Proof. If $\angle ABC = 30^\circ$, then $\angle BCA = 60^\circ$ and $OC = \frac{a}{2} = CA$. Hence $\triangle AOC$ is equilateral and $C\Omega$ is the perpendicular bisector of AO. As shown above, this perpendicular bisector passes through F and I. Thus C, F, I, and Ω are collinear.

In turn, we have

$$IF = (\text{the inradius of } \triangle ABC) = s - a = \frac{1}{2}\left(a + \frac{a}{2} + \frac{a\sqrt{3}}{2}\right) - a = \frac{a\left(\sqrt{3}-1\right)}{4}$$

and

$$CI = C\Omega - I\Omega = (\text{from above}) = \frac{a\sqrt{3}}{4} - \left(\frac{a}{4} - \frac{a\left(\sqrt{3}-1\right)}{4}\right) = \frac{a\left(\sqrt{3}-1\right)}{2},$$

that is,

$$CI = 2 \cdot IF.$$

Since points C, F, I are collinear, we conclude that F bisects segment CI.

Also solved by Daniel Văcaru, Pitești, Romania; Nandan Sai Dasireddy, Hyderabad, India; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

U599. Evaluate

$$\int_0^\infty \frac{\ln x}{1+x+x^2+x^3+x^4+x^5}\,dx.$$

Proposed by Ankush Kumar Parcha, Indira Gandhi National Open University,India

Solution by Matthew Too, Brockport, New York, USA

We have

$$\begin{aligned}
&\int_0^\infty \frac{\ln x}{1+x+x^2+x^3+x^4+x^5}\,dx \\
&= \int_0^1 \frac{\ln x}{1+x+x^2+x^3+x^4+x^5}\,dx + \underbrace{\int_1^\infty \frac{\ln x}{1+x+x^2+x^3+x^4+x^5}\,dx}_{x\mapsto 1/x} \\
&= \int_0^1 \frac{(1-x^3)\ln x}{1+x+x^2+x^3+x^4+x^5}\,dx = \int_0^1 \frac{(1-x)(x^2+x+1)\ln x}{(x+1)(x^2-x+1)(x^2+x+1)}\,dx \\
&= \int_0^1 \frac{(1-x)\ln x}{(x+1)(x^2-x+1)}\,dx = \int_0^1 \frac{(x^2-x+1)\ln x}{(x+1)(x^2-x+1)}\,dx - \int_0^1 \frac{x^2\ln x}{(x+1)(x^2-x+1)}\,dx \\
&= \int_0^1 \frac{\ln x}{x+1}\,dx - \underbrace{\int_0^1 \frac{x^2\ln x}{x^3+1}\,dx}_{x^3\mapsto x} = \frac{8}{9}\int_0^1 \frac{\ln x}{x+1}\,dx \overset{\text{I.B.P.}}{=} \frac{8}{9}\Big[\ln(x)\ln(x+1)|_0^1 - \underbrace{\int_0^1 \frac{\ln(x+1)}{x}\,dx}_{x\mapsto -x}\Big] \\
&= \frac{8}{9}\left[-\lim_{x\to 0^+}\ln(x)\ln(x+1) - \int_0^{-1}\frac{\ln(1-x)}{x}\,dx\right] = \frac{8}{9}\left[-\lim_{x\to 0^+}\ln(x)\ln(x+1) + \mathrm{Li}_2(-1)\right] \\
&= \frac{8}{9}\left[-\lim_{x\to 0^+}\ln(x)\ln(x+1) - \frac{\pi^2}{12}\right]
\end{aligned}$$

where $\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-x)}{x}\,dx, z\in\mathbb{C}$ is the dilogarithm function. To evaluate the limit above, we use L'Hôpital's rule to get

$$\lim_{x\to 0^+}\ln(x)\ln(x+1) = \lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{\ln(x+1)}} = \lim_{x\to 0^+}\frac{-(x+1)\ln^2(x+1)}{x} = \lim_{x\to 0^+} -2\ln(x+1) - \ln^2(x+1) = 0.$$

Thus,

$$\int_0^\infty \frac{\ln x}{1+x+x^2+x^3+x^4+x^5}\,dx = -\frac{2\pi^2}{27}.$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Theo Koupelis, Cape Coral, FL, USA; Yunyong Zhang, Chinaunicom; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania.

U600. We say that a positive integer k is *good* if there is a non-constant polynomial $P(x)$ such that

$$P(n^k) = P(n)P(n-1)\cdots P(n-k+1) \tag{$*$}$$

for all positive integers n. Find all *good* integers k.

Proposed by Kaan Bilge, Ataturk High School of Science, Turkey

Solution by Daniel Pascuas, Barcelona, Spain

First of all observe that if a polynomial P satisfies condition $(*)$, for all positive integers n, then the polynomial $Q(x) := P(x^k) - P(x)P(x-1)\cdots P(x-k+1)$ has infinitely may roots and so $Q \equiv 0$, which means that

$$P(x^k) = P(x)P(x-1)\cdots P(x-k+1). \tag{$\star$}$$

Thus a positive integer k is good if there is a non-constant polynomial P satisfying $(\star)$.

It is clear that $k = 1$ is good (just take $P(x) = x$). Moreover, $k = 2$ is also good since the polynomial $P(x) = x^2 + x + 1$ satisfies $(\star)$ for $k = 2$:

$$P(x)P(x-1) = (x^2+x+1)(x^2-x+1) = (x^2+1)^4 - x^2 = x^4 + x^2 + 1 = P(x^2).$$

Now we are going to show that there are no more good integers. To do that we need the following

Remark: Any complex root of a non-constant polynomial P satisfying $(\star)$, for some integer $k > 1$, is either zero or unimodular.

To prove this remark just note that if a is a non-zero complex root of P with largest (smallest) modulus, then $(\star)$ shows that a^k is also a root of P and so $|a|^k \le |a|$ ($|a|^k \ge |a|$), which means that $|a| \le 1$ ($|a| \ge 1$).

Assume that P is a non-constant polynomial satisfying $(\star)$, for an integer $k > 2$. If a is root of P, then $a+2$ is a root of $P(x-2)$, and so $(\star)$ implies that $(a+2)^k$ is a root of P. By the remark, $|a| = 0, 1$ and $|a+2|^k = 0, 1$. It follows that $|a| = 1 = |a+2|$, which means that $a = -1$. Therefore the only root of P is $a = -1$, and hence all the roots of $P(x^k)$ must be unimodular. But, since $P(0-1) = P(-1) = 0$, $(\star)$ shows that $P(0^k) = 0$, which is absurd.

Summarizing, the only good integers are 1 and 2.

O595. Let A be a set of integers greater than 1 such that all the positive divisors greater than 1 of $a_1 \cdots a_n - 1$ belong to A whenever $a_1, \ldots, a_n$ are distinct elements from A and $n \geq 2$. We also assume that A has at least two elements. Prove that A contains all the integers greater than 1.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

Solution by the authors

If a_1 and a_2 belong to A, then $a_1 a_2 - 1$ is greater than 1, hence it has at least one prime divisor p, and p belongs to A by hypothesis. Moreover, p cannot be equal to a_1 (since p and a_1 are relatively prime and greater than 1), so a prime divisor q of $pa_1 - 1$ also belongs to A (and q cannot equal p). Now, if p and q are distinct primes from A, then $pq - 1 (> 1)$ has a prime divisor which is clearly different from p and from q, and which, by hypothesis, belongs to A. So there are at least three distinct primes in A. In general, if $p_1, \ldots, p_n \in A$ $(n \geq 2)$ are distinct primes from A, then $p_1 \cdots p_n - 1 > 1$, hence this number has a prime divisor p_{n+1} which is (necessarily) different from any of $p_1, \ldots, p_n$, and which belongs to A. We conclude that A contains infinitely many primes.

Now let $p_1, p_2, \ldots$ be an infinite sequence of (distinct) primes in A. There are among them two odd primes, so the divisor 2 of their product minus 1 belongs to A. Further, let $m \geq 3$ be any positive integer (arbitrarily chosen, but fixed for the moment). Let $q_1, q_2, \ldots$ be the (also infinite) sequence of those primes among $p_1, p_2, \ldots$ that do not divide m. Among $q_1, q_2, \ldots$ there are infinitely many (denote them by $r_1, r_2, \ldots$) that give the same remainder r at division by m. Since we have $r_i \equiv r \bmod m$ for all $i \geq 1$, and each of $r_1, r_2, \ldots$ is relatively prime to m, it follows that r is relatively prime to m, too. Moreover, we have (φ being Euler's totient function)

$$r_1 \cdots r_{\varphi(m)} \equiv r^{\varphi(m)} \equiv 1 \bmod m,$$

by Euler's theorem. Consequently, m divides $r_1 \cdots r_{\varphi(m)} - 1$ (and $r_1, \ldots, r_{\varphi(m)}$ are from A, and $\varphi(m) \geq 2$), thus, by hypothesis, m belongs to A. As $m \geq 3$ is arbitrary and 2 belongs to A, we just got the desired conclusion.

O596. Let $a \ge b \ge c \ge 0$ such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\sqrt{3abc(a+b+c)} + 2(a-c)^2 \ge 3.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

The inequality can be rewritten as

$$\sqrt{3abc(a+b+c)} + 2(a-c)^2 \ge a^2 + b^2 + c^2,$$

or with the following substitutions $a \to a^2$, $b \to b^2$, $c \to c^2$ Denote

$$f(a,b,c) = abc\sqrt{3(a^2+b^2+c^2)} + 2(a^2-c^2)^2 - a^4 - b^4 - c^4.$$

Using the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}
f(a,b,c) - &f(a,a,c)\\
&= abc\sqrt{3(a^2+b^2+c^2)} - a^2c\sqrt{3(2a^2+c^2)} + a^4 - b^4\\
&= a^4 - b^4 - \frac{\sqrt{3}ac[a^2(2a^2+c^2) - b^2(a^2+b^2+c^2)]}{a\sqrt{2a^2+c^2} + b\sqrt{a^2+b^2+c^2}}\\
&= a^4 - b^4 - \frac{3ac(a^2-b^2)(2a^2+b^2+c^2)}{a\sqrt{3(2a^2+c^2)} + b\sqrt{3(a^2+b^2+c^2)}}\\
&\ge a^4 - b^4 - \frac{3ac(a^2-b^2)(2a^2+b^2+c^2)}{a(2a+c) + b(a+b+c)}\\
&= \frac{(a^2-b^2)[(a^2+b^2)(2a^2+b^2+bc+ab+ac) - 3ac(2a^2+b^2+c^2)]}{2a^2+b^2+ab+bc+ca} \ge 0
\end{aligned}$$

because $a^2 + b^2 \ge 2ab \ge 2ac$, $bc \ge c^2$, so

$$(a^2+b^2)(2a^2+b^2+bc) \ge 2ac(2a^2+b^2+c^2)$$

and

$$(a^2+b^2)(ab+ac) - ac(2a^2+b^2+c^2) = a^3(b-c) + a(b^3-c^3) \ge 0.$$

Therefore, it remains to prove that $f(a,a,c) \ge 0$ or

$$a^2c\sqrt{3(2a^2+c^2)} \ge c^2(4a^2-c^2),$$

or noting $t = \frac{a^2}{c^2} \ge 1$ and squaring

$$3t^2(2t+1) \ge (4t-1)^2 \iff (t-1)^2(6t-1) \ge 0,$$

obviously true.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL, USA; Titu Zvonaru, Comănești, Romania.

O597. Let ABC be a triangle and let x, y, z be positive real numbers. Prove that

$$4+\frac{r}{R}+\frac{x}{y+x}(1+\cos A)+\frac{y}{z+x}(1+\cos B)+\frac{z}{x+y}(1+\cos C)\geq(\sin A+\sin B+\sin C)^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author
Using the well-known identity

$$\cos A+\cos B+\cos C=1+\frac{r}{R}$$

and the Cauchy-Schwarz inequality we have

$$\begin{aligned}4+\frac{r}{R}+\sum_{\text{cyc}}\frac{x}{y+x}(1+\cos A)&=\sum_{\text{cyc}}(1+\cos A)+\sum_{\text{cyc}}\frac{x}{y+x}(1+\cos A)\\&=\sum_{\text{cyc}}\left(1+\frac{x}{y+x}\right)(1+\cos A)\\&=2(x+y+z)\sum_{\text{cyc}}\frac{\cos^2\frac{A}{2}}{y+z}\\&\geq 2(x+y+z)\frac{(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2})^2}{(y+z)+(z+x)+(x+y)}\\&=\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)^2\\&\geq(\sin A+\sin B+\sin C)^2.\end{aligned}$$

The last inequality is true because

$$\begin{aligned}\sum_{\text{cyc}}\cos\frac{A}{2}&=\sum_{\text{cyc}}\sin\frac{B+C}{2}\\&\geq\sum_{\text{cyc}}\sin\frac{B+C}{2}\cos\frac{B-C}{2}\\&=\sum_{\text{cyc}}\frac{1}{2}(\sin B+\sin C)\\&=\sin A+\sin B+\sin C.\end{aligned}$$

The conclusion follows.

Also solved by Dao Van Nam, Phúớng Nam High School, Hoáng Mai, Ha Noi, Vietnam; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA.

O598. Let $a_1, a_2, \ldots a_n$ be real numbers such that

$$a_1 + a_2 + \ldots + a_n = a_1^2 + a_2^2 + \ldots + a_n^2 = n - 1$$

Prove that

$$n - 1 \le a_1^3 + a_2^3 + \ldots + a_n^3 < n + 1 - \frac{6n-4}{n^2}$$

Proposed by Josef Tkadlec, Czech Republic

Solution by the author

The lower bound follows immediately from Cauchy-Schwarz inequality in the form $(\sum_i a_i^3)(\sum_i a_i) \ge (\sum_i a_i^2)^2$.

For the upper bound, let $b_i = a_i - 1$. Then

$$\sum_{i=1}^n b_i = (n-1) - n = -1 \quad \text{a} \quad \sum_{i=1}^n b_i^2 = \sum_{i=1}^n (a_i^2 - 2a_i + 1) = (n-1) - 2(n-1) + n = 1.$$

At the same time

$$\sum_{i=1}^n b_i^3 = \left(\sum_{i=1}^n a_i^3\right) - 3(n-1) + 3(n-1) - n = \left(\sum_{i=1}^n a_i^3\right) - n,$$

hence we need to show $\sum_{i=1}^n b_i^3 \in [-1, 1)$. Note that since $\sum_{i=1}^n b_i^2 = 1$, we have $b_i \in [-1, 1]$.

For $x \in [-1, 1]$ we have $x^3 \le x^2$ (with equality if and only if $x \in \{0, 1\}$), hence

$$\sum_{i=1}^n b_i^3 \le \sum_{i=1}^n b_i^2 \le 1.$$

To show that the inequality is in fact sharp, suppose $b_i \in \{0, 1\}$. Then we have $a_i = b_i + 1 \ge 1$, hence $a_1 + a_2 + \cdots + a_n \ge n > n - 1$, a contradiction.

Now, we will show that the maximum is $M_n = n + 1 - \frac{6n-4}{n^2}$. It is straightforward to check that this value is attained for $a_1 = \cdots = a_{n-1} = 1 - 2/n$, $a_n = 2 - 2/n$ (and its permutations).

To prove the bound, shift to new variables $c_i = a_i + 2/n$. Then the conditions become

$$\sum_{i=1}^n c_i = (n-1) + n \cdot \frac{2}{n} = n + 1,$$
$$\sum_{i=1}^n c_i^2 = (n-1) + \frac{4}{n}(n-1) + n \cdot \frac{4}{n^2} = n + 3.$$

Since $\sum_{i=1}^n (c_i - 1)^2 = (n+3) - 2(n+1) + n = 1$, we have $c_i \in [0, 2]$. For any $x \in [0, 2]$ we have $(x-1)^2(x-2) \le 0$, which is equivalent with $x^3 \le 4x^2 - 5x + 2$. Applying this bound to all c_i we get

$$\sum_{i=1}^n c_i^3 \le 4(n+3) - 5(n+1) + 2n = n + 7.$$

Reverting back to a_i this rewrites as the desired

$$\sum_{i=1}^n a_i^3 = \sum_{i=1}^n (c_i - 2/n)^3 \le (n+7) - \frac{6}{n}(n+3) + \frac{12}{n^2}(n+1) - \frac{8}{n^3}n = n + 1 - \frac{6n-4}{n^2}.$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Adam John Frederickson, Utah Valley University, UT, USA.

O599. There are n children in a school. They form groups with each other, of various sizes, in a such a way that no child is left alone. Then, all of these children go to a park, where they have to sit around circular tables, each group around its table. Both the order and sense of the seating arrangements matter. Find in terms of n a closed formula for the number of ways this whole thing can be orchestrated; i.e breaking up into groups together with their seating arrangement around circles.

Proposed by Arpon Basu, AECS-4 School, Mumbai, India

First solution by the author

Since any permutation can be broken down into disjoint cycles (of possibly size 1) of numbers (Why? Begin with the first element, say a_0, and keep applying the permutation σ to it until $\sigma^k(a_0) = a_0$. Then the numbers $\{a_0, \sigma(a_0), \sigma^2(a_0), ...,$
$\sigma^{k-1}(a_0)\}$ form a cycle. Remove these numbers from the set being permuted, and repeat the process again until every member of the set is the member of some cycle), had not the "no child should be left alone" condition been imposed, the total number of ways would have had a bijection with the number of permutations, ie:- $n!$.

But due to the "no child should be left alone" clause, the number of ways has a one-to-one relation with the number of derangements of set of size n (because cycles of size 1 correspond to fixed points of a permutation). The answer is thus $D_n = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$.

Second solution by Polyahedra, Polk State College, FL, USA

Let a_n be the number of such orchestrations. Notice that $a_1 = 0$ and assume that $a_0 = 1$. We claim that $a_{n+1} = n(a_n + a_{n-1})$ for all $n \geq 1$.

Indeed, we can think recursively by adding Alice to the n children in the school. If Alice is seated with at least two other children at a table, then we can view her as being placed in one of the n spaces between two children in an existing arrangement of n children. This gives the term na_n. If Alice sits with only one other child at a table, then there are n choices of this other child, and the remaining $n-1$ children form their required arrangement. This gives the term na_{n-1} and proves the claimed recurrence. Since this recurrence is the same as the well-known one for derangement, with the same initial values, we have

$$a_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor$$

for all $n \geq 1$.

O600. Prove that in any triangle ABC the following inequality holds:

$$\frac{\sin A}{1+\cos^2 B+\cos^2 C}+\frac{\sin B}{\cos^2 C+\cos^2 A}+\frac{\sin C}{1+\cos^2 A+\cos^2 B}\leq\sqrt{3}$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Marian Ursărescu, Roman-Vodă National College, Roman, Romania

$$1+\cos^2 B+\cos^2 C=\sin^2 A+\cos^2 A+\cos^2 B+\cos^2 C$$

We must show:

$$\sum_{cyc}\frac{\sin A}{\sin^2 A+\cos^2 A+\cos^2 B+\cos^2 C}\leq\sqrt{3} \tag{1}$$

But in any triangle ABC we have: $\cos^2 A+\cos^2 B+\cos^2 C\geq\frac{3}{4}$ then:

$$\sin^2 A+\cos^2 A+\cos^2 B+\cos^2 C\geq\frac{3}{4}+\sin^2 A$$

Hence,

$$\frac{\sin A}{\sin^2 A+\cos^2 A+\cos^2 B+\cos^2 C}\leq\frac{\sin A}{\frac{3}{4}+\sin^2 A} \tag{2}$$

From (1) and (2) we must show:

$$\sum_{cyc}\frac{\sin A}{\sin^2 A+\frac{3}{4}}\leq\sqrt{3} \tag{3}$$

$$A\in(0,\pi)\Rightarrow\sin A>0\Rightarrow\sin^2 A+\frac{3}{4}\geq 2\sqrt{\frac{3}{4}\sin^2 A}\Rightarrow\sin^2 A+\frac{3}{4}\geq\sqrt{3}\sin A$$

$$\frac{1}{\sin^2 A+\frac{3}{4}}\leq\frac{1}{\sqrt{3}\sin A}\Rightarrow\frac{\sin A}{\sin^2 A+\frac{3}{4}}\leq\frac{1}{\sqrt{3}}$$

$$\sum_{cyc}\frac{\sin A}{\sin^2 A+\frac{3}{4}}\leq\sqrt{3}$$

and then (3) its true.
Observation:

$$\cos^2 A+\cos^2 B+cos^2 C=1-\frac{s^2-(4R^2+4Rr+3r^2)}{2R^2} \tag{4}$$

From Gerretsen's inequality:

$$s^2-4R^2-4Rr-r^2\leq 2r^2 \tag{5}$$

From (4) and (5), it follows:

$$\cos^2 A+\cos^2 B+\cos^2 C\geq 1-\frac{r^2}{R^2}\overset{Euler}{\geq}\frac{3}{4}$$

Also solved by Le Hoang Bao, Tien Giang, Vietnam; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA.