Junior Problems

J601. Let a, b, c, d be real numbers such that $7 > a \geq b+1 \geq c+3 \geq d+4$. Prove that

$$\frac{1}{7-a}+\frac{4}{6-b}+\frac{9}{4-c}+\frac{16}{3-d} \geq a+b+c+d.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J602. Prove that for any positive real number x,

$$\sqrt{x}+\frac{1}{\sqrt{x}} \geq \sqrt{1+\frac{1}{x+1}+\frac{1}{\sqrt{x+1}}}.$$

When does equality hold?

Proposed by An Zhenping, Xianyang Normal University, China

J603. Let ABC be a triangle with centroid G and M, N, P, Q be the midpoints of the segments AB, BC, CA, AG, respectively. Prove that if

$$\sin(A-B)\sin C = \sin(C-A)\sin B,$$

then points M, N, P, Q lie on a circle.

Proposed by Mihaela Berindeanu, Bucharest, România

J604. Let m, n, p be odd positive integers such that $(m-n)(n-p)(p-m) = 0$ and

$$\frac{7}{m}+\frac{8}{n}+\frac{9}{p} = 2.$$

Find the least possible value of mnp.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J605. Prove that

(a) for an arbitrary triangle ABC,

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab} \geq 2+\frac{r}{2R},$$

(b) if triangle ABC is acute, then

$$\frac{m_a^2}{b^2+c^2}+\frac{m_b^2}{c^2+a^2}+\frac{m_c^2}{a^2+b^2} \leq 1+\frac{r}{4R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J606. Let a, b, c be real numbers in the interval $[0, 1]$. Prove that

$$2 \leq \frac{b+c}{1+a}+\frac{c+a}{1+b}+\frac{a+b}{1+c}+2(1-a)(1-b)(1-c) \leq 3.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S601. Solve in integers the equation

$$16x^2y^2(x^2+1)(y^2-1)+4(x^4+y^4+x^2-y^2)=2023^2-1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S602. For every integer $n>1$, let

$$A=\frac{1}{n+1}\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3}+\frac{1}{5}\right)\cdots\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)$$

and

$$B=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)\cdots\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right).$$

Compare A and B.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S603. Let ABC be a triangle with incenter I and centroid G. Line AG intersects BC in E and line AI intersects BC in D and the circumcircle in M. Point P is the orthogonal projection of E onto AM. Prove that if $(AB+AC)^2=16AP\cdot DM$ then GI is parallel to BC.

Proposed by Mihaela Berindeanu, Bucharest, România

S604. Let x,y,z be real numbers such that $-1\le x,y,z\le 1$ and $x+y+z+xyz=0$. Prove that

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\le\sqrt{9+xyz}.$$

Proposed by Marius Stănean, Zalău, România

S605. Let ABC be an acute triangle with circumcenter O and circumradius R. Let R_a,R_b,R_c be the circumradii of triangles OBC, OCA, OAB, respectively. Prove that triangle ABC is equilateral if and only if

$$R^3+R^2(R_a+R_b+R_c)=4R_aR_bR_c.$$

Proposed by Marian Ursărescu, Roman, România

S606. Let a,b,c be positive real numbers. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+abc\ge a+b+c+1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Undergraduate Problems

U601. Let a, b, c, d be real numbers such that all roots of the polynomial $P(x) = x^5 - 10x^4 + ax^3 + bx^2 + cx + d$ are greater than 1. Find the minimum possible value of $a + b + c + d$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U602. Let $a, b > 0$. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \frac{x^n}{\sqrt{ax^{2n} + b}} \mathrm{d}x.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Cluj-Napoca, România

U603. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the conditions:

(a) $f(x + 1)f(y) - f(xy) = (2x + 1)f(y)$ for all $x, y \in \mathbb{R}$,

(b) $f(x) > 2x$ for all $x > 2$.

Proposed by Mircea Becheanu, Canada

U604. If

$$f(m) = \int_0^1 \frac{(x^m - 1)\ln(1 + x)}{x \ln(x)}\, \mathrm{d}x,$$

then evaluate

$$\lim_{m\to 0} \left(\frac{2\pi^2}{9m} - \frac{8f(m)}{3m^2} \right).$$

Proposed by Ty Halpen, Florida, USA

U605. Let a, b, c be nonzero complex numbers having the same modulus for which

$$a^3 + b^3 + c^3 = rabc,$$

where r is a real number.

(i) Prove that $-1 \le r \le 3$.

(ii) Prove that if $r < 3$ then one and only one of the equations $ax^2 + bx + c = 0$, $bx^2 + cx + a = 0$, $cx^2 + ax + b = 0$ has a root of modulus 1.

Proposed by Florin Stănescu, Găeşti, România

U606. Determine all bijective functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f^{(mn+1)}(m + n) = f(m)f(n),$$

for all positive integers m, n such that $|m-n|$ is odd. Here $f^{(k)}$ denotes $k-$times composition $f \circ f \circ \ldots. \circ f$.

Proposed by Valentio Iverson, Waterloo, Canada and Stanve Avrilium Widjaja, Singapore

Olympiad Problems

O601. Find all integers n for which $(n+1)^2 + (2n+1)^2$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O602. Let a, b, c, d be positive real numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+d} = 1.$$

Prove that

$$ab + ac + ad + bc + bd + cd + 6 \geq 5(a+b+c+d).$$

Proposed by Marius Stănean, Zalău, România

O603. Find all triples (x, y, z) of nonnegative integers such that

$$x^2 - y^2 = 2^z + 2022.$$

Proposed by Todor Zaharinov, Sofia, Bulgaria

O604. Let $n > 2$ be an integer. For every nonempty subset $A = \{a_1, \ldots, a_k\}$ of the set $\{1, 2, \ldots, n\}$ we denote by m_A the arithmetic mean of the elements of A. Find the least possible value of the difference $|m_A - m_B|$ when A and B run over all subsets of $\{1, 2, \ldots, n\}$.

Proposed by Cristi Săvescu, România

O605. Let a, b, c be positive real numbers and let x, y, z be real numbers. Prove that

$$\frac{abxy}{a+b+2c} + \frac{bcyz}{b+c+2a} + \frac{cazx}{c+a+2b} \leq \frac{1}{4}(ax^2 + by^2 + cz^2).$$

Proposed by An Zhenping, Xianyang University of Science, China

O606. Prove that the product of six consecutive positive integers is not the fifth power of an integer.

Proposed by Titu Andreescu, USA and Marian Tetiva, România