Sperner's Lemma and Brouwer's Fixed Point

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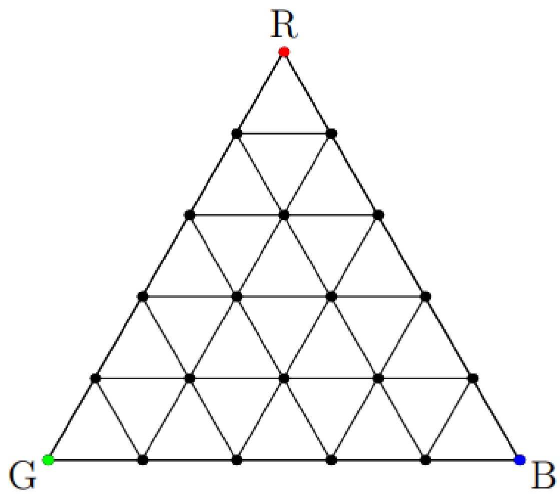
Abstract

In this article we examine and prove the Sperner lemma. If T is a triangulation colored with a Sperner coloring, at least one of the small triangular faces of T will have its three vertices colored with three different colors. In the next part we will examine the fixed point theorem and at the end, we will examine the hex.

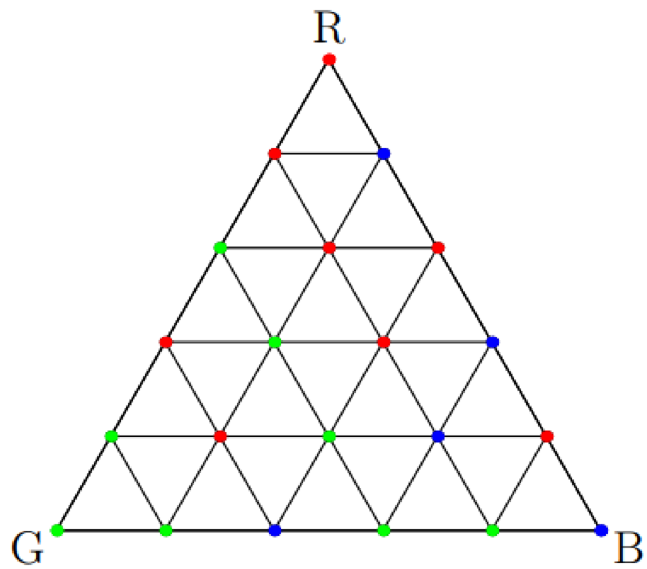
1 Sperner's Lemma

Let T be a graph formed by taking the triangle with vertices R, G, B, and triangulating it, that is, breaking it up into many small triangular pieces, like so:

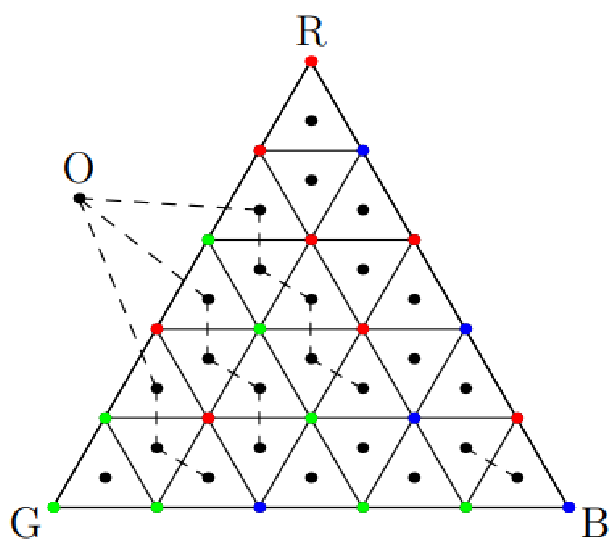


Now color every vertex of T red, green, or blue, subject to the following rules: Color vertex R red, G green, B blue, and every vertex on the side of the outer triangle connecting R and G either red or green, every vertex on the side connecting R and B either red or blue, and every vertex on the side connecting G and B either green or blue. The interior vertices can be colored with any of the 3 colors. This is called a Sperner coloring

Lemma : (Sperner). If T is a triangulation colored with a Sperner coloring, at least one of the small trianglular faces of T will have its three vertices colored with three different colors.



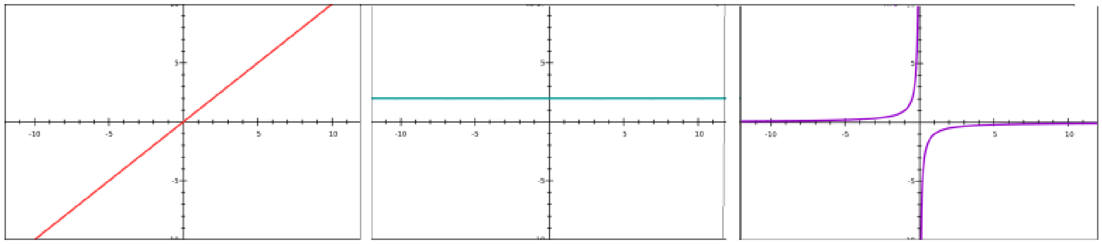
Proof : In order to prove this, we make another graph, $T^{'}$, whose vertices are faces of T. All of the faces inside the triangle RGB will themselves be triangles, but there will be one more face, the outside face O. Note that each pair of neighboring faces of T meet at an edge of T, whose two vertices are colored. We connect these two faces, as vertices in $T^{'}$, if and only if the two vertices of the edge where they meet are colored red and green. Faces which are not adjacent are not connected in $T^{'}$.



2 Fixed Points

For a function $f : X \to X$, a fixed point $c \in X$ is a point where $f(c) = c$.

When a function has a fixed point, c, the point (c, c) is on its graph. The function $f(x) = x$ is composed entirely of fixed points, but it is largely unique in this respect. Many other functions may not even have one fixed point.

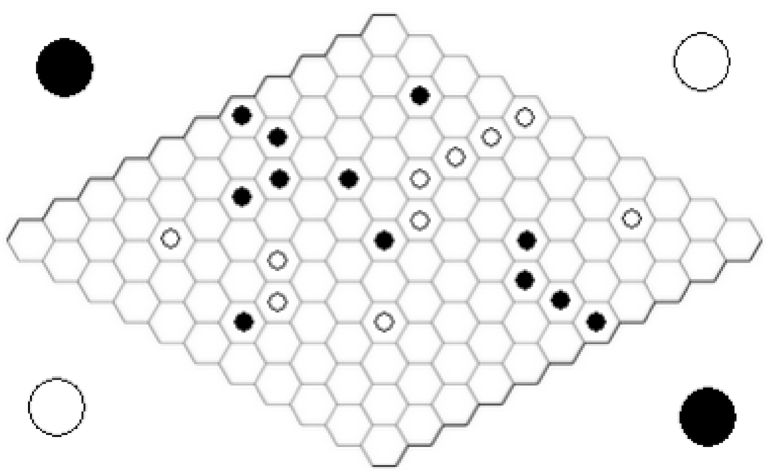


$f(x) = x$, $f(x) = 2$, and $f(x) = \frac{-1}{x}$, respectively. The first is entirely fixed points, the second has one fixed point at 2, and the last has none.

Fixed points came into mathematical focus in the late 19th century. The mathematician Henri Poincar´e began using them in topological analysis of nonlinear problems, moving fixedpoint theory towards the front of topology. Luitzen Egbertus Jan Brouwer, of the University of Amsterdam, worked with algebraic topology. He formulated his fixed point theorem, which was first published relating only to the three dimensional case in 1909, though other proofs for this specific case already existed.

3 Hex Game

Hex is played with two people, on a diamond-shaped board made up of hexagonal cells. The board dimensions can vary, but the typical size is 11 × 11. Two opposite sides of the board are labeled "black", and the remaining two sides are labeled "white". One of the players has a supply of black tiles while the other player has a supply of white tiles. The players alternate turns to place their tile on any unoccupied space on the game board, with the goal of forming an unbroken chain of tiles (of his own color of course) linking his two regions. Figure below shows a Hex game board with the black and white regions. Some game pieces have already been played. Although the rules of Hex are simple, the game can provide insights for many mathematical concepts. The Hex Theorem states that a game of Hex cannot end in a draw. The only way to keep the opponent from building a winning chain is to build a winning chain first. Although the Hex Theorem is intuitive, proving it can require invoking complicated topological results. John Nash is said to have proven the Hex Theorem, but he may not have bothered to publish the proof. David Gale gave a simple proof of the Hex Theorem based on graph theory and also showed the equivalence between the Hex Theorem and the Brouwer Fixed Point Theorem



(1) Having extra/random pieces of your own color lying on the board cannot hurt you.

(2) The game therefore admits a winning strategy for one of the players, and indeed it is the first player who has a winning strategy.

To prove the first lemma, suppose that there is an extra piece at position x on the board. If x is part of your winning strategy, then on the turn when you should be playing at position x, you could instead lay down another piece somewhere else. If x is not part of your winning strategy, then you would not care that it is occupied. We defer the proof of the second lemma to the next section. To prove that the first player must win Hex, let there be two players A and B: player A goes first and B goes second. Suppose for the sake of contradiction that B, the second player, has a winning strategy. Then A can just play a random move somewhere on the board. When B plays, he is effectively the first player. Now the original first player, A, is effectively the "second" player and can play the winning strategy for the rest of the game. By Lemma 1, having an extra piece from that first random move will never hurt A. Since A has "stolen" the winning strategy from B, B must not win. By Lemma 2, Hex cannot end in a draw, and since B cannot win, then A must be the winner. Then the first player must win Hex.

- Someone in class asked, what is keeping B from stealing the strategy back? As in, what if B plays another bogus move, and we're back at A being the effective first player? We are assuming perfect play, so if A's winning strategy beats B when B is playing the best he can, that strategy will also beat B when he is playing less than perfectly.

- Given this, we would then wonder, how can A steal the strategy? Player B had the winning strategy, so if A started out playing a random move, then B can just keep being smart and beat A, right? This is the heart of the contradiction and our proof. Given that A played a random move and has effectively made B the first player, B is no longer in the position to use the winning strategy even though our assumption would imply that B can just keep playing perfectly and beat A. Hence, our assumption that B had the winning strategy must have been flawed.

- This strategy-stealing argument can be applied to any other symmetric game where having an extra move or game piece on the board can never hurt you. An example would be tic-tac-toe, though that game is strongly solved anyway, so we would not need strategy-stealing to know who will win.

- We also needed Lemma 2 to rule out the case of a tie, because otherwise, knowing that the second player will not win does not imply that the first player will win.

- Knowing that the first player wins is great, but unfortunately, this proof doesn't give us an actual winning strategy that we can use.

In this proof, we skipped over the proof of Lemma 2, but for good reason, because it is a theorem all in its own, and will take some work to prove

4 Problems

Problem 1 . In $T^{'}$, show that the degree of O is odd.

Problem 2 . Show that the number of small triangles whose vertices are colored red, green, blue is odd, and thus greater than 0. This proves Sperner's Lemma

Problem 3 . What should the 1-dimensional version of Sperner's Lemma look like, or the 3- dimensional version Prove your $1D$ version.

Problem 4 . Prove your $3D$ version of Sperner's Lemma, and see if you can extend it to arbitrary dimensions.

Problem 5 . Prove Sperner's Lemma from Brouwer's Fixed Point Theorem

References

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