Junior problems

J601. Let a, b, c, d be real numbers such that $7 > a \geq b + 1 \geq c + 3 \geq d + 4$. Prove that

$$\frac{1}{7-a} + \frac{4}{6-b} + \frac{9}{4-c} + \frac{16}{3-d} \geq a + b + c + d.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, CA, USA

Let $s := a + b + c + d$. Since $a < 7, b < 6, c < 4, d < 3$ then $s < 20$. By Cauchy-Schwarz Inequality

$$(7 - a + 6 - b + 4 - c + 3 - d)\left(\frac{1^2}{7-a} + \frac{2^2}{6-b} + \frac{3^2}{4-c} + \frac{4^2}{3-d}\right) \geq (1+2+3+4)^2 = 100.$$

Hence, $\frac{1}{7-a} + \frac{4}{6-b} + \frac{9}{4-c} + \frac{16}{3-d} \geq \frac{100}{20-s}$ and $\frac{100}{20-s} - s = \frac{(s-10)^2}{20-s} \geq 0$.

Equality iff $(7-a, 6-b, 4-c, 3-d) = (1, 2, 3, 4) \iff (a, b, c, d) = (6, 4, 1, -1)$.

Also solved by Polyahedra, Polk State College, USA; Ivan Hadinata, Jember, Indonesia; Alex Grigoryan, Quantum College, Yerevan, Armenia; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Hyunbin Yoo, South Korea; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Henry Ricardo, Westchester Area Math Circle, USA; Sundaresh H R, Shivamogga, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Muhammad Thoriq, Yogyakarta, Indonesia; Israel Castillo Pilco, Huaral, Peru; Adam John Frederickson, Utah Valley University, UT, USA; Dao Van Nam, MAY High School, Hoáng Mai, Ha Noi, Vietnam.

J602. Prove that for any positive real number x,

$$\sqrt{x} + \frac{1}{\sqrt{x}} \geq \sqrt{1 + \frac{1}{x+1}} + \frac{1}{\sqrt{x+1}}.$$

When does equality hold?

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Polyahedra, Polk State College, USA

Rationalizing a numerator, we have

$$\sqrt{x} + \frac{1}{\sqrt{x}} - \sqrt{1 + \frac{1}{x+1}} - \frac{1}{\sqrt{x+1}} = \frac{x+1}{\sqrt{x}} - \frac{x+1}{\sqrt{x+1}(\sqrt{x+2}-1)}.$$

By the Cauchy-Schwarz inequality,

$$\sqrt{(x+1)(x+2)} = \sqrt{\left(\left(\sqrt{x}\right)^2 + 1^2\right)\left(1^2 + \left(\sqrt{x+1}\right)^2\right)} \geq \sqrt{x} + \sqrt{x+1},$$

with equality if and only if $1/\sqrt{x} = \sqrt{x+1}$, that is, if and only if $x = (\sqrt{5}-1)/2$.

Also solved by Ivan Hadinata, Jember, Indonesia; Alex Grigoryan, Quantum College, Yerevan, Armenia; Muhammad Thoriq, Yogyakarta, Indonesia; Israel Castillo Pilco, Huaral, Peru; Adam John Frederickson, Utah Valley University, UT, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; G. C. Greubel, Newport News, VA, USA; Marin Chirciu,Colegiul Național Zinca Golescu, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Soham Dutta, DPS, Ruby Park, West Bengal, India; Sundaresh H R, Shivamogga, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Prajnanaswaroopa S, Bangalore, India; Daniel Pascuas, Barcelona, Spain; Arkady Alt, San Jose, CA, USA.

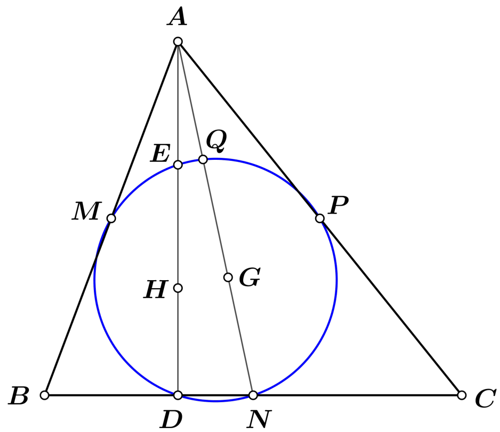
J603. Let ABC be a triangle with centroid G and M, N, P, Q be the midpoints of the segments AB, BC, CA, AG, respectively. Prove that if

$$\sin(A-B)\sin C = \sin(C-A)\sin B,$$

then points M, N, P, Q lie on a circle.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Kousik Sett, India



Draw $AD \perp BC$. H is the orthocenter. AD cuts the nine-point circle at E which is the midpoint of AH. The nine-point circle pass through the points M, N, P, Q, and E.

We have

$$AN = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}, \quad AQ = \frac{1}{3}AN, \quad AE = \frac{1}{2}AH = R\cos A, \quad \text{and} \quad AD = \frac{bc}{2R}.$$

As $\sin C = \sin(A+B)$ and $\sin B = \sin(A+C)$, the given condition

$$\sin(A-B)\sin C = \sin(C-A)\sin B,$$

is equivalent to

$$\begin{aligned}
&\sin(A-B)\sin(A+B) = \sin(C-A)\sin(C+A)\\
\Longrightarrow\ &\sin^2 A - \sin^2 B = \sin^2 C - \sin^2 A\\
\Longrightarrow\ &a^2 - b^2 = c^2 - a^2\\
\Longrightarrow\ &b^2 + c^2 = 2a^2. \qquad (1)
\end{aligned}$$

By power of point theorem, we get

$$AQ \cdot AN = AE \cdot AD,$$

which implies

$$\frac{1}{3}\times\frac{1}{4}(2b^2+2c^2-a^2) = \frac{b^2+c^2-a^2}{4} \Longrightarrow b^2+c^2 = 2a^2,$$

which is true by (1).

Also solved by Alex Grigoryan, Quantum College, Yerevan, Armenia; Polyahedra, Polk State College, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Theo Koupelis, Cape Coral, FL, USA.

J604. Let m, n, p be odd positive integers such that $(m-n)(n-p)(p-m) = 0$ and

$$\frac{7}{m} + \frac{8}{n} + \frac{9}{p} = 2.$$

Find the least possible value of mnp.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by the author

If $m = n$, then $\frac{15}{m} + \frac{9}{p} = 2$, implying

$$(2m-15)(2p-9) = 135.$$

We get $m = n = 9$, $p = 27$ or $m = n = 15$, $p = 9$ or $m = n = 21$, $p = 7$ or $m = n = 75$, $p = 5$, with the least mnp being 2025. If $m = p$, then $\frac{16}{m} + \frac{8}{n} = 2$, implying

$$mn = 4(m+2n),$$

a contradiction. Finally, if $n = p$, then $\frac{7}{m} + \frac{17}{n} = 2$, implying

$$(2m-7)(2n-17) = 119.$$

We get $m = 7$, $n = p = 17$ or $m = 63$, $n = p = 9$, with the least mnp being 2023. So, the least possible value of mnp is 2023.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Daniel Văcaru, Pitești, Romania; Ivan Hadinata, Jember, Indonesia; Alex Grigoryan, Quantum College, Yerevan, Armenia; Muhammad Thoriq, Yogyakarta, Indonesia; Polyahedra, Polk State College, USA; Israel Castillo Pilco, Huaral, Peru; Prajnanaswaroopa S, Bangalore, India; Emon Suin, Ramakrishna Mission Vidyalaya, Narendrapur, India; Sundaresh H R, Shivamogga, India.

J605. Prove that

(a) for an arbitrary triangle ABC,

$$\frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} \geq 2 + \frac{r}{2R},$$

(b) if triangle ABC is acute, then

$$\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} \leq 1 + \frac{r}{4R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Theo Koupelis, Cape Coral, FL, USA

(a) Let s, E be the triangle's semiperimeter and area, respectively. Then we have $4m_a^2 = 2(b^2+c^2) - a^2 \geq (b+c)^2 - a^2 = (a+b+c)(b+c-a) = 4s(s-a)$, and similarly $4m_b^2 \geq 4s(s-b)$, and $4m_c^2 \geq 4s(s-c)$. Thus,

$$\begin{aligned}\frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} &\geq \frac{s}{abc}\left[a(s-a)+b(s-b)+c(s-c)\right] = \frac{s}{abc}\cdot(2s^2-a^2-b^2-c^2)\\ &= \frac{2s}{abc}\cdot(ab+bc+ca-s^2) = 2 + \frac{2(s-a)(s-b)(s-c)}{abc}\\ &= 2 + \frac{E}{2s}\cdot\frac{4E}{abc} = 2 + \frac{r}{2R}.\end{aligned}$$

Equality occurs when $a=b=c$.

(b) For an acute triangle the quantities $\cos A, \cos B, \cos C$ are positive. We have $4m_a^2 = 2(b^2+c^2) - a^2 = b^2+c^2+2bc\cos A \leq (b^2+c^2)(1+\cos A)$, and similarly $4m_b^2 \leq (a^2+c^2)(1+\cos B)$, and $4m_c^2 \leq (a^2+b^2)(1+\cos C)$. Thus,

$$\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} \leq \frac{3}{4} + \frac{1}{4}\sum_c \cos A = \frac{3}{4} + \frac{1}{4}\left(1+\frac{r}{R}\right) = 1 + \frac{r}{4R},$$

where we used the well-known expression

$$\begin{aligned}\cos A + \cos B + \cos C - 1 &= \frac{b^2+c^2-a^2}{2bc} + \frac{a^2+c^2-b^2}{2ac} + \frac{a^2+b^2-c^2}{2ac} - 1\\ &= \frac{4(s-a)(s-b)(s-c)}{abc} = \frac{r}{R}.\end{aligned}$$

Equality occurs when $a=b=c$.

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Alex Grigoryan, Quantum College, Yerevan, Armenia; Polyahedra, Polk State College, USA; Israel Castillo Pilco, Huaral, Peru; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Marin Chirciu,Colegiul Național Zinca Golescu, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Scott H. Brown, Auburn University Montgomery, AL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Dao Van Nam, MAY High School, Hoáng Mai, Ha Noi, Vietnam; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, România.

J606. Let a, b, c be real numbers in the interval $[0, 1]$. Prove that

$$2 \leq \frac{b+c}{1+a} + \frac{c+a}{1+b} + \frac{a+b}{1+c} + 2(1-a)(1-b)(1-c) \leq 3.$$

Proposed by Marius Stănean, Zalău, România

Solution by Polyahedra, Polk State College, USA

The inequalities are equivalent to

$$2abc \leq \frac{a^2(b+c)}{1+a} + \frac{b^2(c+a)}{1+b} + \frac{c^2(a+b)}{1+c} \leq 1 + 2abc.$$

By the AM-GM inequality,

$$\frac{a^2(b+c)}{1+a} + \frac{b^2(c+a)}{1+b} + \frac{c^2(a+b)}{1+c} \geq \frac{2a^2\sqrt{bc}}{2} + \frac{2b^2\sqrt{ca}}{2} + \frac{2c^2\sqrt{ab}}{2} \geq 3abc \geq 2abc.$$

For the upper bound, we see that

$$\frac{a^2(b+c)}{1+a} + \frac{b^2(c+a)}{1+b} + \frac{c^2(a+b)}{1+c} \leq \frac{a^2(b+c)}{2a} + \frac{b^2(c+a)}{2b} + \frac{c^2(a+b)}{2c} = ab + bc + ca.$$

Since $1 + bc - (b+c) = (1-b)(1-c) \geq 0$, $1 + bc \geq b + c$. Therefore,

$$\begin{aligned} 1 + 2abc &= a(1+bc) + b(1+ca) + c(1+ab) + (1-a)(1-b)(1-c) - (ab+bc+ca) \\ &\geq a(b+c) + b(c+a) + c(a+b) - (ab+bc+ca) = ab+bc+ca, \end{aligned}$$

completing the proof.

Also solved by Alex Grigoryan, Quantum College, Yerevan, Armenia; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL, USA.

Senior problems

S601. Solve in integers the equation

$$16x^2y^2(x^2+1)(y^2-1)+4(x^4+y^4+x^2-y^2)=2023^2-1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

The equation can be rewritten as

$$(4x^4+4x^2)(4y^4-4y^2)+(4x^4+4x^2)+(4y^4-4y^2)+1=2023^2,$$

which is equivalent to

$$(4x^4+4x^2+1)(4y^4-4y^2+1)=2023^2.$$

It follows that $(2x^2+1)(2y^2-1)=\pm(7\cdot 17^2)$. An easy check shows that there are no solutions for the equation

$$(2x^2+1)(2y^2-1)=-(7\cdot 17^2).$$

The equation

$$(2x^2+1)(2y^2-1)=7\cdot 17^2$$

implies $x^2=144$ and $y^2=4$. The solutions (x,y) are $(12,2)$, $(12,-2)$, $(-12,2)$, $(-12,-2)$.

Also solved by Ivan Hadinata, Jember, Indonesia; Israel Castillo Pilco, Huaral, Peru; Adam John Frederickson, Utah Valley University, UT, USA; Le Hoang Bao, Tien Giang, Vietnam; Sundaresh H R, Shivamogga, India; Theo Koupelis, Cape Coral, FL, USA.

S602. For every integer $n > 1$ let

$$A = \frac{1}{n+1}\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3}+\frac{1}{5}\right)\cdots\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)$$

and

$$B = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)\cdots\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

Compare A and B.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Theo Koupelis, Cape Coral, FL, USA

We have $A_2 = \frac{1}{3}\cdot\left(1+\frac{1}{3}\right) = \frac{4}{9}$, and $B_2 = \frac{1}{2}\cdot\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{3}{8}$, and thus $A_2 > B_2$. Also,

$$\frac{A_{n+1}}{A_n} = \frac{n+1}{n+2}\cdot\left(1+\frac{1}{3}+\cdots+\frac{1}{2n+1}\right), \quad \text{and} \quad \frac{B_{n+1}}{B_n} = \frac{1}{2}\cdot\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right),$$

for all $n \geq 2$.

When n is odd, the number of terms inside the parentheses for both A_{n+1}/A_n and B_{n+1}/B_n is even. For A_{n+1}/A_n, the average value of the sum of two terms that are symmetrically positioned in the summation is given by

$$\frac{1}{2}\left(\frac{1}{m}+\frac{1}{2n+2-m}\right) = \frac{n+1}{m(2n+2-m)},$$

where m is an odd number from 1 to n. For B_{n+1}/B_n, the average value of the corresponding two terms is

$$\frac{1}{2}\left(\frac{1}{(m+1)/2}+\frac{1}{n+2-(m+1)/2}\right) = \frac{2(n+2)}{(m+1)(2n+3-m)}.$$

We now have

$$\frac{n+1}{n+2}\cdot\frac{n+1}{m(2n+2-m)} \geq \frac{1}{2}\cdot\frac{2(n+2)}{(m+1)(2n+3-m)} \iff (2n+3)(n-m+1)^2 \geq 0,$$

which is obvious.

When n is even, the number of terms inside the parentheses for both A_{n+1}/A_n and B_{n+1}/B_n is odd. In this case, the middle term in the summation for A_{n+1}/A_n is $1/(n+1)$, and the middle term in the summation for B_{n+1}/B_n is $2/(n+2)$, and thus

$$\frac{n+1}{n+2}\cdot\frac{1}{n+1} = \frac{1}{2}\cdot\frac{2}{n+2}.$$

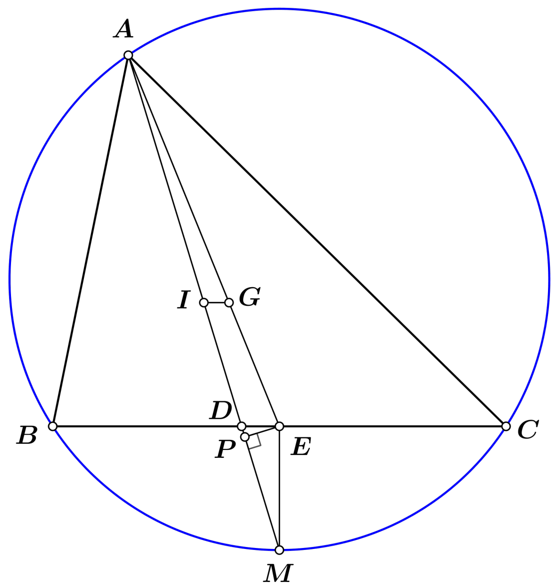
Therefore, for all $n \geq 2$ we have $A_{n+1}/A_n > B_{n+1}/B_n$; taking into account that $A_2 > B_2$, we get $A_n > B_n$ for all integer $n \geq 2$.

Also solved by G. C. Greubel, Newport News, VA, USA; Dao Van Nam, MAY High School, Hoáng Mai, Ha Noi, Vietnam; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S603. Let ABC be a triangle with incenter I and centroid G. Line AG intersects BC in E and line AI intersects BC in D and the circumcircle in M. Point P is the orthogonal projection of E onto AM. Prove that if $(AB+AC)^2 = 16AP \cdot DM$ then GI is parallel to BC.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Kousik Sett, India



M is the midpoint of minor arc BC. Join M, E. Then $\angle MED = 90°$.
We have

$$BD = \frac{ca}{b+c} \quad \text{and} \quad DC = \frac{ba}{b+c}.$$

By intersecting chord theorem, we get

$$AD \cdot DM = BD \cdot DC = \frac{a^2bc}{(b+c)^2}$$

$$\Longrightarrow (AP - DP) \cdot DM = \frac{a^2bc}{(b+c)^2}$$

$$\Longrightarrow AP \cdot DM = DP \cdot DM + \frac{a^2bc}{(b+c)^2}. \tag{1}$$

We have

$$DE = BE - BD = \frac{a}{2} - \frac{ca}{b+c} = \frac{a(b-c)}{2(b+c)}.$$

Since $\angle MED = 90°$ and $EP \perp DM$, we get $ME^2 = MP \cdot MD$.
Again, since $EP \perp DM$, we get

$$MP^2 - DP^2 = ME^2 - DE^2$$

$$\Longrightarrow MD \cdot (MP - DP) = ME^2 - \frac{a^2(b-c)^2}{4(b+c)^2}$$

$$\Longrightarrow MD \cdot DP = \frac{a^2(b-c)^2}{4(b+c)^2}. \tag{2}$$

By (1) and (2), we get

$$AP \cdot DM = \frac{a^2(b-c)^2}{4(b+c)^2} + \frac{a^2bc}{(b+c)^2} = \frac{a^2}{4}.$$

The given condition can be written as

$$(c+b)^2 = AP \cdot DM \implies (b+c)^2 = 4a^2 \implies \frac{b+c}{a} = \frac{2}{1} \implies \frac{AI}{ID} = \frac{AG}{GE}.$$

Hence GI parallel to BC.

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Theo Koupelis, Cape Coral, FL, USA.

S604. Let x, y, z be real numbers such that $-1 \le x, y \le 1$ and $x + y + z + xyz = 0$. Prove that

$$\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} \le \sqrt{9 + xyz}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Without loss of generality, we may assume that

$$z = \min\{x, y, z\}.$$

We have two cases:

Case 1: If $x + y + z \le 0$, then by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} &\le \sqrt{3(1 + x + 1 + y + 1 + z)}\\ &= \sqrt{9 + 3(x + y + z)}\\ &\le \sqrt{9 - x - y - z} = \sqrt{9 + xyz},\end{aligned}$$

with equality when $x = y = z = 0$.

Case 2: If $x + y + z > 0$, then since $xyz = -(x + y + z) < 0$ it follows that $z < 0 < x, y$. Setting $t = \dfrac{x+y}{2} \le 1$, by the Cauchy-Schwarz Inequality, we have

$$\sqrt{x+1} + \sqrt{y+1} \le \sqrt{2(x + y + 2)} = 2\sqrt{1+t},$$

and using the AM-GM Inequality, we have

$$\sqrt{1+z} = \sqrt{1 - \frac{x+y}{1+xy}} \le \sqrt{1 - \frac{2t}{1+t^2}} = \frac{1-t}{\sqrt{1+t^2}}.$$

Also, we have

$$\sqrt{9 + xyz} = \sqrt{9 - x - y - z} = \sqrt{9 - 2t + \frac{x+y}{1+xy}} \ge \sqrt{9 - 2t + \frac{2t}{1+t^2}}$$

We write this inequality as $f(t) \le 0$, where

$$f(t) = 2\sqrt{1+t} + \frac{1-t}{\sqrt{1+t^2}} - \sqrt{9 - 2t + \frac{2t}{1+t^2}},$$

or

$$2\sqrt{(1+t)(1+t^2)} + 1 - t \le \sqrt{9 + 9t^2 - 2t^3}.$$

Squaring both sides of the inequality, expanding and reducing terms, it becomes

$$2(1-t)\sqrt{(1+t)(1+t^2)} \le (1-t)(3t^2 + t + 2),$$

or

$$\frac{t^2(1-t)(9t^2 + 2t + 9)}{3t^2 + t + 2 + 2\sqrt{(1+t)(1+t^2)}} \ge 0,$$

clearly true. The equality holds when $t = 1$ which means $x = y = 1$ and $z = -1$.

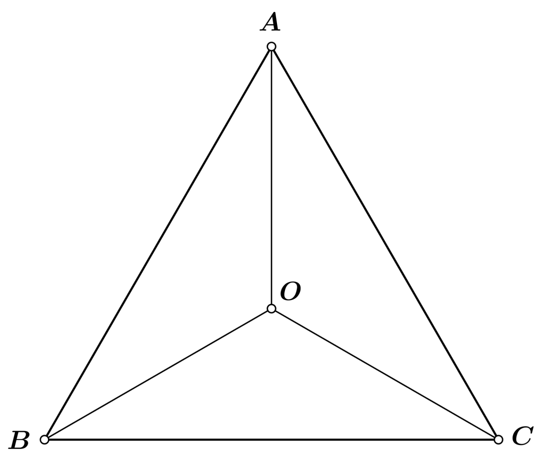
Also solved by Daniel Văcaru, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S605. Let ABC be an acute triangle with circumcenter O and circumradius R. Let R_a; R_b; R_c be the circumradii of triangles OBC, OCA, OAB, respectively. Prove that triangle ABC is equilateral if and only if

$$R^3 + R^2(R_a + R_b + R_c) = 4R_aR_bR_c.$$

Proposed by Marian Ursărescu, Roman, România

Solution by Kousik Sett, India



If triangle ABC is equilateral then

$$AB = BC = CA = \sqrt{3}R = a \text{ and } R_a = R_b = R_c = \frac{aR^2}{4 \times \frac{1}{2}R^2 \sin 120°} = R.$$

Hence

$$R^3 + R^2(R_a + R_b + R_c) = 4R^3 = 4R_aR_bR_c.$$

If the condition

$$R^3 + R^2(R_a + R_b + R_c) = 4R_aR_bR_c \tag{1}$$

is given then we have

$$\frac{R_a}{R} = \frac{aR}{4 \times \frac{1}{2}R^2 \sin 2A} = \frac{1}{2\cos A}.$$

Similarly,

$$\frac{R_b}{R} = \frac{1}{2\cos B} \quad \text{and} \quad \frac{R_c}{R} = \frac{1}{2\cos C}.$$

Therefore, condition (1) can be written as

$$\cos A \cos B + \cos B \cos C + \cos C \cos A = 1 - 2\cos A \cos B \cos C. \tag{2}$$

Using the following identity

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - \cos A \cos B \cos C$$

for any triangle ABC, (2) can be written as

$$\cos^2 A + \cos^2 B + \cos^2 C = \cos A \cos B + \cos B \cos C + \cos C \cos A,$$

which implies

$$\sum(\cos A - \cos B)^2 = 0,$$

which gives

$$\cos A = \cos B = \cos C \implies A = B = C$$

as triangle ABC is acute and hence triangle ABC is equilateral.

Also solved by Daniel Văcaru, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

S606. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + abc \geq a + b + c + 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Marius Stănean, Zalău, România

Using Weighted AM-GM Inequality, we have

$$\frac{2}{3} \cdot \frac{a^2}{b^2} + \frac{1}{6} \cdot \frac{b^2}{c^2} + \frac{1}{6} \cdot \frac{c^2}{a^2} \geq \frac{a^{\frac{4}{3}}}{b^{\frac{4}{3}}} \cdot \frac{b^{\frac{2}{6}}}{c^{\frac{2}{6}}} \cdot \frac{c^{\frac{2}{6}}}{a^{\frac{2}{6}}} = \frac{a}{b}$$

and from AM-GM Inequality

$$\frac{1}{2} \cdot \frac{b^2}{c^2} + \frac{1}{2} \cdot \frac{c^2}{a^2} \geq \frac{b}{a},$$

after summing up yields that

$$\frac{2}{3} \cdot \frac{a^2}{b^2} + \frac{2}{3} \cdot \frac{b^2}{c^2} + \frac{2}{3} \cdot \frac{c^2}{a^2} \geq \frac{a}{b} + \frac{b}{a}.$$

Similarly we have

$$\frac{2}{3} \cdot \frac{a^2}{b^2} + \frac{2}{3} \cdot \frac{b^2}{c^2} + \frac{2}{3} \cdot \frac{c^2}{a^2} \geq \frac{b}{c} + \frac{c}{b}$$

$$\frac{2}{3} \cdot \frac{a^2}{b^2} + \frac{2}{3} \cdot \frac{b^2}{c^2} + \frac{2}{3} \cdot \frac{c^2}{a^2} \geq \frac{c}{a} + \frac{a}{c},$$

which means

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{1}{2} \sum_{cyc} \left(\frac{a}{b} + \frac{b}{a} \right) \geq \frac{1}{3} \sum_{cyc} \left(\frac{a}{b} + \frac{b}{a} \right) + 1.$$

By AM-GM Inequality we have

$$\frac{a}{b} + \frac{a}{c} + abc \geq 3a \Longleftrightarrow \frac{1}{3} \left(\frac{a}{b} + \frac{a}{c} \right) + \frac{1}{3} abc \geq a,$$

and similarly we have

$$\frac{1}{3} \left(\frac{b}{c} + \frac{b}{a} \right) + \frac{1}{3} abc \geq b,$$

$$\frac{1}{3} \left(\frac{c}{a} + \frac{c}{b} \right) + \frac{1}{3} abc \geq c,$$

and all these 3 inequalities summed up lead us to the desired result.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Israel Castillo Pilco, Huaral, Peru; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Undergraduate problems

U601. Let a, b, c, d be real numbers such that all roots of the polynomial $P(x) = x^5 - 10x^4 + ax^3 + bx^2 + cx + d$ are greater than 1. Find the minimum possible value of $a + b + c + d$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

By the AM-GM Inequality we have

$$\prod_{k=1}^{5}(x_k - 1) \le \left(\frac{1}{5}\sum_{k=1}^{5}(x_k - 1)\right)^5$$

implying $-P(1) \le \left(\dfrac{10-5}{5}\right)^5$. Hence, $-1 + 10 - (a + b + c + d) \le 1$, yielding $a + b + c + d \ge 8$. The minimum is 8, achieved for $P(x) = (x - 2)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$.

Second solution by Theo Koupelis, Cape Coral, FL, USA

Let x_i, $i = 1, \dots, 5$ be the real roots of the polynomial, with $x_i > 1$. Using Vieta's formulas we get

$$\begin{aligned}\prod_{i=1}^{5}(x_i - 1) &= -1 + \sum_{i=1}^{5} x_i - \sum_{\substack{i,j=1\\ i\neq j}}^{5} x_i x_j + \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{5} x_i x_j x_k - \sum_{\substack{i,j,k,\ell=1\\ i\neq j\neq k\neq i\\ \ell\neq i,j,k}}^{5} x_i x_j x_k x_\ell + x_1 x_2 x_3 x_4 x_5\\ &= -1 + 10 - a - b - c - d = 9 - (a + b + c + d).\end{aligned}$$

The function $f(x) = \ln(x - 1)$ is concave when $x > 1$ because $f'(x) = \frac{1}{x-1}$ and $f''(x) = -\frac{1}{(x-1)^2} < 0$. Thus, using Jensen's inequality we get

$$\frac{1}{5} \cdot \sum_{i=1}^{5} \ln(x_i - 1) \le \ln\left(\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} - 1\right) = \ln 1 = 0.$$

Therefore, $\prod_{i=1}^{5}(x_i - 1) \le 1$, and thus $a + b + c + d \ge 8$.

Also solved by Daniel Pascuas, Barcelona, Spain; Ivan Hadinata, Jember, Indonesia.

U602. Evaluate

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n \frac{x^n}{\sqrt{ax^{2n}+b}}\mathrm{d}x.$$

Proposed by Ovidiu Furdui and Alina Sîntămărian, Cluj-Napoca, România

Solution by the authors

The limit equals to $\frac{1}{\sqrt{a}}$. We have, since $\sqrt{ax^{2n}+b} > \sqrt{a}x^n$, that

$$\frac{1}{n}\int_0^n \frac{x^n}{\sqrt{ax^{2n}+b}}\mathrm{d}x \le \frac{1}{n}\int_0^n \frac{1}{\sqrt{a}}\mathrm{d}x = \frac{1}{\sqrt{a}}. \tag{1}$$

On the other hand

$$\begin{aligned}\frac{1}{n}\int_0^n \frac{x^n}{\sqrt{ax^{2n}+b}}\mathrm{d}x &\ge \frac{1}{n}\int_1^n \frac{x^n}{\sqrt{ax^{2n}+b}}\mathrm{d}x \\ &= \frac{1}{\sqrt{a}}\cdot\frac{1}{n}\int_1^n \frac{x^n}{\sqrt{x^{2n}+\frac{b}{a}}}\mathrm{d}x \\ &\overset{(*)}{\ge} \frac{1}{\sqrt{a}}\cdot\frac{1}{n}\int_1^n \frac{x}{\sqrt{x^2+\frac{b}{a}}}\mathrm{d}x \\ &= \frac{1}{\sqrt{a}}\cdot\frac{1}{n}\sqrt{x^2+\frac{b}{a}}\,\Bigg|_1^n \\ &= \frac{1}{\sqrt{a}}\cdot\frac{1}{n}\left(\sqrt{n^2+\frac{b}{a}}-\sqrt{1+\frac{b}{a}}\right).\end{aligned} \tag{2}$$

We used at step (*) the inequality $\frac{x^n}{\sqrt{x^{2n}+\frac{b}{a}}} \ge \frac{x}{\sqrt{x^2+\frac{b}{a}}}$, which holds for all $x \ge 1$.

It follows, based on (1) and (2) that

$$\frac{1}{\sqrt{a}}\cdot\frac{1}{n}\left(\sqrt{n^2+\frac{b}{a}}-\sqrt{1+\frac{b}{a}}\right) \le \frac{1}{n}\int_0^n \frac{x^n}{\sqrt{ax^{2n}+b}}\mathrm{d}x \le \frac{1}{\sqrt{a}}, \quad \forall n \ge 1.$$

Passing to the limit, as n tends to ∞, in the above inequalities we get based on the Squeeze Theorem that the desired limit holds and the problem is solved.

Also solved by Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Seán M. Stewart, Thuwal, Saudi Arabia; Matthew Too, Brockport, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Soham Dutta, DPS, Ruby Park, West Bengal, India; Sundaresh H R, Shivamogga, India; Theo Koupelis, Cape Coral, FL, USA; Yunyong Zhang, Chinaunicom; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U603. Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ which satisfy the conditions:

(a) $f(x+1)f(y)-f(xy)=(2x+1)f(y)$, for all $x,y\in\mathbb{R}$,

(b) $f(x)>2x$ for all $x>2$.

Proposed by Mircea Becheanu, Canada

Solution by the author

It is easy to see that $f=0$ and $f(x)=x^2$ satisfy condition (a). The constant function $f=0$ does not satisfy condition (b) and we will prove that only $f(x)=x^2$ is a solution for our problem.

For the pair $(x,0)$ we have $f(x+1)f(0)-f(0)=(2x+1)f(0)$. If $f(0)\neq 0$ we get $f(x+1)=2(x+1)$, for all x. This means $f(x)=2x$ for all x. But $f(x)=2x$ does not verify the equation. Therefore we have $f(0)=0$.

For the pair $(0,1)$ we have $f(1)^2-f(0)=f(1)\Leftrightarrow f(1)^2=f(1)$. Hence, $f(1)=1$ or $f(1)=0$.

Assume that $f(1)=0$. For that pair $(x,1)$ we obtain $f(x+1)f(1)-f(x)=(2x+1)f(1)$, giving the solution $f(x)=0$, for all x.

So, we are in the case $f(0)=0$ and $f(1)=1$. Plugging the pair $(-1,y)$ in the condition (a) one obtains $f(0)f(y)-f(-y)=-f(y)$, which shows that $f(-y)=f(y)$. Hence, the function f is even and we will study it for $x>0$. For the pair $(1,1)$ we obtain $f(2)f(1)-f(1)=3f(1)$, then $f(2)=4$. For the pair $(2,1)$ we obtain $f(3)f(1)-f(2)=5f(1)$, showing that $f(3)=9$.

We prove by induction that $f(n)=n^2$ for all positive integers n. Assume that $f(n)=n^2$ and plug the pair $(n,1)$. We obtain $f(n+1)f(1)=f(n+1)=f(n)+(2n+1)f(1)=n^2+(2n+1)=(n+1)^2$.

For the pair $(n,\frac{1}{n})$ we obtain

$$f(n+1)f\left(\frac{1}{n}\right)=f(1)+(2n+1)f\left(\frac{1}{n}\right)\Rightarrow\left((n+1)^2-(2n+1)\right)f\left(\frac{1}{n}\right)=1,$$

giving that

$$f\left(\frac{1}{n}\right)=\frac{1}{n^2} \tag{1}$$

Plugging in condition (a) the pair $(\frac{1}{n},1)$ we obtain

$$f\left(\frac{1}{n}+1\right)f(1)=f\left(\frac{n+1}{n}\right)=f\left(\frac{1}{n}\right)+\frac{2}{n}+1,$$

and using (1) one obtains

$$f\left(\frac{n+1}{n}\right)=\left(\frac{n+1}{n}\right)^2. \tag{2}$$

Plugging the pair $(\frac{1}{n},m)$ and using (1) and (2) one obtains

$$f\left(\frac{n+1}{n}\right)f(m)=f\left(\frac{m}{n}\right)+\left(\frac{2}{n}+1\right)f(m)\Rightarrow$$

$$f\left(\frac{m}{n}\right)=\left(\frac{n+1}{n}\right)^2m^2-\left(\frac{n+2}{n}\right)m^2=\frac{m^2}{n^2}.$$

This shows that $f(a)=a^2$ for every rational number a.

From condition (a) we obtain

$$f(xy)=f(y)[f(x+1)-(2x+1)] \tag{3}$$

Taking $x>1$ we have $xy>y$ and every positive real number $z>y$ cand be written under the form $z=xy$ for some $x>1$. If $x>1$ we have $x+1>2$ and by condition (b), $f(x+1)>2(x+1)$ giving that $f(x+1)-(2x+1)>1$. Going back in the equation (3) we obtain

$$f(xy)>f(y) \tag{4}$$

showing that the function f is monotonic increasing on $[0,+\infty)$.

We will show now that that $f(x)=x^2$ for all $x>0$. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of positive rational numbers converging to x and such that

$$....a_n < a_{n+1} < ... < x < ... < b_{n+1} < b_n < ....$$

Then $\lim_{n\to\infty}(b_n - a_n) = 0$. Since f is monotonic increasing we have the inequalities

$$....f(a_n) < f(a_{n+1}) < ... < f(x) < ... < f(b_{n+1}) < f(b_n) < ....$$

This gives

$$f(b_n) - f(a_n) = b_n^2 - a_n^2 = (b_n - a_n)(b_n + a_n).$$

Let $M>0$ be a real number such that $a_n < M$ and $b_n < M$. For every real number $\varepsilon > 0$ there exists a positive integer N such that for $n > N$ we have $b_n - a_n < \varepsilon/2M$. Then $f(b_n) - f(a_n) < (\varepsilon/2M)(b_n + a_n) < \varepsilon$ for all $n > N$. This shows that $\lim_{n\to\infty}(f(b_n) - f(a_n)) = 0$,which means that $f(b_n)_{n\geq 1}$ and $f(a_n)_{n\geq 1}$ have the same limit. We obtain that $\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} b_n^2 = x^2 = f(x)$. Since f is even we have $f(x) = x^2$ for all $x \in \mathbb{R}$.

Remark: One can prove that the function is monotonic without using condition (b).

Also solved by Theo Koupelis, Cape Coral, FL, USA.

U604. If

$$f(m) := \int_0^1 \frac{(x^m - 1)\ln(1+x)}{x\ln(x)}\,\mathrm{d}x$$

then evaluate,

$$\lim_{m\to 0}\left(\frac{2\pi^2}{9m} - \frac{8f(m)}{3m^2}\right)$$

Proposed by Ty Halpen, Florida, USA

Solution by the author

$$\begin{aligned}
f(m) &= \int_0^1 \frac{(x^m - 1)\ln(1-x)}{x\ln(x)}\,\mathrm{d}x \\
&= \int_0^1 \int_0^m x^{y-1}\ln(1-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \int_0^m x^{y-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}\,\mathrm{d}y\,\mathrm{d}x \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_0^m \int_0^1 x^{y+k-1}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_0^m \frac{1}{y+k}\,\mathrm{d}x \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ln\left(1+\frac{m}{k}\right)}{k} \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j+1}(-1)^{k+1}m^j}{jk^{j+1}} \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}m^j}{j} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{j+1}}
\end{aligned}$$

Recognize the inner sum as the Dirichlet eta function, whose representation in terms of the zeta function form will be briefly derived:

$$\zeta(j+1) = \frac{1}{1^{j+1}} + \frac{1}{2^{j+1}} + \frac{1}{3^{j+1}} + \frac{1}{4^{j+1}} + \frac{1}{5^{j+1}} + \frac{1}{6^{j+1}} + \dots$$

Notice that $\frac{1}{2^{j+1}}$ can factor out of the even terms:

$$\begin{aligned}
\text{even terms} &= \frac{1}{2^{j+1}}\left(\frac{1}{1^{j+1}} + \frac{1}{2^{j+1}} + \frac{1}{3^{j+1}} + \dots\right) \\
&= \frac{\zeta(j+1)}{2^{j+1}}
\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{S} &= \frac{1}{1^{j+1}} - \frac{1}{2^{j+1}} + \frac{1}{3^{j+1}} - \frac{1}{4^{j+1}} + \frac{1}{5^{j+1}} - \frac{1}{6^{j+1}} + \dots \\
&= \left(\frac{1}{1^{j+1}} + \frac{1}{2^{j+1}} + \frac{1}{3^{j+1}} + \frac{1}{4^{j+1}} + \frac{1}{5^{j+1}} + \dots\right) - 2\left(\frac{1}{2^{j+1}} + \frac{1}{4^{j+1}} + \frac{1}{6^{j+1}} + \dots\right) \\
&= \zeta(j+1) - 2\left(\frac{\zeta(j+1)}{2^{j+1}}\right) \\
&= \zeta(j+1)\left(1 - 2^{-j}\right)
\end{aligned}$$

Thus,

$$f(m) = \sum_{j=1}^{\infty} \frac{\zeta(j+1)\left(1-2^{-j}\right)(-1)^{j+1}m^j}{j}$$

Now, apply the limit and notice that the terms after $j = 2$ will vanish in the limit:

$$\begin{aligned}
\lim_{m\to 0}\left(\frac{2\pi^2}{9m} - \frac{8f(m)}{3m^2}\right) &= \lim_{m\to 0}\left(\frac{2\pi^2}{9m} - \frac{8}{3m^2}\left(\frac{m\zeta(2)}{2} - \frac{3m^2\zeta(3)}{8} + o(m^2)\right)\right)\\
&= \lim_{m\to 0}\left(\frac{2\pi^2}{9m} - \frac{4\zeta(2)}{3m} + \zeta(3) - o(1)\right)\\
&= \lim_{m\to 0}\left(\frac{2\pi^2}{9m} - \frac{2\pi^2}{9m} + \zeta(3) - o(1)\right)\\
&= \zeta(3)
\end{aligned}$$

Also solved by Daniel Pascuas, Barcelona, Spain; Seán M. Stewart, Thuwal, Saudi Arabia; Matthew Too, Brockport, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Brian Bradie, Christopher Newport University, Newport News, VA, USA; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Cape Coral, FL, USA; Yunyong Zhang, Chinaunicom.

U605. Let a, b, c be nonzero complex numbers having the same modulus for which

$$a^3 + b^3 + c^3 = rabc,$$

where r is a real number.

(i) Prove that $-1 \le r \le 3$.

(ii) Prove that if $r < 3$ then one and only one of the equations $ax^2 + bx + c = 0$, $bx^2 + cx + a = 0$, $cx^2 + ax + b = 0$ has a root of modulus 1.

Proposed by Florin Stănescu, Găeşti, România

Solution by Theo Koupelis, Cape Coral, FL, USA

(i) Let $a = \rho e^{i\theta}$, $b = \rho e^{i\phi}$, and $c = \rho e^{i\omega}$. Then the condition $a^3+b^3+c^3 = rabc$ leads to $e^{i3\theta}+e^{i3\phi}+e^{i3\omega} = re^{i(\theta+\phi+\omega)}$ or $r = e^{i(2\theta-\phi-\omega)} + e^{i(2\phi-\theta-\omega)} + e^{i(2\omega-\theta-\phi)}$. Setting $\alpha = 2\theta - \phi - \omega$, $\beta = 2\phi - \theta - \omega$, and $\gamma = 2\omega - \theta - \phi$ we get

$$\begin{aligned} \cos\alpha + \cos\beta + \cos\gamma &= r, \\ \sin\alpha + \sin\beta + \sin\gamma &= 0, \\ \alpha + \beta + \gamma &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} \sin\alpha + \sin\beta = -\sin\gamma = \sin(\alpha+\beta) &\Longrightarrow 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2} \\ &\Longrightarrow \sin\frac{\alpha+\beta}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2} = 0. \end{aligned}$$

If $\alpha + \beta = 2k\pi$, where $k \in \mathbb{Z}$, we get $r = 2\cos(k\pi)\cos(k\pi - \beta) + 1 = 1 + 2\cos\beta$. If $\alpha = 2k\pi$ we get $r = 1 + \cos\beta + \cos(2k\pi + \beta) = 1 + 2\cos\beta$. Similarly, if $\beta = 2k\pi$, we get $r = 1 + 2\cos\alpha$. In all three cases we have $-1 \le r \le 3$. Also, $r = 3$ only when α, β, γ are even multiples of π.

(ii) Without loss of generality, let the equation $ax^2 + bx + c = 0$ have a root of modulo 1, that is, $x = e^{i\lambda}$. Then we have

$$e^{i(\theta+2\lambda)} + e^{i(\phi+\lambda)} + e^{i\omega} = 0 \Longrightarrow e^{i(\theta+2\lambda-\omega)} + e^{i(\phi+\lambda-\omega)} = -1.$$

Thus, we have

$$\begin{aligned} \cos(\theta+2\lambda-\omega) + \cos(\phi+\lambda-\omega) = -1 &\Longrightarrow \cos\frac{3\lambda-\gamma}{2}\cos\frac{\theta-\phi+\lambda}{2} = -\frac{1}{2}, \\ \sin(\theta+2\lambda-\omega) + \sin(\phi+\lambda-\omega) = 0 &\Longrightarrow \sin\frac{3\lambda-\gamma}{2}\cos\frac{\theta-\phi+\lambda}{2} = 0. \end{aligned}$$

Therefore, $3\lambda - \gamma = 4k\pi$ and $\theta - \phi + \lambda = 4n\pi \pm \frac{4\pi}{3}$, or $3\lambda - \gamma = (4k+2)\pi$ and $\theta - \phi + \lambda = 4n\pi \pm \frac{2\pi}{3}$; in both cases, after eliminating λ, we find that β is an even multiple of π. If another quadratic (say $bx^2 + cx + a = 0$) also had a root of modulus 1, in a similar way we find that γ is an even multipe of π, and therefore so would be α and r would be equal to 3, which contradicts the condition $r < 3$. Thus, when $r < 3$, one and only one of the equations $ax^2 + bx + c = 0$, $bx^2 + cx + a = 0$, $cx^2 + ax + b = 0$ has a root of modulus 1.

Also solved by Theo Koupelis, Cape Coral, FL, USA.

U606. Determine all bijective functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$f^{mn+1}(m+n)=f(m)f(n)$$

for all integers m,n such that $|m-n|$ is odd. Here $f^{(k)}$ denotes k-times composition $f\circ f\circ\cdots\circ f$.

Proposed by Valentio Iverson, Waterloo, Canada and Stanve Avrilium Widjaja, Singapore

Solution by the authors

The answer is only $f(x)=x+1$ for all $x\in\mathbb{Z}$, which satisfies the functional equation since

$$f^{mn+1}(m+n)=m+n+(mn+1)=(m+1)(n+1)=f(m)f(n)$$

Now, we'll prove that there are no other functions that satisfy the given functional equation.
Let $P(x,y)$ be the assertion of x and y to the given functional equation. For simplicity, in the following solution, x will always be the odd input and y will always be the even input.

Lemma: $f(0)=1$ and $f^{-1}(0)\in\{-1,2\}$.

Proof: $P(n,0)$ implies

$$f(n)f(0)=f(n)$$

Therefore, $f(0)=1$ or $f(n)=0$ for all odd n, from which the latter case contradicts the bijectivity of f.
$P(-1,2)$ implies

$$f(2)f(-1)=f^{-1}(1)=0$$

Therefore, either $f(2)=0$ or $f(-1)=0$.

First Case: $f(-1)=0$. $f(-1,k)$ with even input k gives us

$$f^{-k+1}(k-1)=0$$

for all even k. Thus, we conclude that $f^{a}(-a)=0$ for all odd a. There are two ways to continue: Since $f^{-a}(a)=0$ for all odd a, we can form the infinite chain

$$\cdots\to-5\to c_{-4}\to-3\to c_{-2}\to-1\to0\to1\to c_2\to3\to c_4\to5\to\dots$$

where $a\to b$ implies $f(a)=b$. Formally, define the directed graph $G(V,E)$ where each edge corresponds to the assertion of $f(a)=b$. This graph consists of connected components, and the component which contains all odd numbers must be a path, since it obviously cannot form a cycle due to the infinite number of the components, or a non-path tree due to the bijectivity of f. To fill the empty spaces, we let c_{2k} to be the space which is reserved for $2k$. Note that $f^k(m)=c_{m+k}$ for m odd, since $f^k(m)$ for the LHS of the original equation will result in a non-odd number.
$P(m,-2)$ implies

$$c_{-m-1}=c_{m+1}\cdot f(-2)$$

which gets us $f(-2)=-1$, and so since $c_N=-c_{-N}$, we also get $c_2=-c_{-2}=2$. Then, we will induct left/right wise: Suppose we have figured out the values of $c_{-2k},c_{-2k+2},\dots,c_{2k}$. We will divide into two cases:

- If $k+1$ is odd, We take $P(f^{-1}(2)=1,f^{-1}(k+1)=k)$, and $P(1,-k)$.
- If $k+2$ is odd, we take $P(1,k+1)$ and $P(1,-k-1)$ to get the values of c_{2k+4} and c_{-2k-4}. $P(1,-2k-2)$ implies the identification of c_{-4k-6} and c_{4k+6}. Finally, using c_{4k+6}, then the value of $f^{-1}(2k+3)=c_{2k+2}$ is determined, and so is c_{-2k-2}.

Now,

$$f^{a+2}(-a-2) = f^a(-a) \Rightarrow f^2(-a-2) = -a \Rightarrow f^2(x) = x+2 \quad \forall x \text{ odd}$$

Consider $P(1,n)$ where n is even. Since $f^2(x) = x+2$ for all odd x, we have

$$f(2n+1) = f^{n+1}(n+1) = f(n)f(1)$$

This gives us $f(4a+1) = f(1)f(2a)$ for all integers a. Since m and n are different in parity, then mn is even and $m+n$ is odd. Therefore, we can rewrite $P(x,y)$ into

$$Q(m,n) : f(mn+m+n) = f^{mn+1}(m+n) = f(m)f(n) \ \forall m,n \text{ distinct in parity}$$

Consider $Q(-2,n)$, this gives us $f(-n-2) = f(n)f(-2)$ for all odd n. Thus, $Q(m,n)$ could be rewritten as:

$$f(2a)f(2b+1) = f(4ab+4a+2b+1)$$

Let $b = 2c$, using $f(4c+1) = f(1)f(2c)$, we have

$$f(4ac+2a+2c) = f(2a)f(2c)$$

Let $a = -1$,

$$f(-2c-2) = f(-2)f(2c)$$

Let $c = -1$, we have $f(-2)^2 = 1$. Since f is injective, $f(-2) = -1$. Thus, we have $f(-2c-2) = -f(2c)$. Notice that $f(4a+1) = f(1)f(2a)$. Let $a = -1$, we have $f(-3) + f(1) = 0$. Notice that $f(f(-3)) = -1 = f(-2) \rightarrow f(-3) = -2$. Therefore, $f(1) = 2$. Then, we also have $f(2) = f^2(1) = 3$. We will rephrase what we have again:

$$f(-3) = -2, f(-2) = -1, f(-1) = 0, f(0) = 1, f(1) = 2, f(2) = 3$$

$$f(4a+1) = 2f(2a)$$

$$f(-2a-2) + f(2a) = 0$$

Let S be the set of integers such that the value of f is uniquely determined here, which is $f(x) = x+1$. We wanted to prove that $S = \mathbb{Z}$.

Claim: If $2a+1 \in S$ if and only if $2a+2 \in S$.
Proof: Notice that if $f(2a+1) = 2a+2$, $f(2a+2) = f^2(2a+1) = 2a+3$. Suppose otherwise, notice that $f(2a+2) = 2a+3 = f(f(2a+1))$. By injectivity we are done.

Claim: If $2a \in S$, then $4a+2 \in S$.
Proof: We know that $f(4a+1) = 2f(2a)$. By the above claim, we are done.

Claim: If $2a \in S$, then $-2-2a \in S$.
Proof: $f(2a) + f(-2-2a) = 0$.

Claim: All even numbers are members in S.
Proof: We'll prove this by induction. We know that $\{-2,0,2\} \in S$. Thus, $\{-4,-2,0,2\} \in S$ since $-4+2 = -2$. Base induction. $-6 = 2(-4)+2$, and $4 = -2-(-6)$. Now, to prove this, suppose that $\{-2k-2, \dots, 0, \dots, 2k\} \in S$. Then we'll prove that both $2k+2$ and $-2k-4$ is in S. One of them must be divisible by 4 since they add up to 2 modulo 4. Thus, the one which is 2 modulo 4, if it is $2k+2 = 4\ell+2$. Notice that by induction $2\ell \in S$, we are done. If it is $-2k-4 = -4\ell-6$, then notice that $-(2\ell+4) \in S$ by induction, we are done. If we get one, we get the other as they sum to -2. Thus, by the claim before, we get that $S = \mathbb{Z}$, so $f(x) = x+1$.

Second Case: $f(2) = 0$. $P(2, k-2)$ for odd input k gives us

$$f^{2k-3}(k) = 0 \implies f^{3-2k}(0) = k$$

Now, as $m+n$ and m are odd,

$$f^{mn+1}(m+n) = f(m)f(n)$$
$$f^{mn+4-2(m+n)}(0) = f^{4-2m}(0) \cdot f(n)$$

Lemma: $f^{2i}(0) = -f^{-2i}(0)$ for all $i \in \mathbb{Z}$.
Proof: $P(1, 2k)$ and $P(3, 2k)$ gives us

$$f^2(0) \cdot f(2k) = f^{2-2k}(0)$$
$$f^{-2}(0) \cdot f(2k) = f^{2k-2}(0)$$

This implies

$$\frac{f^{2-2k}(0)}{f^2(0)} = \frac{f^{2k-2}(0)}{f^{-2}(0)}$$

Plugging $k = 2$ gives us $\frac{f^{-2}(0)}{f^2(0)} \in \{1, -1\}$, and since f is bijective, we conclude that $f^{-2}(0) = -f^2(0)$. Therefore, if $k \neq 1$, we have

$$\frac{f^{2-2k}(0)}{f^{2k-2}(0)} = \frac{f^2(0)}{f^{-2}(0)} = -1$$

which is what we wanted.

This also gives us

$$f(2k) = \frac{f^{2k-2}(0)}{f^{-2}(0)}$$

for any $k \in \mathbb{Z}$. Therefore, we have

$$f(4) = \frac{f^2(0)}{f^{-2}(0)} = -1$$

Since $f^5(0) = -1$, by bijectivity it follows that $f^4(0) = 4$. This means that by taking $k = 3$, we have

$$f(6) \cdot f^{-2}(0) = f^4(0) = 4$$

We know that $f(0) = 1$ and $f^5(0) = f(4) = -1$. Therefore, $f(6)$ and $f^{-2}(0)$ must both be 2 or both be -2. However, this implies $f^{-3}(0) = 6$ by bijectivity, which contradicts the fact that $f^{-3}(0) = 3$, a contradiction.

Olympiad problems

O601. Find all integers n for which $(n+1)^2+(2n+1)^2$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Let

$$(n+1)^2+(2n+1)^2=t^2$$

for some positive integer t. The above is a quadratic equation in n: $5n^2+6n+2-t^2=0$. Call its discriminant Δ. Then $\frac{\Delta}{4}=9-5(2-t^2)=5t^2-1$. In order for the quadratic equation to have an integer n as solution, $5t^2-1$ must be a perfect square. So $5t^2-1=m^2$ for some positive integer m. We obtained the negative Pell's equation

$$m^2-5t^2=-1$$

with minimal solution $(m_1,t_1)=(2,1)$. Its resolvent equation is $u^2-5v^2=1$, whose minimal solution is $(u_1,v_1)=(9,4)$. All positive integer solutions to the resolvent equation are given by $u_k+v_k\sqrt{5}=(9+4\sqrt{5})^k$, $k=1,2,3,\ldots$ and the solutions to the equation $m^2-5t^2=-1$ are (m_k,t_k), where

$$m_k=2u_k+5v_k$$

and

$$t_k=u_k+5v_k,\qquad k=1,2,3,\ldots$$

Hence

$$n_k=\frac{-3\pm m_k}{5},\qquad k=1,2,3,\ldots$$

and the desired solutions are all integers among $\frac{-3\pm m_k}{5}$, $k=1,2,3,\ldots$.

For example,

$$n_1=\frac{-3+38}{5}=7$$

and

$$(7+1)^2+(2\cdot7+1)^2=17^2,$$

$$n_2=\frac{-3-682}{5}=-137$$

and

$$(-137+1)^2+(2(-137)+1)^2=305^2.$$

Also solved by Ivan Hadinata, Jember, Indonesia; Adam John Frederickson, Utah Valley University, UT, USA; Puneeth A, Hmr International School, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; G. C. Greubel, Newport News, VA, USA; Theo Koupelis, Cape Coral, FL, USA.

O602. Let a, b, c, d be positive real numbers such that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=1$$

Prove that

$$ab+ac+ad+bc+bd+cd+6 \geq 5(a+b+c+d)$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

With the substitutions $\frac{1}{1+a} \to a$, $\frac{1}{1+b} \to b$, $\frac{1}{1+c} \to c$, $\frac{1}{1+d} \to d$, it follows that $a+b+c+d=1$ and the inequality is equivalent with

$$\sum_{cyc} \frac{(1-a)(1-b)}{ab} - 5\sum_{cyc} \frac{1-a}{a} \geq -6,$$

$$ab+ac+ad+bc+bd+cd-8(abc+bcd+cda+dab)+32abcd \geq 0.$$

We homogenize the inequality and we get the following inequality

$$\left(\sum_{cyc} a\right)^2 \sum_{cyc} ab - 8\sum_{cyc} a \sum_{cyc} abc + 32abcd \geq 0.$$

Without loss of generality assume that $a+b+c+d=4$. The inequality becomes

$$\sum_{cyc} ab - 2\sum_{cyc} abc + 2abcd \geq 0,$$

$$6-\sum_{cyc} ab + 2\left(1-\sum_{cyc} a + \sum_{cyc} ab - \sum_{cyc} abc + abcd\right) \geq 0,$$

$$-\sum_{cyc}(a-1)(b-1)+2(a-1)(b-1)(c-1)(d-1) \geq 0.$$

Let $x=a-1$, $y=b-1$, $z=c-1$, $t=d-1$, $x+y+z+t=0$, $x,y,z,t \in [-1,3]$. We need to prove that

$$x^2+y^2+z^2+t^2+4xyzt \geq 0.$$

Assuming that $x \leq y \leq z \leq t$ it is clearly that $x \leq 0 \leq t$. We have the following cases

1. $0 \leq y$, then $y+z+t=-x \leq 1 \Longrightarrow yzt<1$, we need to prove that

$$(y+z+t)^2+y^2+z^2+t^2 \geq 4yzt(y+z+t).$$

By AM-GM Inequalituy, we have

$$(y+z+t)^2+y^2+z^2+t^2 \geq 4\sqrt{yzt(y+z+t)} > 4yzt(y+z+t).$$

2. $y \leq 0 \leq z$, clearly true.

3. $z \leq 0$, then $-x-y-z=t \leq 3$ and $-xyz \leq 1$, we need to prove that

$$(x+y+z)^2+x^2+y^2+z^2 \geq 4xyz(x+y+z).$$

But

$$\begin{aligned}(x+y+z)^2+x^2+y^2+z^2 \geq &(x+y+z)^2+\frac{(x+y+z)^2}{3}=\frac{4(x+y+z)^2}{3}\\ \geq &4\sqrt[3]{-xyz}(-x-y-z) \geq 4xyz(x+y+z).\end{aligned}$$

The equality holds when $x=y=z=t=0$ or $x=y=z=-1$, $t=3$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Israel Castillo Pilco, Huaral, Peru; Adam John Frederickson, Utah Valley University, UT, USA; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O603. Find all triples $(x;y;z)$ of nonnegative integers such that

$$x^2 - y^2 = 2^z + 2022.$$

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution by the author

It is clear that $x > y$. If $2^z + 2022 = m.n$ for some integers $m, n > 0$,

$$x^2 - y^2 = (x-y)(x+y) = m.n$$
$$x + y = n; \quad x - y = m; \quad n > m$$
$$x = \frac{n+m}{2}; \quad y = \frac{n-m}{2}$$

Hence m, n have the same parity.

Case 1: $z = 0$

$$(x-y)(x+y) = 2023 = 7.17^2$$

m	n $(n > m)$	x	y
1	2023	1012	1011
7	17^2	148	141
17	7.17	68	51

Case 2: $z = 1$

$$(x-y)(x+y) = 2024 = 2^3.11.23$$

hence m, n are even.

m	n $(n > m)$	x	y
2	1012	507	505
4	506	255	251
2.11	4.23	57	35
4.11	2.23	45	1

Case 3: $z \geq 2$

$$2^z + 2022 \equiv 0 + 2 \equiv 2 \pmod 4$$
$$x^2 \equiv 0, 1 \pmod 4$$
$$x^2 - y^2 \equiv 0, 1, -1 \not\equiv 2 \pmod 4$$

In this case $x^2 - y^2 = 2^z + 2022$ no solutions.

Finally the solutions are the triples $(x;y;z)$: $(1012;1011;0)$, $(148;141;0)$, $(68;51;0)$, $(507;505;1)$, $(255;251;1)$, $(57;35;1)$, $(45;1;1)$.

Also solved by Israel Castillo Pilco, Huaral, Peru; Theo Koupelis, Cape Coral, FL, USA; Emon Suin, Ramakrishna Mission Vidyalaya, Narendrapur, West Bengal, India; Puneeth A, Hmr International School, India; Sundaresh H R, Shivamogga, India.

O604. Let $n > 2$ be an integer. For every nonempty subset $A = \{a_1, \ldots, a_k\}$ of the set $\{1, 2, \ldots, n\}$ we denote by m_A the arithmetic mean of the elements of A. Find the least possible value of the difference $|m_A - m_B|$ when A and B run over all subsets of $\{1, 2, \ldots, n\}$.

Proposed by Cristi Săvescu, România

Solution by Emon Suin, Ramakrishna Mission Vidyalaya, Narendrapur, West Bengal, India

We consider the subset $A = \{1, 3\}$ and $B = \{1, 2, 3\}$. Then, we have, $m_A = \frac{1+3}{2} = 2 = \frac{1+2+3}{3} = m_B$, thus implying that $m_A - m_B = 0$.

Hence, the minimum possible value of $|m_A - m_B|$ is 0, as it cannot be negative, due to its absolute value.

Also solved by Theo Koupelis, Cape Coral, FL, USA.

O605. Let a, b, c be positive real numbers and let x, y, z be real numbers. Prove that

$$\frac{abxy}{a+b+2c} + \frac{bcyz}{b+c+2a} + \frac{cazx}{c+a+2b} \le \frac{1}{4}(ax^2 + by^2 + cz^2).$$

Proposed by An Zhenping, Xianyang University of Science, China

Solution by Theo Koupelis, Cape Coral, FL, USA

f $z = 0$ we have

$$\frac{abxy}{a+b+2c} \le \frac{abxy}{2\sqrt{ab}+2c} < \frac{abxy}{2\sqrt{ab}} = \frac{1}{2}\sqrt{ab}xy \le \frac{1}{4}(ax^2 + by^2).$$

Similarly for $y = 0$ or $x = 0$. Let $xyz \neq 0$. We rewrite the given inequality as a quadratic in x:

$$ax^2 - 4ax\left(\frac{by}{a+b+2c} + \frac{cz}{c+a+2b}\right) + by^2 + cz^2 - \frac{4bcyz}{b+c+2a} \ge 0.$$

Because $a > 0$, it is sufficient to show that the discriminant of the quadratic is nonpositive, that is

$$4a\left[\frac{by}{a+b+2c} + \frac{cz}{a+c+2b}\right]^2 - \left[by^2 + cz^2 - \frac{4bcyz}{b+c+2a}\right] \le 0,$$

or after expanding,

$$\left[1 - \frac{4ab}{(a+b+2c)^2}\right]by^2 - 4bcyz\left[\frac{2a}{(a+b+2c)(a+c+2b)} + \frac{1}{b+c+2a}\right] + \left[1 - \frac{4ac}{(a+c+2b)^2}\right]cz^2 \ge 0.$$

Clearly $(a+b+2c)^2 > 4ab$ and thus it is sufficient to show that the discriminant of the above quadratic in y is nonpositive, that is

$$4bc\left[2a(b+c+2a) + (a+b+2c)(a+c+2b)\right]^2 \le \left[(a+b+2c)^2 - 4ab\right]\left[(a+c+2b)^2 - 4ac\right](b+c+2a)^2.$$

After expanding we rewrite the above inequality as

$$\begin{aligned} &3\left[a^3 + b^3 + c^3 + 3abc - a^2(b+c) - b^2(c+a) - c^2(a+b)\right]^2 \\ &\quad + \left[a^6 + b^6 + c^6 + 3a^2b^2c^2 - a^4(b^2+c^2) - b^4(a^2+c^2) - c^4(a^2+b^2)\right] \\ &\quad + 6abc\left[a^3 + b^3 + c^3 + 2a^2(b+c) + 2b^2(c+a) + 2c^2(a+b) - 15abc\right] \\ &\quad + \sum_c 18ab(a^2-b^2)^2 + \sum_c a^2b^2(a-b)^2, \end{aligned}$$

which is obvious by Schur and AM-GM. Equality occurs when $a = b = c$ and $x = y = z$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O606. Prove that the product of six consecutive positive integers is not the fifth power of an integer.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by the authors

Assume that, for some positive integer n, we have that $n(n+1)(n+2)(n+3)(n+4)(n+5)$ is the fifth power of a (positive) integer. Note first that the longest sequence of consecutive integers none of which is relatively prime to 30 has 5 members (the numbers being congruent modulo 30 either to 2, 3, 4, 5, 6, or to 24, 25, 26, 27, 28). Consequently, among the six consecutive integers n, $n+1$, $n+2$, $n+3$, $n+4$, and $n+5$ there is (at least) one which is relatively prime to 30, therefore one which is not divisible by any of 2, 3, or 5. On the other hand, any prime $p > 5$ can appear in the factorization of at most one of the six consecutive numbers. So, if it appears in one of the six factorizations, p must have an exponent divisible by 5. All these facts lead to the conclusion that the factor among n, $n+1$, $n+2$, $n+3$, $n+4$, $n+5$ that is not divisible by 2, 3, and 5 appears with an exponent that is a multiple of 5, therefore that factor is a perfect fifth power. Together with the (assumed) hypothesis this implies that the product of the five other among the six consecutive factors is also a perfect fifth power. So, we finish our proof by showing that none of the products $n(n+1)(n+2)(n+3)(n+4)$, $n(n+1)(n+2)(n+3)(n+5)$, $n(n+1)(n+2)(n+4)(n+5)$, $n(n+1)(n+3)(n+4)(n+5)$, $n(n+2)(n+3)(n+4)(n+5)$, and $(n+1)(n+2)(n+3)(n+4)(n+5)$ is the fifth power of a positive integer.

Indeed, we have

$$(n+1)^5 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1,$$

$$(n+2)^5 = n^5 + 10n^4 + 40n^3 + 80n^2 + 80n + 32,$$

and

$$n(n+1)(n+2)(n+3)(n+4) = n^5 + 10n^4 + 35n^3 + 50n^2 + 24n,$$

implying

$$(n+1)^5 < n(n+1)(n+2)(n+3)(n+4) < (n+2)^5$$

for all positive integers n. Of course, this means that $n(n+1)(n+2)(n+3)(n+4)$ is not the fifth power of an integer, and, clearly, this also settles the case of $(n+1)(n+2)(n+3)(n+4)(n+5)$.

Further we have

$$(n+3)^5 = n^5 + 15n^4 + 90n^3 + 270n^2 + 405n + 243,$$

$$n(n+1)(n+2)(n+3)(n+5) = n^5 + 11n^4 + 41n^3 + 61n^2 + 30n,$$

$$n(n+1)(n+2)(n+4)(n+5) = n^5 + 12n^4 + 49n^3 + 78n^2 + 40n,$$

$$n(n+1)(n+3)(n+4)(n+5) = n^5 + 13n^4 + 59n^3 + 107n^2 + 60n,$$

and

$$n(n+2)(n+3)(n+4)(n+5) = n^5 + 14n^4 + 71n^3 + 154n^2 + 120n.$$

Thus, it immediately follows that

$$(n+2)^5 < n(n+1)(n+3)(n+4)(n+5) < (n+3)^5$$

and

$$(n+2)^5 < n(n+2)(n+3)(n+4)(n+5) < (n+3)^5$$

showing that $n(n+1)(n+3)(n+4)(n+5)$ and $n(n+2)(n+3)(n+4)(n+5)$ are not perfect fifth powers for any positive integer n.

We also have, by the above, that

$$n(n+1)(n+2)(n+3)(n+5) < (n+3)^5$$

and

$$n(n+1)(n+2)(n+4)(n+5) < (n+3)^5$$

for any positive integer n. Moreover,

$$\begin{aligned} n(n+1)(n+2)(n+4)(n+5)-(n+2)^5 &= \\ &= 2n^4+9n^3-2n^2-40n-32 \\ &\geq 15n^3-2n^2-40n-32 \\ &\geq 43n^2-40n-32 \\ &\geq 89n-32>0 \end{aligned}$$

for $n \geq 3$. So we have

$$(n+2)^5 < n(n+1)(n+2)(n+4)(n+5) < (n+3)^5$$

for all $n \geq 3$ therefore $n(n+1)(n+2)(n+4)(n+5)$ is not a perfect power for any integer $n \geq 3$. But $n(n+1)(n+2)(n+4)(n+5)$ is not a perfect fifth power (or of any kind) for $n \in \{1,2\}$, so we are done with this case, too. Finally

$$\begin{aligned} n(n+1)(n+2)(n+3)(n+5)-(n+2)^5 &= \\ &= n^4+n^3-19n^2-50n-32 \\ &\geq 6n^3-2n^2-40n-32 \\ &\geq 28n^2-40n-32 \\ &\geq 100n-32>0 \end{aligned}$$

for all $n \geq 5$, and one easily checks that $n(n+1)(n+2)(n+4)(n+5)$ is not a perfect fifth power for $n \in \{1,2,3,4\}$, completing our proof.

Remark. A theorem of Erdős and Selfridge (1974) shows that no product of $k \geq 2$ consecutive positive integers can be the lth power ($l \geq 2$) of a positive integer, so this is not at all new. Nevertheless, some particular cases have simple and elementary proofs, as this one from our problem. What we do here is to adapt the proof of the fact that the product of four consecutive positive integers cannot be a perfect cube, which is pretty well known (and rather folkloric). The case from our problem might also be (well) known.

Also solved by Emon Suin, Ramakrishna Mission Vidyalaya, Narendrapur, West Bengal, India; Ivan Hadinata, Jember, Indonesia; Puneeth A, Hmr International School, India; Theo Koupelis, Cape Coral, FL, USA.