An Interesting Generator of Parabolic Systems of Coaxial Circles Using Apollonius’ Problem

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Abstract

Using Apollonius’ classic construction problem, we generate additional tangent circles from the triangle’s incircle, resulting in a newly created triplet of tangent circles. We will use the inverse as the dominant approach to the proof of this result.

1 Introduction

Apollonius’ problem [1, 2, 3, 4, 5, 7] of constructing the tangent circles can be considered one of the most classical problems of elementary Euclidean geometry. Historically, Apollonius’ problem has been of interest to many mathematicians because of its wide range of applications in geometry and other fields.

Figure 1: An illustration of the eight Apollonius circles of three separate circles.

In this paper, we apply Apollonius’ problem in constructing tangent circles. Points and lines can also be considered degenerate circles. Apollonius’ problem also mentioned these cases. Using Apollonius’ problem, we will generate some more circles tangent from a triangle. In the process, we found a pair of three tangent circles. The following theorem is established
**Theorem 1** (Main theorem). Let $\Omega$ be the circumcircle of triangle $ABC$ and $\omega$ be an arbitrary circle touching $AB$, $AC$ and lying inside angle $\angle BAC$. Let $\omega_1$ be a circle passing through $A$, $B$ and tangent to $\omega$; $\omega_2$ be a circle passing through $A$, $C$ and tangent to $\omega$.

(a) Circle $\omega_3$ is internally tangent to $\omega_1$, externally tangent to $\omega_2$ and share $AB$ as the external common tangent with $\omega$.

(b) Circle $\omega_4$ is internally tangent to $\omega_2$, externally tangent to $\omega_1$ and share $AC$ as the external common tangent with $\omega$.

(c) Circle $\omega_5$ is internally tangent to $\omega_3$ and $\omega_4$ and is tangent to the arc $BC$ containing $A$ of $\Omega$.

(d) Circle $\omega_6$ is externally tangent to $\omega_3$ and $\omega_4$ and is tangent to the arc $BC$ containing $A$ of $\Omega$.

(e) Circle $\omega_7$ is internally tangent to $\omega_3$ and $\omega_4$ and is tangent to the arc $BC$ not containing $A$ of $\Omega$.

(f) Circle $\omega_8$ is externally tangent to $\omega_3$ and $\omega_4$ and is tangent to the arc $BC$ not containing $A$ of $\Omega$.

Then,

(g) the circles $\omega_5$ and $\omega_6$ are tangent, and they are both tangent to $\Omega$.

(h) the circles $\omega_7$ and $\omega_8$ are tangent, and they are both tangent to $\Omega$.

Thus, we have two parabolic systems of coaxial circles.

**Remark.** In case $\omega$ is a circle touching to $AB$, $AC$ and $\Omega$ (it is $A$-mixtilinear incircle or $A$-mixtilinear excircle of $ABC$), the statement is trivial since $\omega_5$ or $\omega_6$ coincides $\Omega$.

## 2 Proof of the main Theorem

To prepare for the solution, we would like to recall some concepts.

**Definition 1** (See [6], §§2). If $A$ and $B$ are any two points, $AB$ means the distance from $A$ to $B$, and $BA$ the distance from $B$ to $A$. One of these will be represented by a positive number, the other by the same number with a negative sign. The notations $AB$ and $BA$ are called by signed lengths of segments.

**Definition 2** (See [6], §§16–19). The directed angle from a line $\ell$ to a line $\ell'$ denoted by $(\ell, \ell')$ is that angle through which $\ell$ must be rotated in the positive direction to become parallel to $\ell'$ or to coincide with $\ell'$.

**Some notations used in the article**

- $XY$ denote the signed lengths of the segment $XY$.
- If directed angle from a line $\ell$ to a line $\ell'$ is $\alpha$ modulo $\pi$, we denote it as $(\ell, \ell') = \alpha \mod \pi$.
- $I_p^A$ denote the inversion center $A$ and power $p$ with a real number $p$; see [8].
Lemma 1. With six points $A, B, C, D, P,$ and $Q$ lie on the line $d$ satisfying two equations
\[ DA \cdot DB = DP \cdot DQ \]
and
\[ \frac{AP^2}{AQ^2} = \frac{CP \cdot DP}{CQ \cdot DQ} \]
Then,
\[ AB \cdot AC = AP \cdot AQ. \]

Proof. Let $\omega$ be an arbitrary circle pass through $P, Q$ and $T$ be the midpoint of arc $PQ$ of $\omega$. Let $TA$ meet $\omega$ again at $X$ and $XC$ and $XD$ meet $\omega$ again at $Z, Y$ respectively. Since
\[ \frac{AP^2}{AQ^2} = \frac{CP \cdot DP}{CQ \cdot DQ} \]
we get $XC$ and $XD$ are two isogonal lines in $\angle PXQ$, which means $ZY \parallel PQ$ and $XA$ is the bisector of $\angle DXC$. It is obvious that

$$DA \cdot DB = DP \cdot DQ = DX \cdot DY$$

thus $X$, $Y$, $A$ and $B$ are on a circle. By angle chasing,

$$(BY, BA) = (XY, XA) = (XD, XA) = (XA, XC) = (XT, XZ) = (YT, YZ) \pmod{\pi}$$

then $T$, $Y$ and $B$ are collinear because $YZ \parallel BA$. This leads to

$$(BT, BC) = (YT, YZ) = (XT, XZ) = (XT, XC) \pmod{\pi}$$

or $CXTB$ is cyclic. Then we get

$$AP \cdot AQ = AX \cdot AT = AC \cdot AB.$$ 

This finishes the proof of Lemma 1.

Lemma 2. Let $(O)$ be an arbitrary circle internally tangent to $(O_1)$, $(O_2)$ and meet the external common tangent of these two circles $d$ at $A$ and $B$. Then if $d'$, $(O_3)$ and $(O_4)$ are respectively the images of $(O)$, $(O_1)$ and $(O_2)$ through the inversion $I_p^A$, then $d'$ passes through the exsimilicenter of $(O_3)$ and $(O_4)$.

![Figure 4: Proof of Lemma 2](image-url)
Proof. Let $d$ touch $(O_1)$ and $(O_2)$ at $P$ and $Q$ respectively, $(O)$ touch $(O_1)$ and $(O_2)$ at $E$ and $F$ respectively.

Without loss of generality, we can assume that $p = AP \cdot AQ$.

Since $\mathcal{T}_p^A : P \mapsto Q$, $(O_1) \mapsto (O_3)$, $O_3$ lies on $AO_1$ and $(O_3)$ is tangent to $AB$ at $Q$. Similarly, $O_4P \perp AB$ and $O_4$ lies on $AO_2$.

Let $O_3O_4$ meet $AB$ at $C$ and $O_1O_2$ meet $AB$ at $D$. It is obvious that $D$ is the exsimilicenter of $(O_1)$ and $(O_2)$ then from Monge’s Circle Theorem [9], $D, E$ and $F$ are collinear. Note that $EP$ and $FQ$ are the bisectors of $\angle AEB$ and $\angle AFB$ respectively. This leads to $EP$ meeting $FQ$ at $J$ is the midpoint of arc $AB$ of $(O)$. In triangle $JEF$, $PQ$ is perpendicular to the line connecting $J$ with circumcenter $O$ hence $PQ$ is an anti-parallel line of $EF$ concerning $\angle JEF$ so $EFQP$ is cyclic. Thus,

$$\overrightarrow{DP} \cdot \overrightarrow{DQ} = \overrightarrow{DE} \cdot \overrightarrow{DF} = \overrightarrow{DA} \cdot \overrightarrow{DB}.$$ 

It is obvious that

$$\frac{CP}{CQ} = \frac{O_1P}{O_2Q} = \frac{O_2Q \cdot \overline{AP}}{O_1P \cdot \overline{AQ}} = \frac{O_2Q \cdot \overline{AP^2}}{O_1P \cdot \overline{AQ^2}} = \frac{\overline{DQ} \cdot \overline{AP^2}}{\overline{DP} \cdot \overline{AQ^2}}.$$ 

Then by Lemma [1] we get

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{AP} \cdot \overrightarrow{AQ}.$$ 

This leads to $\mathcal{T}_p^A : C \mapsto B$ which means $d'$ passes through $C$. This finishes the proof of Lemma [2].

**Lemma 3.** Let $X$ be the intersection of two external common tangents of $(O_1)$ and $(O_2)$. An arbitrary line $d$ passes through $X$. Let circle $(O_3)$ touch $d$ at $J$ and externally tangent to $(O_1)$ and $(O_2)$. Let circle $(O_4)$ touch $d$ at $K$ and internally tangent to $(O_1)$ and $(O_2)$. Then if $K$ lies on ray $XJ$, $(O_3)$ is tangent to $(O_4)$. Moreover, we have $XJ = XK$.

**Proof.** Let one external common tangent of $(O_1)$ and $(O_2)$ touch these two circles at $C$ and $D$ respectively. Assume that $(O_3)$ is tangent to $(O_1)$ and $(O_2)$ at $M$ and $N$ respectively. Let $JM$ meet $(O_1)$ again at $A$ and $JN$ meet $(O_2)$ again at $B$.

By angle chasing,

$$(O_1A, AM) = -(O_1M, MA) = (O_3J, JM) \quad (\mod \pi)$$

and

$$(O_2B, BN) = -(O_2N, NB) = (O_3J, JN) \quad (\mod \pi),$$

we get $BO_2 \parallel JO_3 \parallel AO_1$. By Monge’s Circle Theorem [9], $M, N$ and $X$ are collinear.

In $\triangle O_1AM$ and $\triangle O_2NB$, $AM$ meet $BN$ at $J$, $O_1M$ meet $O_2N$ at $O_3$ and $JO_3 \parallel AO_1 \parallel BO_2$ then by Desargues’ theorem, $MN$, $O_1O_2$ and $AB$ concur which means $A, B$ and $X$ are collinear.

Let $AB$ and $MN$ meet $(O_2)$ again at $P$ and $Q$, then $O_2Q \parallel O_3M$ so

$$\frac{NJ}{NB} = \frac{NO_3}{NO_2} = \frac{NM}{NQ}$$

or $JM \parallel QB$, this leads to

$$(PN, PB) = (QN, QB) = (MJ, MN) \quad (\mod \pi)$$
so $MNPA$ is cyclic. Since $X$ is the exsimilicenter of $(O_1)$ and $(O_2)$ and $XD$ touches $(O_2)$. We have

$$XC \cdot XD = XD^2 \cdot \frac{XC}{XD}$$

$$= XB \cdot XP \cdot \frac{XO_1}{XO_2}$$

$$= XP \cdot XB \cdot \frac{XA}{XO_2}$$

$$= XP \cdot XA$$

$$= XM \cdot XN,$$

then $XJ^2 = XM \cdot XN = XC \cdot XD$. Similarly, $XK^2 = XC \cdot XD$ and we get $XJ = XK$. This finishes the proof of Lemma 3.

**Coming back to Theorem** Let $P$ be a We prove that $\omega_7$ touches $\omega_8$ at $P$ Let $T^A$ be the inversion with center $A$ and let $X'$ be the image of point $X$ through $T^A$, $\gamma'$ be the image of $\gamma$ through $T^A$ for arbitrary circle $\gamma$.

From Lemma 2, we have $B'$ is the exsimilicenter of $\omega'$ and $\omega_3'$ since $\omega_1$ is internally tangent to $\omega$ and $\omega_3$. Similarly, $C'$ is the exsimilicenter of $\omega'$ and $\omega_4'$ and from Monge’s Circle Theorem 9, $B'C'$ pass through the exsimilicenter $X$ of $\omega_3'$ and $\omega_4'$.

Note that $\omega_5$ is internally tangent to $\omega_3$ and $\omega_4$ none of these two circles contains the other. Then none of $\omega_3'$ and $\omega_4'$ contain the other so that these two circles have two external common tangents passing through $X$.

From Lemma 3 since $\omega_i'$ touches $B'C'$ and $\omega_3'$ and $\omega_4'$ for $i = 5, 8$, then there is a partition of $\omega_5', \omega_6', \omega_7', \omega_8'$ into two 2-element-subsets such that two circles in each subset tangent to $B'C'$. Assume that two tangent points of these four circles with $B'C'$ is $P'$ and $Q'$, then their images $P$ and $Q$ lies on circle $\Omega$.

Since $\omega_i'$ passes through either $P$ or $Q$, $\omega_i$ passes through either $P$ or $Q$. We know that either $P$ or $Q$ lies on the arc $BC$ not containing $A$ and we can assume that it is $P$. This means $\omega_7$ touches $\Omega$ at $P$ since their tangent point lies on the arc $BC$ not containing $A$. Similarly, $\omega_8$ touches $\Omega$ at $P$. Hence, the other two circles touching $\Omega$ at $Q$ and this finishes the proof of Theorem 1.
Figure 6: Proof of main Theorem.

3 Conclusion

Apollonius’ problem is a classic problem of Euclidean geometry. The above article is just a little application of Apollonius’ problem in constructing more tangent circles from the inscribed circle. However, with the addition of other classical tools in classical Euclidean geometry such as Monge’s Circle Theorem, Desargues’ Theorem, and the inversion, we have managed to reproduce the very flexible coupling of the classical theorems in Euclidean geometry.

References


