An Interesting Generator of Parabolic Systems of Coaxial Circles Using Apollonius' Problem

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Abstract

Using Apollonius' classic construction problem, we generate additional tangent circles from the triangle's incircle, resulting in a newly created triplet of tangent circles. We will use the inverse as the dominant approach to the proof of this result.

1 Introduction

Apollonius' problem [1, 2, 3, 4, 5, 7] of constructing the tangent circles can be considered one of the most classical problems of elementary Euclidean geometry. Historically, Apollonius' problem has been of interest to many mathematicians because of its wide range of applications in geometry and other fields.

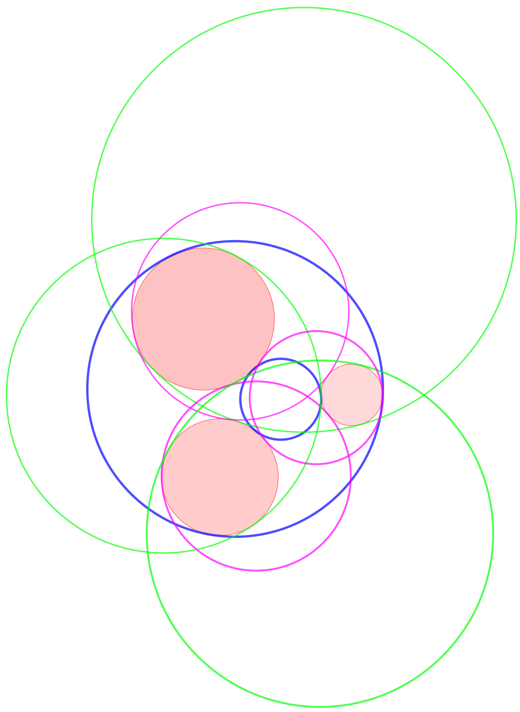

Figure 1: An illustration of the eight Apollonius circles of three separate circles.

In this paper, we apply Apollonius' problem in constructing tangent circles. Points and lines can also be considered degenerate circles. Apollonius' problem also mentioned these cases. Using Apollonius' problem, we will generate some more circles tangent from a triangle. In the process, we found a pair of three tangent circles. The following theorem is established

Theorem 1 (Main theorem)**.** Let Ω be the circumcircle of triangle ABC and ω be an arbitrary circle touching AB, AC and lying inside angle $\angle BAC$. Let ω_1 be a circle passing through A, B and tangent to ω; ω_2 be a circle passing through A, C and tangent to ω.

(a) Circle ω_3 is internally tangent to ω_1, externally tangent to ω_2 and share AB as the external common tangent with ω.

(b) Circle ω_4 is internally tangent to ω_2, externally tangent to ω_1 and share AC as the external common tangent with ω.

(c) Circle ω_5 is internally tangent to ω_3 and ω_4 and is tangent to the arc $\widehat{BC}$ containing A of Ω.

(d) Circle ω_6 is externally tangent to ω_3 and ω_4 and is tangent to the arc $\widehat{BC}$ containing A of Ω.

(e) Circle ω_7 is internally tangent to ω_3 and ω_4 and is tangent to the arc $\widehat{BC}$ not containing A of Ω.

(f) Circle ω_8 is externally tangent to ω_3 and ω_4 and is tangent to the arc $\widehat{BC}$ not containing A of Ω.

Then,

(g) the circles ω_5 and ω_6 are tangent, and they are both tangent to Ω.

(h) the circles ω_7 and ω_8 are tangent, and they are both tangent to Ω.

Thus, we have two parabolic systems of coaxial circles.

Remark. In case ω is a circle touching to AB, AC and Ω (it is A-mixtilinear incircle or A-mixtilinear excircle of ABC), the statement is trivial since ω_5 or ω_6 coincides Ω.

2 Proof of the main Theorem

To prepare for the solution, we would like to recall some concepts.

Definition 1 (See [6], §§2)**.** If A and B are any two points, $\overline{AB}$ means the distance from A to B, and $\overline{BA}$ the distance from B to A. One of these will be represented by a positive number, the other by the same number with a negative sign. The notations $\overline{AB}$ and $\overline{BA}$ are called by signed lengths of segments.

Definition 2 (See [6], §§16–19)**.** The directed angle from a line ℓ to a line ℓ' denoted by (ℓ, ℓ') is that angle through which ℓ must be rotated in the positive direction to become parallel to ℓ' or to coincide with ℓ'.

Some notations used in the article

- $\overline{XY}$ denote the signed lengths of the segment XY.
- If directed angle from a line ℓ to a line ℓ' is α modulo π, we denote it as $(\ell, \ell') = \alpha \pmod{\pi}$.
- $\mathcal{I}_p^A$ denote the inversion center A and power p with a real number p; see [8].

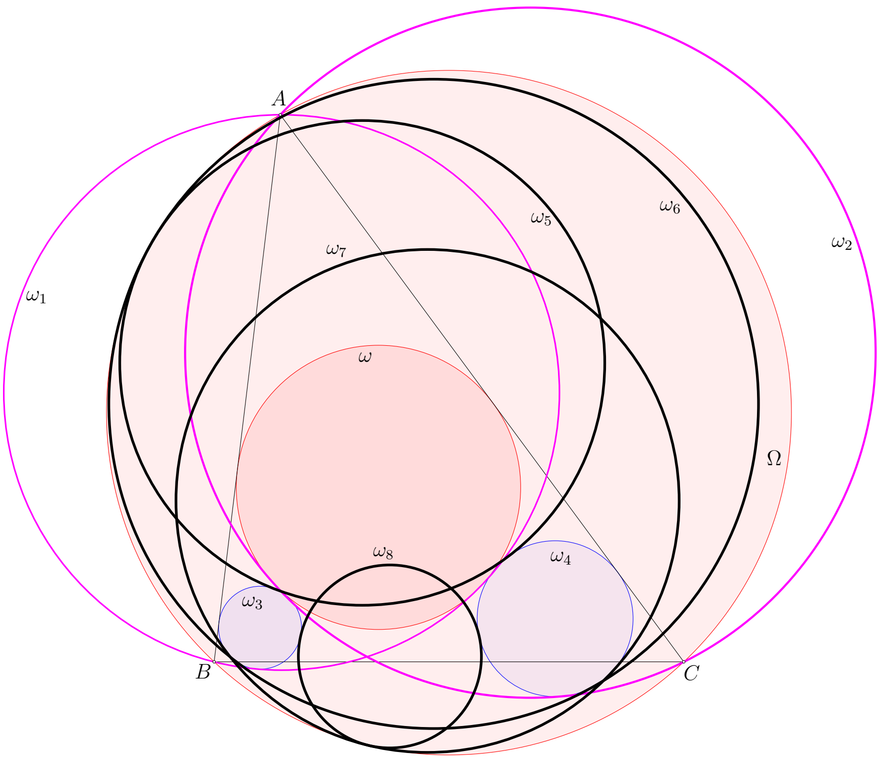

Figure 2: Two new parabolic systems of coaxial circles.

Lemma 1. With six points A, B, C, D, P, and Q lie on the line d satisfying two equations

$$\overline{DA} \cdot \overline{DB} = \overline{DP} \cdot \overline{DQ}$$

and

$$\frac{AP^2}{AQ^2} = \frac{\overline{CP} \cdot \overline{DP}}{\overline{CQ} \cdot \overline{DQ}}.$$

Then,

$$\overline{AB} \cdot \overline{AC} = \overline{AP} \cdot \overline{AQ}.$$

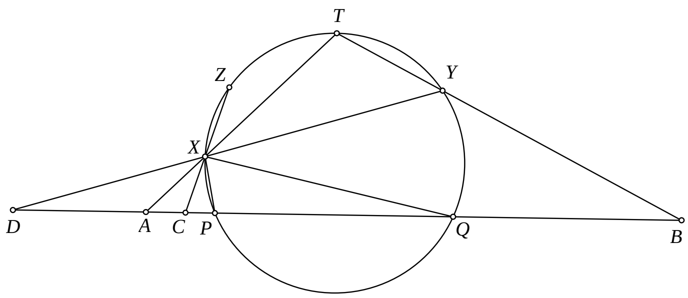

Figure 3: Proof of Lemma 1.

Proof. Let ω be an arbitrary circle pass through P, Q and T be the midpoint of arc PQ of ω. Let TA meet ω again at X and XC and XD meet ω again at Z, Y respectively. Since

$$\frac{AP^2}{AQ^2} = \frac{\overline{CP} \cdot \overline{DP}}{\overline{CQ} \cdot \overline{DQ}}$$

we get XC and XD are two isogonal lines in $\angle PXQ$, which means $ZY \parallel PQ$ and XA is the bisector of $\angle DXC$. It is obvious that

$$\overline{DA} \cdot \overline{DB} = \overline{DP} \cdot \overline{DQ} = \overline{DX} \cdot \overline{DY}$$

thus X, Y, A and B are on a circle. By angle chasing,

$$\begin{aligned}(BY, BA) &= (XY, XA)\\ &= (XD, XA)\\ &= (XA, XC)\\ &= (XT, XZ)\\ &= (YT, YZ) \pmod{\pi}\end{aligned}$$

then T, Y and B are collinear because $YZ \parallel BA$. This leads to

$$(BT, BC) = (YT, YZ) = (XT, XZ) = (XT, XC) \pmod{\pi}$$

or $CXTB$ is cyclic. Then we get

$$\overline{AP} \cdot \overline{AQ} = \overline{AX} \cdot \overline{AT} = \overline{AC} \cdot \overline{AB}.$$

This finishes the proof of Lemma 1. □

Lemma 2. Let (O) be an arbitrary circle internally tangent to (O_1), (O_2) and meet the external common tangent of these two circles d at A and B. Then if d', (O_3) and (O_4) are respectively the images of (O), (O_1) and (O_2) through the inversion $\mathcal{I}_p^A$, then d' passes through the exsimilicenter of (O_3) and (O_4).

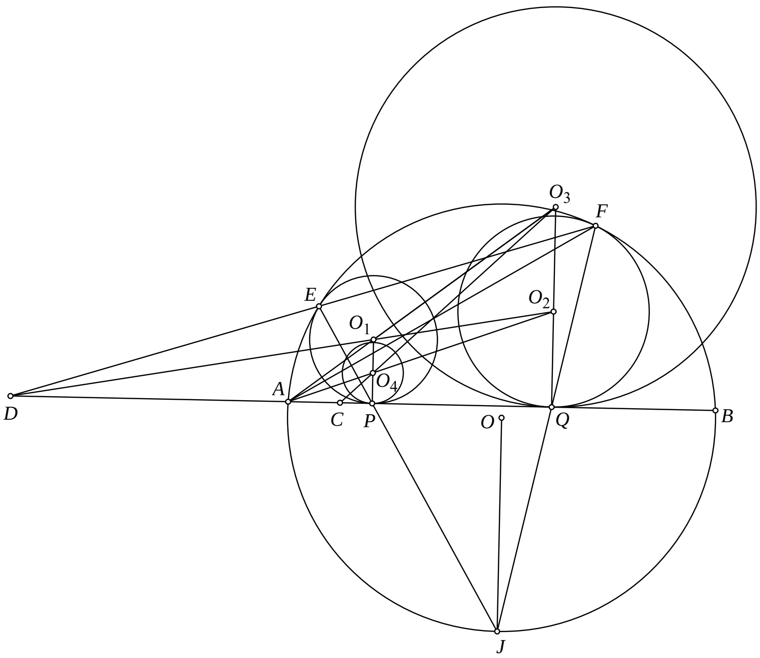

Figure 4: Proof of Lemma 2.

Proof. Let d touch (O_1) and (O_2) at P and Q respectively, (O) touch (O_1) and (O_2) at E and F respectively.

Without loss of generality, we can assume that $p = \overline{AP} \cdot \overline{AQ}$.

Since $\mathcal{I}_p^A : P \mapsto Q$, $(O_1) \mapsto (O_3)$, O_3 lies on AO_1 and (O_3) is tangent to AB at Q. Similarly, $O_4P \perp AB$ and O_4 lies on AO_2.

Let O_3O_4 meet AB at C and O_1O_2 meet AB at D. It is obvious that D is the exsimilicenter of (O_1) and (O_2) then from Monge's Circle Theorem [9], D, E and F are collinear. Note that EP and FQ are the bisectors of $\angle AEB$ and $\angle AFB$ respectively. This leads to EP meeting FQ at J is the midpoint of arc AB of (O). In triangle JEF, PQ is perpendicular to the line connecting J with circumcenter O hence PQ is an anti-parallel line of EF concerning $\angle JEF$ so $EFQP$ is cyclic. Thus,

$$\overline{DP} \cdot \overline{DQ} = \overline{DE} \cdot \overline{DF} = \overline{DA} \cdot \overline{DB}.$$

It is obvious that

$$\begin{aligned}
\frac{\overline{CP}}{\overline{CQ}} &= \frac{\overline{O_4P}}{\overline{O_3Q}} \\
&= \frac{\overline{O_2Q} \cdot \frac{\overline{AP}}{\overline{AQ}}}{\overline{O_1P} \cdot \frac{\overline{AQ}}{\overline{AP}}} \\
&= \frac{\overline{O_2Q}}{\overline{O_1P}} \cdot \frac{AP^2}{AQ^2} \\
&= \frac{\overline{DQ}}{\overline{DP}} \cdot \frac{AP^2}{AQ^2}.
\end{aligned}$$

Then by Lemma 1, we get

$$\overline{AB} \cdot \overline{AC} = \overline{AP} \cdot \overline{AQ}.$$

This leads to $\mathcal{I}_p^A : C \mapsto B$ which means d' passes through C. This finishes the proof of Lemma 2. □

Lemma 3. Let X be the intersection of two external common tangents of (O_1) and (O_2). An arbitrary line d passes through X. Let circle (O_3) touch d at J and externally tangent to (O_1) and (O_2). Let circle (O_4) touch d at K and internally tangent to (O_1) and (O_2). Then if K lies on ray XJ, (O_3) is tangent to (O_4). Moreover, we have $XJ = XK$.

Proof. Let one external common tangent of (O_1) and (O_2) touch these two circles at C and D respectively. Assume that (O_3) is tangent to (O_1) and (O_2) at M and N respectively. Let JM meet (O_1) again at A and JN meet (O_2) again at B.

By angle chasing,

$$(O_1A, AM) = -(O_1M, MA) = (O_3J, JM) \pmod{\pi}$$

and

$$(O_2B, BN) = -(O_2N, NB) = (O_3J, JN) \pmod{\pi},$$

we get $BO_2 \parallel JO_3 \parallel AO_1$. By Monge's Circle Theorem [9], M, N and X are collinear.

In $\triangle O_1AM$ and $\triangle O_2NB$, AM meet BN at J, O_1M meet O_2N at O_3 and $JO_3 \parallel AO_1 \parallel BO_2$ then by Desargues' theorem, MN, O_1O_2 and AB concur which means A, B and X are collinear.

Let AB and MN meet (O_2) again at P and Q, then $O_2Q \parallel O_3M$ so

$$\frac{\overline{NJ}}{\overline{NB}} = \frac{\overline{NO_3}}{\overline{NO_2}} = \frac{\overline{NM}}{\overline{NQ}}$$

or $JM \parallel QB$, this leads to

$$(PN, PB) = (QN, QB) = (MJ, MN) \pmod{\pi}$$

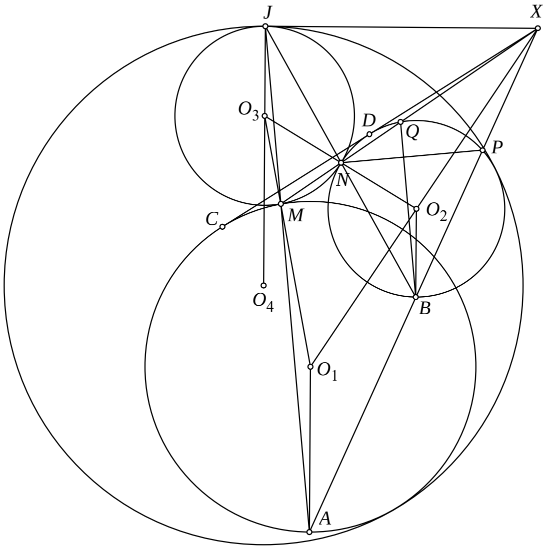

Figure 5: Proof of Lemma 3.

so $MNPA$ is cyclic. Since X is the exsimilicenter of (O_1) and (O_2) and XD touches (O_2). We have

$$\begin{aligned}\overline{XC}\cdot\overline{XD} &= XD^2\cdot\frac{\overline{XC}}{\overline{XD}}\\ &= \overline{XB}\cdot\overline{XP}\cdot\frac{\overline{XO_1}}{\overline{XO_2}}\\ &= \overline{XP}\cdot\overline{XB}\cdot\frac{\overline{XA}}{\overline{XB}}\\ &= \overline{XP}\cdot\overline{XA}\\ &= \overline{XM}\cdot\overline{XN},\end{aligned}$$

then $XJ^2 = \overline{XM}\cdot\overline{XN} = \overline{XC}\cdot\overline{XD}$. Similarly, $XK^2 = \overline{XC}\cdot\overline{XD}$ and we get $XJ = XK$. This finishes the proof of Lemma 3. $\square$

Coming back to Theorem 1. Let P be a We prove that ω_7 touches ω_8 at P Let $\mathcal{I}^A$ be the inversion with center A and let X' be the image of point X through $\mathcal{I}^A$, γ' be the image of γ through $\mathcal{I}^A$ for arbitrary circle γ.

From Lemma 2, we have B' is the exsimilicenter of ω' and ω'_3 since ω_1 is internally tangent to ω and ω_3. Similarly, C' is the exsimilicenter of ω' and ω'_4 and from Monge's Circle Theorem [9], $B'C'$ pass through the exsimilicenter X of ω'_3 and ω'_4.

Note that ω_5 is internally tangent to ω_3 and ω_4 none of these two circles contains the other. Then none of ω'_3 and ω'_4 contain the other so that these two circles have two external common tangents passing through X.

From Lemma 3, since ω'_i touches $B'C'$ and ω'_3 and ω'_4 for $i = \overline{5,8}$, then there is a partition of $\omega'_5, \omega'_6, \omega'_7, \omega'_8$ into two 2-element-subsets such that two circles in each subset tangent to $B'C'$. Assume that two tangent points of these four circles with $B'C'$ is P' and Q', then their images P and Q lies on circle Ω.

Since ω'_i passes through either P' or Q', ω_i passes through either P or Q. We know that either P or Q lies on the arc BC not containing A and we can assume that it is P. This means ω_7 touches Ω at P since their tangent point lies on the arc BC not containing A. Similarly, ω_8 touches Ω at P. Hence, the other two circles touching Ω at Q and this finishes the proof of Theorem 1.

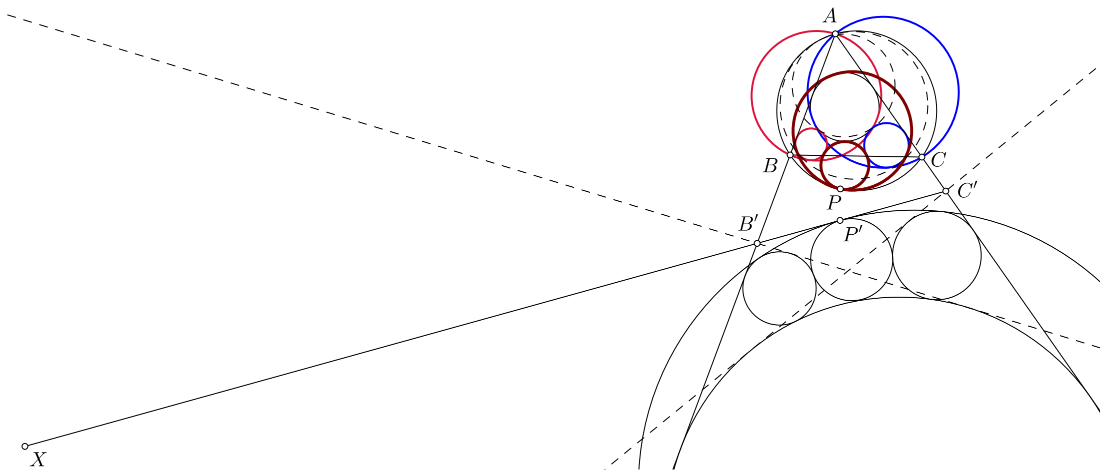

Figure 6: Proof of main Theorem.

□

3 Conclusion

Apollonius' problem is a classic problem of Euclidean geometry. The above article is just a little application of Apollonius' problem in constructing more tangent circles from the inscribed circle. However, with the addition of other classical tools in classical Euclidean geometry such as Monge's Circle Theorem, Desargues' Theorem, and the inversion, we have managed to reproduce the very flexible coupling of the classical theorems in Euclidean geometry.

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