Junior Problems

J607. Find all integers k such that

$$(3m^2 - 4n^2)^2 + (4m^2 - 3n^2)^2 + 7mn(10m^2 + kmn + 10n^2)$$

is a perfect square for all integers m and n.

Proposed by Adrian Andreescu, Dallas, USA

J608. Let a, b, c, d be integers such that ad is odd and bc is even. Prove that if all solutions to the equation

$$ax^3 + bx^2 + cx + d = 0$$

are real numbers then at least one of them is irrational.

Proposed by Mihaela Berindeanu,Bucharest, România

J609. Find all triples (x, y, z) of real numbers for which

$$x^2 - 9y - 2\sqrt{z-4} + 22 = y^2 - 9z - 2\sqrt{x-4} + 22 = z^2 - 9x - 2\sqrt{y-4} + 22 = 0.$$

Proposed by Mihaly Bencze, Braşov, România and Neculai Stanciu, Buzău, România

J610. Let a, b, c be pairwise distinct real numbers in the interval $[0, 1]$. Find the minimum value of

$$\frac{1}{\sqrt{|a-b|}} + \frac{1}{\sqrt{|b-c|}} + \frac{1}{\sqrt{|c-a|}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J611. Let a, b, c be positive real numbers. Prove that

$$3(ab + bc + ca) \leq a\sqrt{b^2 + 8c^2} + b\sqrt{c^2 + 8a^2} + c\sqrt{a^2 + 8b^2} \leq (a + b + c)^2.$$

Proposed by Tran Tien Manh, Vinh City, Vietnam

J612. Let $ABCD$ be a parallelogram. The tangent at C to he circumcircle of the triangle BCD intersects line AB at E and line AD at F. Let O be the center of the circumcircle of the triangle AEF. Prove that AO is perpendicular to BD.

Proposed by Mihai Miculiţa, Oradea, România

Senior Problems

S607. Let k be a positive integer, $a = 1 + k^2(2k^2+1)(2k^2+2k+1)$, and let $b = 2k^2$. Prove that $ab^n + 1$ is composite for any positive integer n.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

S608. Let a, b, c be positive real numbers. Prove that

$$\frac{3}{a^3+b^3+c^3} \leq \frac{1}{a(a^2+2bc)} + \frac{1}{b(b^2+2ca)} + \frac{1}{c(c^2+2ab)} \leq \frac{1}{abc}.$$

Proposed by An Zhenping, Xianyang Normal University, China

S609. Let ABC be a triangle with medians m_a, m_b, m_c and area Δ. Prove that

$$\sqrt{2\left(m_am_b + m_bm_c + m_cm_a - 3\sqrt{3}\Delta\right)} \geq \max\{m_a, m_b, m_c\} - \min\{m_a, m_b, m_c\}.$$

Proposed by Marius Stănean, Zalău, România

S610. Prove that in any triangle ABC,

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \frac{1}{\sqrt{3}}\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S611. Let p and q be two prime numbers such that $p \equiv -23 \pmod{60}$, $q \equiv -47 \pmod{120}$ and $q = 2p - 1$. Prove that the number $n = pq$ satisfies the following relations.

$$n | 2^n - 2,\ n | 3^n - 3,\ \ n \nmid 5^n - 5.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

S612. Given are triangle ABC and a point O on the plane. Define $D = AO \cap BC, E = BO \cap AC, F = CO \cap AB, Q = DF \cap AC, R = DE \cap AB$, and $S = AD \cap RQ$. Let the line passing through B and parallel to DE intersects AC at X and the line passing through C and parallel to DF intersects AB at Y. Let $XY \cap BC = T$. If SB, SC, SE, SF intersect AT at K, L, M, N, respectively, prove that $|KN| = |NA| = |AM| = |ML|$

Proposed by Bariş Koyunku, INKA Schools, Istanbul, Turkey

Undergraduate Problems

U607. Let p be a prime and let m, n be integers with $m \geq n \geq 0$. Prove that

$$\binom{2p+m}{2p+n} + \binom{m}{n} + \binom{m}{2p+n} + 2\binom{m}{p+n} \equiv \left[2\binom{p+m}{p+n} + 2\binom{p+m}{2p+n}\right] \pmod{p^2}.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, România

U608. Consider the formal power series

$$f(x) = x + x^2 + x^4 + x^8 + x^{16} + \cdots$$

Find the coefficient of x^{10} in $f(f(x))$.

Proposed by Treanungkul Mal, Ideal Public School, West Bengal, India

U609. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(x) \geq 2x$ for all $x \in [0, 1]$. We denote

$$I_f = \int_0^1 x f(x)^3 dx \text{ and } I_g = \int_0^1 x^3 f(x) dx.$$

Find the minimum of $I_f - 3I_g$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U610. If n is a positive integer find the value of

$$\int_0^{\infty} x^{n-1} e^{-x} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cos\left(x + \frac{k\pi}{2}\right) \right\} dx.$$

Proposed by Seán M. Stewart, King Abdullah Univ. of Science and Technology, Thuwal, Saudi Arabia

U611. We call a set A of real numbers *nice* if there are non-zero distinct numbers $x_1, \ldots, x_n \in A$ and integers $y_1, \cdots, y_n$ (not all zero) such that

$$x_1^{y_1} \ldots x_n^{y_n} = 1.$$

Let $P(x)$ be a second degree polynomial with rational coefficients. Prove that the set

$$\{P(n) | n \in \mathbb{N}, n > 0\}$$

is *nice.*

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U612. If n is a nonnegative integer and $m \in \{0, 1, 2, \ldots, n\}$, evaluate

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n+m}.$$

Proposed by Seán M. Stewart, King Abdullah Univ. of Science and Technology, Thuwal, Saudi Arabia

Olympiad Problems

O607. Let $a_1, \dots, a_k$ and b be positive integers and let $f(t) = (t+a_1)\dots(t+a_k)$. Prove that there are infinitely many positive integers n such that $P(f(n)) \geq P(f(n+b))$, where $P(m)$ denotes the greatest prime divisor of the integer $m \geq 2$. (For instance, $P(81) = P(3) = 3$ or $P(56) = P(14) = 7$.)

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O608. Find the minimum possible value of the positive integer k such that the equation

$$x^2 + kxy + y^2 = 2022$$

is solvable in positive integers x and y.

Proposed by Todor Zaharinov, Sofia, Bulgaria

O609. Let ABC be a triangle with circumcircle $\Gamma(O)$ and points D, E on the side BC such that $\angle BAD = \angle CAE$. The perpendicular lines through D to AD and E to AE intersects AB, AC at K, L and M, N, respectively. The circumcircles of the triangles AMN and AKL intersect again Γ at F and G, respectively. Prove that the points D, E, F, G are concyclic.

Proposed by Marius Stănean, Zalău, România

O610. Let a, b, c be positive real numbers. Prove that

$$a^2\sqrt{\frac{a+b}{a+c}} + b^2\sqrt{\frac{b+c}{b+a}} + c^2\sqrt{\frac{c+a}{c+b}} \geq ab + bc + ca.$$

Proposed by An Zhenping, Xianyang Normal University, China

O611. Let $m \geq 2$ and $a_1, \dots, a_m$ be positive integers such that $a_1, \dots, a_m$ are not all equal. Prove that $(n+a_1)\dots(n+a_m)$ is the m^{th} power of a positive integer for at most finitely many positive integers n.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O612. Let a, b, c, d, e be positive real numbers such that

$$a^3 + b^3 + c^3 + d^3 + e^3 + abcde = 6.$$

Prove that $a + bcde \leq 2$.

Proposed by An Zhenping, Xianyang Normal University, China