A few generalizations of the Euler's Totient Function

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Abstract

The goal of this paper is to present a few generalizations of Euler's totient function which can be successfully used in solving combinatorial and number theory problems.

1 Introduction

Undoubtedly, the Euler's Totient Function (shortly ETF) is paramount in number theory and a very helpful tool in problem-solving. Recall that the ETF $\varphi(n)$ is the number of coprimes of the positive integer n which are lesser or equal to n. Different generalizations of the ETF are known (see [3]-[6]), but ETF continues to remain an important source of further developments. In this paper, we will introduce a few generalizations of the ETF and applications that emerge from number sense arguments.

2 Preliminaries

Unless otherwise stated, the numbers used throughout this paper are positive integers. The following known results will be used throughout this paper:

Lemma 2.1. *The number of multiples of p bounded by b, denoted as M_p^b, is given by $M_p^b = \left[\dfrac{b}{p}\right]$, where $[x]$ denotes the integer part of x.*

Definition 2.1. *For any n, the ETF $\varphi(n)$ is defined as the number of coprimes with n from the interval $[1, n]$, i.e.*

$$\varphi(n) = card\{1 \leq k \leq n \,|\, (k, n) = 1\}$$

.

Lemma 2.2. *If the decomposition of $n \geq 2$ into prime factors is*

$$n = p_1^{a_1} p_2^{a_2} \dots p_l^{a_l}$$

where l is a positive integer, $p_1, p_2, \dots p_l$ are primes such that $p_1 < p_2 < \dots < p_l$, and $a_1, a_2, \dots, a_l$ are positive integers, then the ETF $\varphi(n)$ can be calculated using the following formulas:

$$\varphi(n) = n - \left(\sum_i \left[\frac{n}{p_i}\right] - \sum_{i<j} \left[\frac{n}{p_i p_j}\right] + \dots + (-1)^{k-1} \sum_{i_1<i_2<\dots<i_k} \left[\frac{n}{p_1 p_2 \dots p_k}\right] + \dots + (-1)^{l-1} \left[\frac{n}{p_1 p_2 \dots p_l}\right]\right)$$

and

$$\varphi(n) = p_1^{a_1-1}\, p_2^{a_2-1} \ldots p_l^{a_l-1}(p_1-1)(p_2-1)\ldots(p_l-1)$$

3 The bounded-ETF

In this section, we will introduce the bounded-ETF $\varphi^b(n)$ representing the number of coprimes of n from the interval $[1, b]$, where $1 \le b \le n$.

Definition 3.1. *The number $\varphi^b(n)$ represents the number of coprimes of n which are bounded by 1 and b, where $1 \le b \le n$, is given by*

$$\varphi^b(n) = card\{1 \le k \le b \,|\, (k, n) = 1\}$$

Example: The number of coprimes of 12 bounded by 7 is given by $\varphi^7(12) = 3$

Theorem 1. *The properties of the bounded-ETF $\varphi^b(n)$, $1 \le b \le n$:*
(i)$\varphi^n(n) = \varphi(n)$
(ii) if $1 \le b \le c \le k$, then $\varphi^b(k) \le \varphi^c(k)$
(iii) if $1 \le b \le k$ and $k|h$, then $\varphi^b(k) \ge \varphi^b(h)$
(iv) if the decomposition of n into prime factors is

$$n = p_1^{a_1}\, p_2^{a_2} \ldots p_l^{a_l}$$

where $p_1, p_2, \ldots p_l$ are primes such that $p_1 < p_2 < \ldots < p_l$, then $\varphi^b(n)$ is given by

$$\varphi^b(n) = [b] - \left(\sum_i \left[\frac{b}{p_i}\right] - \sum_{i<j} \left[\frac{b}{p_i p_j}\right] + \ldots + (-1)^{k-1} \sum_{i_1<i_2<\ldots<i_k} \left[\frac{b}{p_1 p_2 \ldots p_k}\right] + \ldots + (-1)^{l-1} \left[\frac{b}{p_1 p_2 \ldots p_l}\right] \right) \quad (*)$$

Proof. (i)-(iii) result immediately from the Definitions 2.1 and 3.1

(iv) Note that if a number k, $1 \le k \le b$ is coprime with n, then k is not divisible by any of the primes $p_1, p_2, \ldots, p_l$. The proof is similar to the known proof of Lemma 2.2, i.e. it results from Lemma 2.1 and the Principle of Inclusion-Exclusion. □

4 The generalized-ETF

In this section, we will introduce the generalized-ETF $\psi(n)$.
We denote by C_n the set of the coprimes of n from the interval $[1, n]$, i.e.

$$C_n = \{1 \le k \le n \,|\, (k, n) = 1\}$$

Note that $\varphi(n) = card\, C_n$ (1)

Definition 4.1. *We define the generalized-ETF $\psi(n)$ as*

$$\psi(n) = card \left(\bigcup_{d|n} C_d \right)$$

Example: Calculate $\psi(12)$.
We will find first the sets C_d, where d is a divisor of 12:

$$C_1 = C_2 = \{1\}, C_3 = \{1,2\}, C_4 = \{1,3\}, C_6 = \{1,5\}, C_{12} = \{1,5,7,11\}$$

Next, the union of the sets C_d, where d is a divisor of 12 is given by

$$\bigcup_{d|n} C_d = \{1,2,3,5,7,11\}$$

It follows $\psi(12) = 6$.

The next theorem enables a nice application of the generalized-ETF $\psi(n)$ and a shorter way to determine the values of the generalized-ETF $\psi(n)$ for small values of n.

Theorem 2. *Let be $n \geq 2$. The generalized-ETF $\psi(n)$ represents the number of distinct numerators which appears after completely simplifying the fractions*

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

Proof. $\diamond$ Simplifying the fractions from theorem and maintaining them in the same order, results $n-1$ irreducible fractions as below

$$\frac{i_1}{j_1}, \frac{i_2}{j_2}, \ldots, \frac{i_{n-1}}{j_{n-1}}$$

where $(i_s, j_s) = 1$ for any $s = 1, 2, \ldots, n-1$. Note that the denominators j_s are divisors of n.

$\diamond$ Regroup the fractions with the same denominator, writing the groups of fractions in the increasing order of the denominators and, inside each group, arrange the fractions in increasing order. It will result in a sequence of $n-1$ fractions as below

$$\frac{h_1}{k_1}, \frac{h_2}{k_2}, \ldots, \frac{h_{n-1}}{k_{n-1}} \tag{2}$$

For instance $h_{n-1} = n-1$, $k_{n-1} = n$, $h_1 = 1, k_1 = e$, where $e \neq 1$ is the smallest divisor of n.

In the same group of fractions with a common denominator d, we have the following:

$\triangleright$ d is a divisor of n

$\triangleright$ all the numerators are prime with the common denominator d

$\triangleright$ all the irreducible fractions are smaller than 1

Note that the group of fractions with the common denominator d described above is given by the set C_d and $card\, C_d = \varphi(d)$.

Finally, considering the union of the family of sets C_d, where d is a divisor of n, we get a set that contains all the numerators of the irreducible fractions from (2). On the other hand, according to Definition 4.1, the union of the family of sets C_d, where d is a divisor of n, is the generalized-ETF $\psi(n)$. From the last two statements, it results the conclusion of Theorem 2. $\square$

Example: Find $\psi(12)$ using Theorem 2.

◇ Consider the sequence of fractions lesser than 1 having the same denominator 12

$$\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12},$$

◇ Simplify the fractions

$$\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12},$$

◇ Regroup the fractions with the same denominator keeping the increasing order of the numerators in each group. Arrange the groups in the increasing order of their denominators.

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}, \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$$

◇ $\varphi(12)$ represent the number of distinct numerators in the list above, i.e. $\varphi(12) = 6$.

The first 30 values of the $\psi(n)$ are given in the table below:

n	$\psi(n)$	n	$\psi(n)$	n	$\psi(n)$
1	1	11	10	21	14
2	1	12	6	22	15
3	2	13	12	23	22
4	2	14	9	24	10
5	4	15	9	25	20
6	3	16	8	26	18
7	6	17	16	27	18
8	4	18	9	28	15
9	6	19	18	29	28
10	5	20	10	30	15

5 The computational formula of the generalized-ETF

Theorem 2. is the key to the proof for the computational formula of the generalized-ETF $\psi(n)$. Consider the decomposition of $n, n \geq 2$ into prime factors

$$n = p_1^{a_1} p_2^{a_2} \ldots p_l^{a_l}$$

where $p_1, p_2, \ldots p_l$ are primes such that $p_1 < p_2 < \ldots < p_l$.

The previous example finding $\psi(12)$ as well as other similar examples, might be helpful in understanding the next proof. Basically, in order to count the number of distinct numerators that appear in (2), we need to consider the following categories of numerators:

(C) distinct numerators provided by the *coprimes* of n smaller than n, i.e. the numerators of the fractions $\frac{k}{n}$, where $(k,n) = 1$, $1 \le k \le n-1$. The number of them is $\varphi(n)$. Outside these numerators coprime with n, 'new numerators' containing prime factors of n appear. These new numerators are described in the next categories.

(P) 'new numerators' provided by the simplified fractions from (2), except the fractions with the denominator n mentioned at (C), i.e. new numerators which contain one or more of the *prime* factors $p_1, p_2, \dots p_l$ appearing in the decomposition of n.
The challenge is how to count these new numerators. They are of different types, according to the prime factors of n that might appear in their decomposition into prime factors.

(P1) new numerators containing *1 prime* factor, for instance, p_1 but not containing $p_2, p_3, \dots, p_l$, i.e. new numerators provided by the irreducible fractions

$$\frac{p_1 k}{p_2^{a_2} \dots p_l^{a_l}} \tag{3}$$

where $(k, p_2^{a_2} \dots p_l^{a_l}) = 1$ and $k \le \dfrac{p_2^{a_2} \dots p_l^{a_l}}{p_1}$.
Note that $(k, p_2^{a_2} \dots p_l^{a_l}) = 1$ is equivalent to $(k, p_2 \dots p_l) = 1$.

The number of these new numerators, i.e. containing p_1, but coprimes with $p_2, p_3, \dots, p_l$, and bounded by the value $k \le \dfrac{p_2^{a_2} \dots p_l^{a_l}}{p_1}$, is given by the the bounded-ETF $\varphi^{\mathbf{b}}(\mathbf{n})$ introduced in section 3, for the particular values $\mathbf{n}$ and $\mathbf{b}$ described in (3), i.e.

$$\mathbf{n} = p_2^{a_2} \dots p_l^{a_l}, \quad \mathbf{b} = \frac{p_2^{a_2} \dots p_l^{a_l}}{p_1} \tag{4}$$

In order to uniformize the expressions of type (3) and to get an uniform expression of the bounded-ETF $\varphi^{\mathbf{b}}(\mathbf{n})$, we will write the irreducible fractions from (3) in the equivalent form:

$$\frac{p_1^{a_1+1} k}{p_1^{a_1} p_2^{a_2} \dots p_l^{a_l}} \tag{5}$$

where $(k, p_2 \dots p_l) = 1$ and $k \le \dfrac{p_1^{a_1} p_2^{a_2} \dots p_l^{a_l}}{p_1^{a_1+1}}$
or, even shortly:

$$\frac{p_1^{a_1+1} k}{n} \tag{6}$$

where $(k, p_2 \dots p_l) = 1$ and $k \le \dfrac{n}{p_1^{a_1+1}}$.

In this way,

$$\mathbf{n} = \frac{n}{p_1^{a_1}}, \quad \mathbf{b} = \frac{n}{p_1^{a_1+1}}.$$

Therefore, the number of the new numerators containing p_1, but coprimes with $p_2, p_3, \ldots, p_l$, and bounded by the value $\frac{n}{p_1^{a_1+1}}$ is given by the the bounded-ETF:

$$\varphi^{\frac{n}{p_1^{a_1+1}}}\left(\frac{n}{p_1^{a_1}}\right) \tag{7}$$

Similarly, the number of the new numerators containing p_k, but coprimes with the other $l-1$ primes from the decomposition of n, and bounded by the value $\frac{n}{p_k^{a_k+1}}$ is given by the the bounded-ETF:

$$\varphi^{\frac{n}{p_k^{a_k+1}}}\left(\frac{n}{p_k^{a_k}}\right) \tag{8}$$

Finally, the number $\phi_1(n)$ of new numerators of type (P1) is given by the sum of the bounded-ETF's as in (8), i.e.

$$\phi_1(n) = \sum_{k=1}^{l} \varphi^{\frac{n}{p_k^{a_k+1}}}\left(\frac{n}{p_k^{a_k}}\right) \tag{9}$$

In the same way as in the type (P1), we compute the number of new numerators in the next types (P2), (P3), ... , (Pl).

(P2) the number of the new numerators containing *2 prime* factors, for instance, $p_1.p_2$ but not containing the other factors, and bounded by the value $\frac{n}{p_1^{a_1+1}p_2^{a_2+1}}$, is given by:

$$\varphi^{\frac{n}{p_1^{a_1+1}p_2^{a_2+1}}}\left(\frac{n}{p_1^{a_1}p_2^{a_2}}\right) \tag{10}$$

Finally, the number $\phi_2(n)$ of new numerators of type (P2) is given by the sum of the bounded-ETF's as in (10), i.e.

$$\phi_2(n) = \sum_{1\le i<j\le l} \varphi^{\frac{n}{p_i^{a_i+1}p_j^{a_j+1}}}\left(\frac{n}{p_i^{a_i}p_j^{a_j}}\right) \tag{11}$$

$\vdots$

(Pk) the number of the new numerators containing *k prime* factors, for instance, $p_1, \ldots, p_k$, but not containing the other factors, and bounded by the value $\frac{n}{p_1^{a_1+1}\ldots p_k^{a_k+1}}$, is given by:

$$\varphi^{\frac{n}{p_1^{a_1+1}\ldots p_k^{a_k+1}}}\left(\frac{n}{p_1^{a_1}\ldots p_k^{a_k}}\right) \tag{12}$$

Finally, the number $\phi_k(n)$ of new numerators of type (Pk) is given by the sum of the bounded-ETF's as in (12), i.e.

$$\phi_k(n) = \sum_{1\le i_1<\ldots<i_k\le l} \varphi^{\overline{\frac{n}{p_{i_1}^{a_{i_1}+1}\cdots p_{i_k}^{a_{i_k}+1}}}}\left(\frac{n}{p_{i_1}^{a_{i_1}}\cdots p_{i_k}^{a_k}}\right) \tag{13}$$

$\vdots$

(Pl) the number of new numerators containing all l *prime* factors, i.e. $p_1,\ldots,p_l$, and bounded by the value $\dfrac{n}{p_1^{a_1+1}\cdots p_l^{a_l+1}}$, is given by:

$$\varphi^{\overline{\frac{n}{p_1^{a_1+1}\cdots p_l^{a_l+1}}}}\left(\frac{n}{p_1^{a_1}\cdots p_l^{a_l}}\right) \tag{14}$$

Finally, the number $\phi_l(n)$ of new numerators of type (Pl) has one single term:

$$\phi_l(n) = \varphi^{\overline{\frac{n}{p_1^{a_1+1}\cdots p_l^{a_l+1}}}}\left(\frac{n}{p_1^{a_1}\cdots p_l^{a_l}}\right) \tag{15}$$

Note that $\phi_l(n) = 0$, but we just write the type (Pl) to show the last type of new numerators that we count. In fact, for sufficiently big l, many ϕ_h could be 0 when h is getting close to l.

In conclusion, the computational formula of $\psi(n)$ is given by:

$$\psi(n) = \varphi(n) + \phi_1(n) + \phi_2(n) + \ldots + \phi_l(n) \tag{16}$$

For the fluency of the proof above, we will introduce the following terminology and notations related to the bounded-ETF's and generalized-ETF's invoked in the proof.

Definition 5.1. *Consider the decomposition of $n, n \ge 2$ into prime factors:*

$$n = p_1^{a_1}\, p_2^{a_2} \ldots p_l^{a_l}$$

where $p_1, p_2, \ldots p_l$ are primes such that $p_1 < p_2 < \ldots < p_l$

Using the bounded-ETF from Definition 3.1., we define the following:

($\star$) *(i) the 1-bounded-ETF $\varphi(n, p_k)$ associated with 1-prime-factor p_k of the decomposition of n, for $1 \le k \le l$, as below:*

$$\varphi(n, p_k) = \varphi^{\overline{\frac{n}{p_k^{a_k+1}}}}\left(\frac{n}{p_k^{a_k}}\right)$$

(ii) the summative-1-bounded-ETF $\phi_1(n)$ of the 1-prime-factors bounded-ETF $\varphi(n, p_k)$, $k = 1, 2, \ldots, l$, as below:

$$\phi_1(n) = \sum_{k=1}^{n} \varphi(n, p_k)$$

($\star$) *(i) the 2-bounded-ETF* $\varphi(n, p_k, p_h)$ *associated with 2-prime-factors* p_k, p_h *of the decomposition of* n*, for* $1 \le k < h \le l$*, as below:*

$$\varphi(n, p_k, p_h) = \varphi^{\overline{\frac{n}{p_k^{a_k+1} p_h^{a_h+1}}}} \left(\frac{n}{p_k^{a_k} p_h^{a_h}} \right)$$

(ii) the summative-2-bounded-ETF $\phi_2(n)$ *of the 2-prime-factors bounded-ETF* $\varphi(n, p_k, p_h)$*,* $1 \le k < h \le l$*, as below:*

$$\phi_2(n) = \sum_{1 \le k < h \le l} \varphi(n, p_k, p_h)$$

$\vdots$

($\star$) *(i) the k-bounded-ETF* $\varphi(n, p_{i_1}, p_{i_2}, \dots, p_{i_k})$ *associated with the k-prime-factors* $p_{i_1}, p_{i_2}, \dots, p_{i_k}$ *of the decomposition of* n*, for* $1 \le i_1 < i_2 < \cdots < i_k \le l$*, as below:*

$$\varphi(n, p_{i_1}, p_{i_2}, \dots, p_{i_k}) = \varphi^{\overline{\frac{n}{p_{i_1}^{a_{i_1}+1} p_{i_2}^{a_{i_2}+1} \dots p_{i_k}^{a_{i_k}+1}}}} \left(\frac{n}{p_{i_1}^{a_1} p_{i_2}^{a_2} \dots p_{i_k}^{a_k}} \right)$$

(ii) the summative-k-bounded-ETF $\phi_k(n)$ *of the k-prime-factors bounded-ETF* $\varphi(n, p_{i_1}, p_{i_2}, \dots, p_{i_k})$*,* $1 \le i_1 < i_2 < \cdots < i_k \le l$*, as below:*

$$\phi_k(n) = \sum_{1 \le i_1 < i_2 < \dots < i_k \le l} (n, p_{i_1}, p_{i_2}, \dots, p_{i_k})$$

$\vdots$

($\star$) *(i) the l-bounded-ETF* $\varphi(n, p_1, p_2, \dots, p_l)$ *associated with the l-prime-factors* $p_1, p_2, \dots, p_l$ *of the decomposition of* n*, as below:*

$$\varphi(n, p_1, p_2, \dots, p_l) = \varphi^{\overline{\frac{n}{p_1^{a_1+1} p_2^{a_2+1} \dots p_l^{a_l+1}}}} \left(\frac{n}{p_1^{a_1} p_2^{a_2} \dots p_l^{a_l}} \right)$$

(ii) the summative-l-bounded-ETF $\phi_l(n)$ *of the l-prime-factors bounded-ETF* $\varphi(n, p_1, p_2, \dots, p_l)$*, associated with all the factors of* n*, i.e.* $p_1, p_2, \dots, p_l$*, as below:*

$$\phi_l(n) = \varphi(n, p_1, p_2, \dots, p_l)$$

According to the proof at the beginning of section 5, we are finally able to state the computational formula of the generalized-ETF $\psi(n)$ in terms of the summative-k-bounded-ETF's for $k = 1, 2, \dots, l$ enabled by the Definition 5.1.Therefore:

Theorem 3. *The generalized-ETF $\psi(n)$ is given by*

$$\psi(n) = \varphi(n) + \phi_1(n) + \phi_2(n) + \ldots + \phi_l(n)$$

Corollary 3.1. *For $n \geq 2$, the generalized-ETF $\psi(n)$ has the following properties:*

(i) $\varphi(n) \leq \psi(n) \leq n-1$

(ii) $\psi(n) = n-1$ *iff* n *prime*

(iii) $\psi(p^a) = p^a - p^{a-1}$, *for any* p *prime*

(iv) $\psi(n) = \varphi(n)$ *iff* $n = p^a$, p *prime*

Proof. (i)-(ii) Because $\psi(n)$ represents the number of distinct numerators which appears after completely simplifying the fractions

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

it results that the set of the distinct numerators implicitly contains all the coprimes with n which are smaller or equal than $n-1$. In consequence, it results the first inequality $\varphi(n) \leq \psi(n)$.
Similarly, because $\psi(n)$ represents the number of distinct numerators which appears after completely simplifying the fractions

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

it results that the number of distinct numerators is smaller or equal to the number of fractions above, i.e. $n-1$. In consequence, it results the second inequality $\psi(n) \leq n-1$.
The equality $\psi(n) = \varphi(n) = n-1$ appears when when all the numerators $1, 2, \ldots, n-1$ are coprimes with n, i.e. when n is prime.
(iii) $\varphi(p^a) = p^{a-1}(p-1)$, from Lemma 2.2. According to Definition 5.1., there is an 1-bounded-ETF $\varphi(n,p)$ associated with the prime factor p and, according to formula (7), we can farther calculate the bounded-ETF, as below

$$\varphi(n,p) = \varphi^{\frac{n}{p^{a+1}}}\left(\frac{n}{p^a}\right) = \left[\frac{n}{p^{a+1}}\right] = \left[\frac{p^a}{p^{a+1}}\right] = 0$$

It results that the summative-1-bounded-ETF $\phi_1(n) = 0$. From Theorem 3., we have successively $\psi(n) = \varphi(n) + \phi(n) = \varphi(n) + 0 = p^{a-1}(p-1)$.
(iv) according to Theorem 3., $\psi(n) = \varphi(n)$ is equivalent to $\phi_1(n) = \phi_2(n) = \ldots = \phi_l(n) = 0$, where $n = p_1^{a_1} p_2^{a_2} \ldots p_l^{a_l}$ is the decomposition into prime factors, $p_1 < p_2 < \ldots < p_l$. Note that $\psi_1(n) = 0$ is equivalent to all the 1-bounded-ETF $\varphi(n, p_k) = 0$, for any $1 \leq k \leq l$, or equivalently

$$\varphi(n,p_k) = \varphi^{\frac{n}{p_k^{a+1}}}\left(\frac{n}{p_k^a}\right) = \left[\frac{n}{p_k^{a_k+1}}\right] = \left[\frac{p_1^{a_1} \cdots p_{k-1}^{a_{k-1}} p_{k+1}^{a_{k+1}} \cdots p_l^{a_l}}{p_k}\right] = 0 \tag{17}$$

If $l \geq 2$, it results that $\left[\frac{p_2^{a_2} \cdots p_l^{a_l}}{p_1}\right] \neq 0$, because $p_1 < p_2 < \ldots < p_l$ and $a_1, a_2, \ldots, a_l$ are positive integers. Therefore, (17) is equivalent to $l = 1$ and $n = p^a$. There is only 1-bounded-ETF $\varphi(n,p) = 0$ as in (17) and, obviously, $\phi_1 = 0$. In conclusion, (iv) is proved. □

6 Applications

Example 6.1. *1. Solve in the positive integers the equation $\psi(n) = 4$.*

Proof. From Corollary 3.1. (i) it results $\varphi(n) \leq 4$ and $n \geq 5$. If $n = p_1^{a_1} p_2^{a_2} ... p_l^{a_l}$ is the decomposition of n into prime factors, $p_1 < p_2 < ... < p_l$, then we have

$$p_1^{a_1-1} p_2^{a_2-1} \dots p_l^{a_l-1} (p_1 - 1)(p_2 - 1)) \dots (p_l - 1) \leq 4$$

If $l \geq 3$ then $\varphi(n) \geq (2-1)(3-1)(5-1) = 8$ leads to contradiction. It remains to analyze the cases $l = 1, 2$ for $n \geq 5$. Note that prime factors greater than 7 head to $\varphi(n) \geq 6$ and implicitly $\psi(n) \geq 6$. Therefore, we will consider only potential prime factors of 2, 3 or 5 in the decomposition of n.

l	$\varphi(n)$	n	$\psi(n)$	True/False
1	$2^2(2-1)$	2^3	4	T
1	$(3-1)$	3	3	F
1	$(5-1)$	5	4	T
2	$(2-1)(3-1)$	$2 \cdot 3$	3	F
2	$2(2-1)(3-1)$	$2^2 \cdot 3$	6	F
2	$(2-1)(5-1)$	$2 \cdot 5$	5	F

In conclusion, the solutions are $n = 5, 8$. □

Example 6.2. *(Problem 13, AIME I 2022) Let S be the set of all rational numbers that can be expressed as a repeating decimal in the form $0.\overline{abcd}$, where at least one of the digits a, b, c, d is nonzero. Let N be the number of distinct numerators obtained when numbers in S are written as fractions in lowest terms. For example, both 4 and 410 are counted among the distinct numerators for numbers in S because $0.\overline{abcd} = \dfrac{4}{11}$ and $0.\overline{1230} = \dfrac{410}{3333}$. Find the remainder when N is divided by 1000.*

Proof. Reformulating the problem, it is required to find the number of the distinct numerators that remains after transforming the sequence of fractions

$$\frac{1}{9999}, \frac{2}{9999}, \dots \frac{9998}{9999}$$

into irreducible fractions. This number is the generalized- ETF $\psi(9999)$ according to Theorem 2. Using the computational formula of $\psi(9999)$ described in section 5, we have the following steps:
▹ $9999 = 3^2 \cdot 11 \cdot 101$
▹ $\varphi(9999) = 3 \cdot (3-1) \cdot (11-1) \cdot (101-1) = 6000$
▹ the 1-bounded-ETF's $\varphi(n, 3), \varphi(n, 11), \varphi(n, 101)$ are:

$$\varphi(n, 3) = \varphi^{\frac{n}{3^3}}\left(\frac{n}{3^2}\right) = \left[\frac{n}{3^3}\right] - \left(\left[\frac{n}{3^3 \cdot 11}\right] + \left[\frac{n}{3^3 \cdot 101}\right] - \left[\frac{n}{3^3 \cdot 11 \cdot 101}\right]\right) = 370 - (33 + 3) = 334$$

$$\varphi(n, 11) = \varphi^{\frac{n}{11^2}}\left(\frac{n}{11}\right) = \left[\frac{n}{11^2}\right] - \left(\left[\frac{n}{11^2 \cdot 3}\right] + \left[\frac{n}{11^2 \cdot 101}\right] - \left[\frac{n}{11^2 \cdot 3 \cdot 101}\right]\right) = 82 - 27 = 55$$

$$\varphi(n,101)=\varphi^{\frac{n}{101^2}}\left(\frac{n}{101}\right)=\left[\frac{n}{101^2}\right]-\left(\left[\frac{n}{101^2\cdot 3}\right]+\left[\frac{n}{101^2\cdot 11}\right]-\left[\frac{n}{101^2\cdot 3\cdot 11}\right]\right)=0$$

Therefore, the summative-1-bounded-ETF $\phi_1(n)=334+55=389$.
▷ the 2-bounded-ETF's are $\varphi(n,3,11),\varphi(n,3,101),\varphi(n,11,101)$. Similarly,

$$\varphi(n,3,11)=\varphi^{\frac{n}{3^3}\cdot 11^2}\left(\frac{n}{3^2\cdot 11}\right)=\left[\frac{n}{3^3\cdot 11^2}\right]-\left[\frac{n}{3^3\cdot 11^2\cdot 101}\right]=3$$

$$\varphi(n,3,101)=0,\varphi(n,11,101)=0$$

Therefore, the summative-2-bounded-ETF $\phi_2(n)=3$.
▷ The generalized-ETF $\psi(n)$ is given by

$$\psi(n)=\varphi(n)+\phi_1(n)+\phi_2(n)=6000+389+3=6392$$

Finally, $\psi(n)\equiv 392\,(mod\,1000)$

□

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