Solutions for the Frobenius Number in Three Variables

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Abstract

In this paper we investigate a general method of solving for the Frobenius number of three coprime integers, a, b, c, denoted $g(a, b, c)$, which is the largest positive integer that cannot be expressed in the form $ax + by + cz$ for non-negative integers x, y, z. We divide the problem into three main subcases and show how to solve for the Frobenius number in those cases and give complete solutions for all cases, except for the third case where we impose an additional condition.

1 Introduction

The purpose of this paper is to find the solutions to the Frobenius number in three variables. Solutions for the Frobenius Number for any number of coprime integers have been given by Kannan [4], and for three variables they have been given by Tripathi [5]. Kannan [4], and Tripathi et al. [6] also gave generalizations of the problem. Hujter et al. [3] provided an almost-complete algorithm and solution for the problem and Beck et al. [1] gives empirical insight into approximations of the Frobenius Problem for three variables. The purpose of this paper is to find solutions for the Frobenius number in three variables using a different approach, namely, via a combination of Bézout's Lemma and elementary linear Diophantine analysis. All results presented in this paper are new, with the exception of **Lemma 1** and **Theorem 1**.

Definition 1. Given a set of coprime positive integers, $a_1, a_2, \ldots, a_n$, the Frobenius number, denoted $g(a_1, a_2, \ldots, a_n)$, is defined to be the largest integer inexpressible as a linear combination of these integers with non-negative integer coefficients.

In other words, the Frobenius number is the largest integer that does not have a solution to $g(a_1, a_2, \ldots, a_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for $x_1, x_2, \ldots, x_n \in \mathbb{N}^*$ (for non-negative integers $x_1, x_2, \ldots, x_n$).

1.1 Immediate Results

We first present the following Lemma:

Lemma 1 (Bézout)**.** *Given non-zero integers a, b, c, then there exists integers x, y, z, such that $ax + by + cz = \gcd(a, b, c)$. Furthermore, $\gcd(a, b, c)$ is the least natural number with this property.*

Proof. Consider the set $S := \{m = au + bv + cw \mid u, v, w \in \mathbb{Z},\ m > 0\}$. In words, S is the set of all integers $m > 0$ that can be expressed as a linear combination of integers a, b, c with integral coefficients. It follows that S has a minimal element, as it is nonempty (take $u = a,\ v = b,\ w = c$ for example) and bounded from below (i.e. all integers $m > 0$). Let this minimal element be d. Then, $d = ax + by + cz$ for some integral x, y, z. Trivially, $\gcd(a, b, c) \mid d$. If $k \neq d$ and $k \in S$, then $k = dq + r$ for non-negative $q, r \in \mathbb{N}^*$ and $r < d$. In particular, $r = k - dq$. If r is nonzero, then $r \in S$ and hence contradicts the minimality of d. So, $d \mid k$ as $r = 0$. Hence, we find that $d \mid a,\ d \mid b,\ d \mid c$ (as $a, b, c \in S$) so $d \mid \gcd(a, b, c)$. It follows that $d = \gcd(a, b, c)$. □

Remark 1. The above lemma, given by Bézout, generalizes to n positive variables and can be proven similarily as above. The above proof is actually a generalization of Bézout's original statement which stated that there exists $x, y \in \mathbb{Z}$ such that, for positive integral a, b, $ax + by = \gcd(a, b)$ and the $\gcd(a, b)$ is the least such integer that can be expressible in such a fashion.

We first note that by Bézout's Lemma, we may write any positive integer n as

$$ax + by + cz = n \tag{1}$$

for some $x, y, z \in \mathbb{Z}$ as long as $\gcd(a, b, c) = 1$. Specifically, we may write

$$ax + by + cz = g(a, b, c) \tag{2}$$

provided at least one of x, y, z is negative (due to the definition of the Frobenius number). Suppose that $\min(a, b, c) = a$. We now present the following:

Theorem 1. *The Frobenius number of three positive integers a, b, c, with $a = \min(a, b, c)$, may be written in the form*

$$g(a, b, c) = -a + by + cz \tag{3}$$

with $y, z \geq 0$.

Proof. We may note that $g(a, b, c) + a = ax + by + cz$ for some $x, y, z \geq 0$ by definition of the Frobenius number. Thus, $g(a, b, c) = a(x - 1) + by + cz$. By definition of the Frobenius number, one of $x - 1, y, z < 0$ so we conclude that $x = 0$ since we had assumed $x, y, z \geq 0$. □

Remark 2. This Lemma is also found in Brauer and Shockley [2], but in a related form relating to the largest integer f does not have *positive* integeral solutions x, y, z to $f(a, b, c) = ax + by + cz$. The inclined reader may refer to [2] for this alternate formulation.

2 Results

We denote $\gcd(b, c)$ as d. Then, from the two-variable Bézout's Lemma, there exists integers β and γ such that

$$a = \frac{b}{d}\beta + \frac{c}{d}\gamma \tag{4}$$

We define (β_p, γ_p) to be solutions to (4) such that $|\beta_p - \gamma_p|$ is minimized. With this in mind, we look at three different cases for the Frobenius number of a, b, c.

2.1 Immediate Bounds

Theorem 2. *If we write $g(a, b, c) = -a + by + cz$ for $y, z \geq 0$, then we may state the following bounds on y and z:*

$$y < \frac{a}{\gcd(a, b)} \tag{5}$$

$$z < \frac{a}{\gcd(a, c)} \tag{6}$$

Proof. If $y \geq \frac{a}{\gcd(a,b)}$, then we may note that $g(a, b, c) = a\left(\frac{b}{\gcd(a,b)} - 1\right) + b\left(y - \frac{a}{\gcd(a,b)}\right) + cz$. Hence $g(a, b, c)$ could then be expressed in terms of $ax + by + cz$ for whole numbers x, y, z which contradicts the definition of the Frobenius number. The proof for $z < \frac{a}{\gcd(a,c)}$ follows similarly. □

2.2 Case 1: $\beta_p, \gamma_p \geq 0$

We suppose that both of β_p and γ_p are greater than or equal to 0. Then we may write:

$$g(a, b, c) = -a + by + cz \tag{7}$$

$$= a(dk - 1) + b(y - \beta_p k) + c(z - \gamma_p k) \tag{8}$$

In particular, when k is positive, note that $dk - 1 \geq 0$. So one of $y - \beta_p$ or $z - \gamma_p$ is negative, which can be seen when $k = 1$. Furthermore, for some integer $t > 0$

$$a(dk - 1) + b(y - \beta_p k) + c(z - \gamma_p k)$$

$$= a(dk - 1) + b\left(y - \beta_p k \pm \frac{c}{d}t\right) + c\left(z - \gamma_p k \mp \frac{b}{d}t\right) \tag{9}$$

We now suppose that in (8), $y - \beta_p$ is negative (assuming $z - \gamma_p$ is negative will also give similar results). First we note that we wish to minimize the values of k and t, as it is trivial to see that for large values of k and t the coefficient of c will be negative. Hence, to obtain the strictest possible bound on z, we must minimize k, t.

Theorem 3. *We claim that $\beta_p - 1 < \frac{a}{\gcd(a,b)}$ and $\gamma_p - 1 < \frac{a}{\gcd(a,c)}$.*

Proof. Expressing

$$a = d_1 \cdot d_2 \cdot a'$$
$$b = d_1 \cdot d_3 \cdot b'$$
$$c = d_2 \cdot d_3 \cdot c'$$

where d_1, d_2, d_3 are $\gcd(a,b), \gcd(a,c), \gcd(b,c)$ respectively. Now, $a = \frac{b}{d}\beta_p + \frac{c}{d}\gamma_p = d_1 b' \beta_p + d_2 c' \gamma_p$. Thus, $a \geq d_1 b' \beta_p$, since $\gamma_p \geq 0$, so $\frac{a}{\gcd(a,b)} = d_2 a' \geq b' \beta_p > \beta_p - 1$. We can argue for $\gamma_p - 1 < \frac{a}{\gcd(a,c)}$ similarly. □

Thus, the largest value of y is $\beta_p - 1$, which makes the coefficient of b in (8) as -1 when $k = 1$. Thus, in (9), $t = 1$ makes the coefficient of b positive, so $z < \gamma_p + \frac{b}{d}$.

Theorem 4. *In this case, $y = \beta_p - 1$ and if $\gamma_p + \frac{b}{d} - 1 < \frac{a}{\gcd(a,c)}$, then $z = \gamma_p + \frac{b}{d} - 1$. Otherwise, if $\gamma_p + \frac{b}{d} - 1 \geq \frac{a}{\gcd(a,c)}$, then the maximal z is $z = \frac{a}{\gcd(a,c)} - 1$.*

Proof. We know that $k > 0$ by (8).

$$\beta_p - 1 - \beta_p k + \frac{c}{d}t = \beta_p(1-k) + \frac{c}{d}t - 1 > 0$$

Then, the above inequality, if true, implies that $t > 0$.

$$\gamma_p + \frac{b}{d} - 1 - \gamma_p k - \frac{b}{d}t = \gamma_p(1-k) + \frac{b}{d}(1-t) - 1 > 0$$

But the latter inequality is not true for $k, t > 0$. Therefore, there do not exist any integers k, t such that the coefficients of $a,\ b,\ c$ in (9) are all simultaneously positive for our choice of $y = \beta_p - 1$ and $z = \gamma_p + \frac{b}{d} - 1$. Therefore, for all $z \leq \gamma_p + \frac{b}{d} - 1$ there do not exist such k, t such that the coefficients of $a,\ b,\ c$ in (9) are all positive, for $y = \beta_p - 1$. □

Thus, if $\beta_p, \gamma_p > 0$

$$g(a,b,c) = \begin{cases} -a + b(\beta_p - 1) + c(\gamma_p + \frac{b}{d} - 1), & \gamma_p + \frac{b}{d} - 1 < \frac{a}{\gcd(a,c)} \\ -a + b(\beta_p - 1) + c(\frac{a}{\gcd(a,c)} - 1), & \gamma_p + \frac{b}{d} - 1 \geq \frac{a}{\gcd(a,c)} \end{cases}$$

Example 1. Compute $g(21, 35, 45)$

First compute β_p and γ_p. It's trivial to see that $21 = 7(3 \pm 9t) + 9(0 \mp 7t)$, so $\beta_p = 3, \gamma_p = 0$. Thus, using the first formula, we get

$$g(21, 35, 45) = -21 + 35(2) + 45(6) = 319$$

Example 2. Compute $g(290, 483, 851)$

Note that $\gcd(483, 851) = 23$. For larger numbers, we can find β_p, γ_p with the Extended Euclidean Algorithm. We demonstrate this below:

$$37 = 21(1) + 16 \Longrightarrow 16 = 37 - 21(1)$$
$$21 = 16(1) + 5 \Longrightarrow 5 = 21 - 16(1) = 21(2) - 37(1)$$
$$16 = 5(3) + 1 \Longrightarrow 1 = 16 - 5(3) = 21(-7) + 37(4)$$

So 290 = 21(-2030) + 37(1160) = 21(5) + 37(5). So, $\beta_p = 5, \gamma_p = 5$. Again, using the first formula gives

$$g(290, 483, 851) = -290 + 483(4) + 851(25) = 22917$$

Example 3. Compute $g(91, 92, 136)$

Once again, we start by computing $\gcd(92, 136) = 4$. From this, we realize that $91 = 23(1) + 34(2)$ and so we find that $\beta_p = 1, \gamma_p = 2$. Using the upper branch again yields:

$$g(91, 92, 136) = -91 + 92(1-1) + 136(2 + 23 - 1) = 3173$$

2.3 Case 2:

Here we assume that one of β_p or γ_p is negative (from (4)). Thus, to make γ_p unsigned, we will use

$$a = \frac{b}{d}\beta_p - \frac{c}{d}\gamma_p \tag{10}$$

here, in (10), with both $\beta_p, \gamma_p > 0$. We may now write:

$$\begin{aligned} g(a,b,c) &= -a + by + cz && (11)\\ &= a(dk-1) + b(y - \beta_p k) + c(z + \gamma_p k) && (12)\\ &= a(dk-1) + b\left(y - \beta_p k + \frac{c}{d}t\right) + c\left(z + \gamma_p k - \frac{b}{d}t\right) && (13) \end{aligned}$$

We may assume that $y < \beta_p$, in the case of $k = 1$, which is justified by **Theorem** 1 since we know that there exists whole number values for y and z. Based on this:

Theorem 5. *The system*

$$\begin{cases} y - \beta_p k + \frac{c}{d}t & \geq 0 \\ z + \gamma_p k - \frac{b}{d}t & \geq 0 \end{cases} \tag{14}$$

do not have any solutions for $k > 0$ and $t \geq 0$.

Proof. Since $dk - 1$ is clearly greater than or equal to zero for all positive integral k, by (13) one of $y - \beta_p k + \frac{c}{d}t$ or $z + \gamma_p k - \frac{b}{d}t$ must be negative by definition of the Frobenius number. Thus, the above system (14) has no solutions for positive, integral k and t. □

Corollary. It follows from (14) that

$$\frac{\beta_p k - y}{\frac{c}{d}} \leq t \leq \frac{z + \gamma_p k}{\frac{b}{d}} \tag{15}$$

must have no solutions for $k > 0$ and $t \geq 0$, which is yielded after simple algebraic manipulation of (14).

We wish to find integers y, z such that there do not exist positive integers k, t that satisfy (15). Note that if we fix a particular value of t, and there exist two values $k_1 < k_2$ such that

$$t - 1 < \frac{\beta_p k_1 - y}{\frac{c}{d}} < \frac{\beta_p k_2 - y}{\frac{c}{d}} \leq t$$

then because (15) has no solutions for $k > 0$ and $t \geq 0$

$$\frac{z + \gamma_p k_1}{\frac{b}{d}}, \frac{z + \gamma_p k_2}{\frac{b}{d}} < t$$

From these, we get that

$$z < \frac{b}{d}t - \gamma_p k_1 \tag{16}$$

$$z < \frac{b}{d}t - \gamma_p k_2 \tag{17}$$

Since the more restrictive of these is (17), we seek to maximize k for each integer t to obtain the most restrictive of these bounds. It follows that:

Theorem 6. *For each integer t we seek to find the largest integer k such that*

$$\frac{\beta_p k - y}{\frac{c}{d}} \leq t \tag{18}$$

and then compute its corresponding value of z. Then, we wish to find the minimum such value of z (or the tightest such bound for z), as taken amongst the successive minima obtained for each integer t.

Remark 3. Note that the successive values of k that we are interested in are *typically* $\left\lceil \frac{c}{d\beta_p} \right\rceil$ apart.

Definition 2. We define $\Delta k := \left\lceil \frac{c}{d\beta_p} \right\rceil$.

Remark 4. This is the least integer such that $\frac{c}{d} \leq \beta_p \Delta k$.

Definition 3. We define $k_0 := \left\lfloor \frac{\frac{c}{d}+y}{\beta_p} \right\rfloor$.

Remark 5. This is the greatest integer k_0 such that $\frac{\beta_p k_0 - y}{\frac{c}{d}} \leq 1$.

Hence, the values of k that we wish to test are of the form

$$k = k_0 + \Delta k \cdot x, \quad x \in \mathbb{N}^* \tag{19}$$

However, note that (19) holds true only upto a certain point since there exists some value of x such that

$$\frac{\beta_p(k_0 + \Delta k \cdot x) - y}{\frac{c}{d}} > x + 1 \tag{20}$$

and we further note that the minimum such x is precisely

$$x = \left\lfloor \frac{\frac{c}{d} + y - \beta_p k_0}{\beta_p \Delta k - \frac{c}{d}} \right\rfloor + 1 \tag{21}$$

Definition 4. We define $x := \left\lfloor \frac{\frac{c}{d} - \beta_p k_0 + y}{\beta_p \Delta k - \frac{c}{d}} \right\rfloor$.

that is, the greatest integer less than the the value described in (21).

Furthermore, we note that the typical change in successive values of k, Δk, brings about a change in the successive values of z which we define as

Definition 5. $\delta z := \frac{b}{d} - \gamma_p \Delta k$.

Namely an increase in k by Δk increases t by 1 in (15) and hence the bound on the value of z changes by the quantity δz.

We now split the problem into two subcases.

2.3.1 $\delta z \geq 0$

In this case, the bound on the value z increases as k increases by Δk. More specifically, from $t = x$ to $t = x + 1$ (for x as described in **Definition** 4) note that the value of k will increase, for the first time, less than Δk. The value of z then increases by an amount greater than δz; hence, it still increases. Thus the most restrictive bound on z occurs when $t = 0$ and consequently $k = k_0$ (**Definition** 3).

So,

$$z = \frac{b}{d} - \gamma_p k_0 - 1 \tag{22}$$

which follows from (14) since we want

$$1 \leq \frac{z + \gamma_p k_0}{\frac{b}{d}} \tag{23}$$

to be false.

Now it remains to to find the values of y and consequently k_0. The maximum such y is obviously $y_1 = \beta_p - 1$. For this value of y, its k_0 is $\left\lfloor \frac{\frac{c}{d}+\beta_p-1}{\beta_p} \right\rfloor$. Thus, the least value of y that has the same value of k_0 as $\beta_p - 1$ is

$$y = \beta_p \left\lfloor \frac{\frac{c}{d} + \beta_p - 1}{\beta_p} \right\rfloor - \frac{c}{d}$$

So the maximum value of y such that its value for k_0 is $\left\lfloor \frac{\frac{c}{d}+\beta_p-1}{\beta_p} \right\rfloor - 1$ is

$$y_2 = \beta_p \left\lfloor \frac{\frac{c}{d} + \beta_p - 1}{\beta_p} \right\rfloor - \frac{c}{d} - 1 \tag{24}$$

We can further notice that the greatest value of y that has a k_0 of $\left\lfloor \frac{\frac{c}{d}+\beta_p-1}{\beta_p} \right\rfloor - \ell - 1$ for some positive integral ℓ would decrease the value of y_2 in (24) by $\ell \cdot \beta_p$ and increase the corresponding value of z from (22) by $\ell \cdot \gamma_p$. But then, $c \cdot \ell \cdot \gamma_p - b \cdot \ell \cdot \beta_p < 0$ so this pair of y and z only decreases the value of g when compared to values of y_2 in (24) and its corresponding value of z.

Hence, we only need to test the value of g when $y = y_1, y_2$ and compute their corresponding values of z by (22) take the maximum of these to determine the Frobenius number in this case.

Example 4. Compute $g(7, 11, 15)$.

We, as usual, begin by noting that $7 = 11(2) + 15(-1)$. Hence,

$$\begin{aligned} g(7, 11, 15) &= \max(-7 + 11(1) + 15(11 - 8 - 1), -7 + 11(0) + 15(11 - 7 - 1)) \\ &= \max(34, 38) \\ &= 38 \end{aligned}$$

Example 5. Compute $g(91, 156, 196)$.

Here, we begin by noting that $\gcd(156, 196) = 4$. Furthermore, $91 = 49(13) + 39(-14)$. So

$$\begin{aligned} g(91, 156, 196) = \max(&-91 + 196(12) + 156(49 - 14(3) - 1), \\ &-91 + 196(-1) + 156(49 - 14(2) - 1)) \\ = \max&(3197, 2183) \\ = 3197& \end{aligned}$$

Example 6. Compute $g(16, 21, 45)$.

We again note that $16 = 15(2) + 7(-2)$ so it follows that

$$\begin{aligned} g(16, 21, 45) &= \max(-16 + 45(1) + 21(6), -16 + 45(0) + 21(8)) \\ &= \max(155, 152) \\ &= 155 \end{aligned}$$

2.3.2 $\delta z < 0$

In this particular case, we provide an additional restriction that we will introduce below.

In this case, as we increase k by regular intervals of Δk, z decreases. A simple rearrangement of (15) shows that

$$z = \frac{b}{d}t - \gamma_p k - 1$$

But, we can also obtain two useful equations:

$$\begin{aligned} z &= \frac{b}{d}t - \gamma_p k - 1 && (25) \\ &= \frac{b}{d}t - \gamma_p \left\lfloor \frac{\frac{c}{d}t + y}{\beta_p} \right\rfloor - 1 && (26) \\ &= \frac{b}{d} \left\lceil \frac{\beta_p k - y}{\frac{c}{d}} \right\rceil - \gamma_p k - 1 && (27) \end{aligned}$$

Both of the single-variable equations (26) and (27) can be easily derived from the guiding inequality (15).

We first establish a few immediate results pertaining to this case:

Theorem 7. *$\frac{c}{d}$ is strictly greater than β_p.*

Proof. We prove this via another result, namely, we first show that $\gamma_p < \frac{b}{d}$. Suppose that $\frac{b}{d} \leq \gamma_p$. Then $\beta_p \leq \frac{c}{d}$ since otherwise if $\beta_p > \frac{c}{d}$, we note that this contradicts the assumption that $\mid \beta_p - \gamma_p \mid$ is minimal as we can find a new $\beta = \beta_p - \frac{c}{d}$ and $\gamma = -\gamma_p + \frac{b}{d}$ and we note that the absolute difference between β and γ is smaller than that between β_p and $-\gamma_p$. Contradiction. So $\beta_p \leq \frac{c}{d}$ in this case. However, this would then imply that $a \leq 0$ as per equation (10) which is once again a contradiction. So $\gamma_p < \frac{b}{d}$.

But $\delta z = \frac{b}{d} - \gamma_p \Delta k < 0 \Rightarrow \frac{b}{d} < \gamma_p \left\lceil \frac{c}{d\beta_p} \right\rceil$ and $\frac{b}{\gamma_p d} - \frac{c}{d\beta_p} = \frac{a}{\gamma_p \beta_p} > 0$. So $\frac{b}{d\gamma_p} < \left\lceil \frac{c}{d\beta_p} \right\rceil$ and $\frac{b}{d\gamma_p} > \frac{c}{d\beta_p}$. Thus, $\left\lceil \frac{b}{d\gamma_p} \right\rceil = \left\lceil \frac{c}{d\beta_p} \right\rceil = \Delta k$. Furthermore, since $\frac{b}{d\gamma_p} > 1$, we also have that $\Delta k = \left\lceil \frac{c}{d\beta_p} \right\rceil > 1$ which implies that $\frac{c}{d} > \beta_p$. ☐

Remark 6. We observed that $\frac{b}{d} > \gamma_p$ in this case which is curiously analogous to $\frac{c}{d} > \beta_p$.

Remark 7. We also observed here that $\Delta k = \left\lceil \frac{c}{d\beta_p} \right\rceil = \left\lceil \frac{b}{d\gamma_p} \right\rceil$ which is once again curiously symmetrical.

We now note the following restriction that we have for this case: namely, going forward, we assume that

$$\left\lceil \frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}} \right\rceil \leq \left\lfloor \frac{\gamma_p}{\gamma_p \Delta k - \frac{b}{d}} \right\rfloor \tag{28}$$

Remark 8. We may also note that $\frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}} < \frac{\gamma_p}{\gamma_p \Delta k - \frac{b}{d}}$ is a true statement which can be seen by simple algebraic rearrangement.

The above restriction (28) arises from the following:

Increasing k by Δk will typically increase $\left\lceil \frac{\beta_p k - y}{\frac{c}{d}} \right\rceil$ by 1 and hence the value of t in (15) also increases by 1. But, because $\beta_p \Delta k > \frac{c}{d}$, each increase in k by the quantity Δk effectively adds a remainder of $\beta_p \Delta k - \frac{c}{d}$ to the numerator of $\frac{\beta_p k - y}{\frac{c}{d}}$. Hence, we may also note that there exists an integer x such that after x increases of k by Δk (or after increasing k by $\Delta k \cdot x$), an additional increase by Δk will increase $\left\lceil \frac{\beta_p k - y}{\frac{c}{d}} \right\rceil$ by 2. Hence at this point, k must be increased by $\Delta k - 1$ to cause $\left\lceil \frac{\beta_p k - y}{\frac{c}{d}} \right\rceil$ to increase by 1.

The first or initial jump of $\Delta k - 1$ will cause a total increase of $\beta_p \left\lfloor \frac{c}{d\beta_p} \right\rfloor$ in the numerator. The maximum offset in the numerator (prior to the jump of $\Delta k - 1 = \left\lfloor \frac{c}{d\beta_p} \right\rfloor$) is exactly $\beta_p \Delta k - \frac{c}{d}$. Hence, the maximum value for x would be given as the greatest integer less than or equal to

$$\frac{\frac{c}{d} - \left(\beta_p \left\lfloor \frac{c}{d\beta_p} \right\rfloor - \beta_p \Delta k + \frac{c}{d}\right)}{\beta_p \Delta k - \frac{c}{d}} = \frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}}. \tag{29}$$

So we can then express x as

$$x = \left\lceil \frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}} \right\rceil - 1 \tag{30}$$

We now assume that an increase in $x\Delta k$ results in following:

$$(\delta z + \gamma_p) + \delta z \cdot x \geq 0 \tag{31}$$

This would imply that after reaching a minima in the value of z, when we increase the value of k by at most $(\Delta k - 1) + x \cdot \Delta k$ to reach the next minima in the value of z, the overall value of g only increases (the $\delta z + \gamma_p$ appears from the initial increase in the value of k by $\Delta k - 1$). Thus the minimum value of g would be obtained and would occur at $k_0 + \Delta k \cdot x$ where x is as defined in **Definition** 4. So, recalling that δz is negative,

$$x + 1 \leq \frac{\gamma_p}{\gamma_p \Delta k - \frac{b}{d}} \tag{32}$$

and since x is an integer we can write that (32) is true if and only if

$$x + 1 = \left\lceil \frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}} \right\rceil \leq \left\lfloor \frac{\gamma_p}{\gamma_p \Delta k - \frac{b}{d}} \right\rfloor \tag{33}$$

Hence, based on assumption (28), we can then realize that, loosely stated, every "skip" in the values for k (i.e. when the subsequent value of t requires an increase in k by $\Delta k - 1$), results in a higher value for z. So the optimum value for z (i.e. the tightest restriction on z) occurs at

$$k = k_0 + \Delta k \cdot \left\lfloor \frac{\frac{c}{d} - \beta_p k_0 + y}{\beta_p \Delta k - \frac{c}{d}} \right\rfloor \tag{34}$$

and we simply substitute this value in for z in (27), for each y that we test.

Now it remains to find the values of y. We may note that the values of y_1 and y_2 must still be tested as noted in the case $\delta z \geq 0$. However, with subsequently decreasing values of z we can also find "new values" of y that are also local maxima. In particular, by **Theorem** 7 and (27), we seek to minimize the value of k to maximize z.

So an alternate value of $y < y_2$ (this way k_0 remains the same) could be derived as follows: We seek to reduce the value of the value of x in $k = k_0 + \Delta k \cdot x$ in accordance with (34) for x defined in **Definition** 4. We note that the value of x in y_2 is

$$x = \left\lfloor \frac{\frac{c}{d} - \beta_p \left(\left\lfloor \frac{\frac{c}{d}+\beta_p-1}{\beta_p} \right\rfloor - 1 \right) + \beta_p \left\lfloor \frac{\frac{c}{d}+\beta_p-1}{\beta_p} \right\rfloor - \frac{c}{d} - 1}{\beta_p \Delta k - \frac{c}{d}} \right\rfloor \tag{35}$$

$$= \left\lfloor \frac{\beta_p - 1}{\beta_p \Delta k - \frac{c}{d}} \right\rfloor \tag{36}$$

Remark 9. Note that the value of x for $y = y_2$ is exactly the value of x described in (30).

So to reduce this x-value in (36) by 1, we substitute in the value obtained in (36) into the inequality (20) to obtain

$$y_3 = \beta_p \left\lceil \frac{\frac{c}{d} - 1}{\beta_p} \right\rceil - \frac{c}{d} + \left(\beta_p \Delta k - \frac{c}{d} \right) \left\lfloor \frac{\beta_p - 1}{\beta_p \Delta k - \frac{c}{d}} \right\rfloor - 1 \tag{37}$$

From this point on, to reduce the value of x further by 1, it requires that y be reduced by $\beta_p \Delta k - \frac{c}{d}$ as per (21). Doing so will increase z by $\gamma_p \Delta k - \frac{b}{d}$. But then, $\frac{b}{d}\left(\frac{c}{d} - \beta_p \Delta k\right) + \frac{c}{d}\left(\gamma_p \Delta k - \frac{b}{d}\right) = -a\Delta k < 0$ and hence gives a lesser value of g; so y_3 is a relative maxima for the value of g. So we only have to test $y_1,\ y_2,\ y_3$ and the corresponding values of z as in (27) and subsequently find the maximum possible value for $-a + by + cz$ in order to find the Frobenius number of a, b, c in this case (under the assumption that (28) is true).

Example 7. Compute $g(44, 57, 82)$

As always, we begin by noting that $44 = 82(20) - 57(28)$. Furthermore, $\Delta k = 3$ and $\delta z = -2$. Validating the assumption (28) shows that $7 \leq 14$ and thus is true. So we now test $y_1 = 19, y_2 = 2, y_3 = 0$ and consequently the corresponding critical values of k by equation (36) are $k_1 = 18, k_2 = 20, k_3 = 17$. This gives values of z as $z_1 = -13, z_2 = 13, z_3 = 15$ and thus gives values for g as $773, 861, 811$ respectively. The maximum of these is clearly $g(44, 57, 82) = 861$.

Example 8. Compute $g(31, 38, 65)$

A quick check reveals that $31 = 65(11) - 38(18)$. We then note that $\delta z = -7$ and $\Delta k = 4$. Checking to see that the assumption (28) holds, we see that indeed $1 < 2$ and the hence our assumption holds. Then, we find that the possible values for y are $y_1 = 10, y_2 = 5, y_3 = 0$ and their consequent values for z are $z_1 = -8, z_2 = 3, z_3 = 10$. So our possible values for g are $315, 408, 349$ and since 408 is indeed the maximum of these, we conclude that $g(31, 38, 65) = 408$.

Remark 10. It may be noted that the inequality in (28) appears to be true in many cases. However, we note that in cases such as $g(176,\ 360,\ 457)$, the assumption (28) is false. Indeed, in this case, we reach that $\delta z = -29 < 0$ and $176 = 457(128) + 360(-162)$, and yet $\left\lceil \frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}} \right\rceil = 6$ and $\left\lceil \frac{\gamma_p}{\gamma_p \Delta k - \frac{b}{d}} \right\rceil = 5$. Hence, (28) is invalid in this particular instance. Therefore, the next step would be to solve for the Frobenius number of such an instance where the inequality (28) is false, i.e. specifically when $\left\lceil \frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}} \right\rceil = \left\lceil \frac{\gamma_p}{\gamma_p \Delta k - \frac{b}{d}} \right\rceil + 1$, since $\frac{\beta_p}{\beta_p \Delta k - \frac{c}{d}} < \frac{\gamma_p}{\gamma_p \Delta k - \frac{b}{d}}$.

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