Junior Problems

J613. Find all integers $n \geq 2$ for which both $n-1$ and n^2+1 divide n^4+2039.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J614. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{\sqrt{1+b^2c+bc^2}} + \frac{b}{\sqrt{1+c^2a+ca^2}} + \frac{c}{\sqrt{1+a^2b+ab^2}} \geq \frac{a+b+c}{\sqrt{3}}.$$

Proposed by Mircea Becheanu, Canada

J615. Prove that in any triangle ABC,

$$\frac{m_b m_c}{(m_b+m_c)^2} \leq \frac{2a^2+bc}{8a^2+(b+c)^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J616. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a(b+c)^2}{a+3} + \frac{b(c+a)^2}{b+3} + \frac{c(a+b)^2}{c+3} \leq 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J617. Prove that triangle ABC is equilateral if and only if

$$2\left(a^2\cos A + b^2\cos B + c^2\cos C\right) \geq \sqrt{3\left(a^4+b^4+c^4\right)}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

J618. Let ABC be a triangle and let w_a, w_b, w_c be the lengths of its angle bisectors. Prove that

$$w_a\left(bc-a^2\right) + w_b\left(ca-b^2\right) + w_c\left(ab-c^2\right) \geq 0.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S613. Solve the equation

$$2(\sin x + \cos x) + \sec x + \csc x = 4\sqrt{2}.$$

Proposed by Adrian Andreescu, USA

S614. Let a, b, c be positive real numbers such that $(a+b)(b+c)(c+a) = 9abc$. Prove that

$$\sqrt[3]{2abc} \le \max(a, b, c) \le \sqrt[3]{4abc}.$$

Proposed by Marius Stănean, Zalău, România

S615. Let ABC be a triangle. Prove that

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \ge \frac{4\left(a^2+b^2+c^2\right)}{ab+bc+ca} - \frac{2r}{R}$$

Proposed by Titu Zvonaru, Comăneşti, România

S616. Let ABC be a triangle with centroid G. The ray AG intersects the side BC and the circumcircle at A_1, A_2, respectively. Pairs of points B_1, B_2 and C_1, C_2 are defined similarly. Prove that

(i) $A_1A_2 + B_1B_2 + C_1C_2 \ge \dfrac{\sqrt{3}}{2} \cdot \dfrac{a^3+b^3+c^3}{a^2+b^2+c^2}$.

(ii) $\dfrac{A_1A_2}{AA_2 + 2A_1A_2} + \dfrac{B_1B_2}{BB_2 + 2B_1B_2} + \dfrac{C_1C_2}{CC_2 + 2C_1C_2} = \dfrac{1}{2}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S617. Consider the polynomial $f(X) = 1 + X + X^2 + \cdots + X^{2023}$. Find the coefficient of X^{2023} in the polynomial $f(X^5)f(X^9)$.

Proposed by Mircea Becheanu, Canada

S618. Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$\frac{1}{abc} + \frac{4\sqrt{2}}{a^2+b^2+c^2} \ge \frac{9+4\sqrt{2}}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U613. Prove that there are infinitely many polynomials $P(x)$ with real coefficients such that

$$P(x)^2 + P(y)^2 + P(z)^2 + 2P(x)P(y)P(z) = 1,$$

for all real numbers x, y, z which satisfy the condition $x^2 + y^2 + z^2 = xyz + 4$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U614. Let $a_1, a_2, \ldots, a_n, \ldots$ be positive real numbers. Consider the series

$$S_1 = \sum_{k=3}^{\infty} \frac{1}{a_k(\ln(\ln a_k))^r}, \qquad S_2 = \sum_{k=3}^{\infty} \frac{1}{a_k(\ln(\ln k))^s}$$

where $s \geq r > 0$ are real numbers. Prove that if S_1 converges then S_2 converges.

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

U615. Evaluate

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos(\sin(\cos x)) - 1}{\left(\frac{\pi}{2} - x\right)^2}$$

Proosed by Mircea Becheanu, Canada

U616. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that

$$f(f(f(x))) + f(f(x)) - f(x) - x = 0,$$

for all $x \in \mathbb{R}$. Prove that $f(f(x)) = x$ for all $x \in \mathbb{R}$.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

U617. The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$; $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Evaluate

$$\lim_{n \to \infty} \left[\left(1 + \frac{F_1}{F_{2n}}\right) \left(1 + \frac{F_2}{F_{2n}}\right) \cdots \left(1 + \frac{F_n}{F_{2n}}\right) \right]^{L_n}.$$

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

U618. Let d be a positive integer and $n = \binom{d}{2}$. Let $z_1, \ldots, z_d$ be complex numbers on a unit circle. For every integers i, j such that $1 \leq i < j \leq d$ we consider the positive real number $x = |z_i - z_j|^2$. These numbers are arranged in some order to obtain a sequence $x_1, x_2, \ldots, x_n$. Prove that

$$\sum_{1 \leq i < j \leq n} x_i x_j \leq \frac{d^4 - 3d^2}{2}$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Olympiad Problems

O613. Solve in integers the equation
$$x^3 - 7xy + y^3 = 2023.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O614. Let a, b, c, d be positive real numbers such that
$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+d} = 1.$$
Prove that
$$abc + bcd + cda + dab + 36 \geq 12(a + b + c + d).$$

Proposed by Marius Stănean, Zalău, România

O615. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that
$$abc(\sqrt{a^3} + \sqrt{b^3} + \sqrt{c^3}) \leq 3.$$

Proposed by Tran Tien Manh, Vinh City, Vietnam

O616. Let a, b, c be positive numbers such that $a + b + c = 3$. Prove that
$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 2\left(a^2 + b^2 + c^2\right) - (ab + bc + ca).$$

Proposed by An Zhenping, Xianyang Normal University, China

O617. Let $a < b < c < d$ be positive integers. Prove that
$$\gcd(a! + 1, b! + 1, c! + 1, d! + 1) < d^{\frac{d-a}{3}}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O618. Let ABC be an acute triangle. Prove that
$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - \frac{3}{2} \geq k\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - \frac{1}{8}\right),$$
where $k = 4\left(1 + \sqrt{2} - \sqrt{2 + \sqrt{2}}\right)^2$. When does equality hold?

Proposed by Marius Stănean, Zalău, România