Junior problems

J607. Find all integers k such that

$$(3m^2 - 4n^2)^2 + (4m^2 - 3n^2)^2 + 7mn(10m^2 + kmn + 10n^2)$$

is a perfect square for all integers m and n.

Proposed by Adrian Andreescu, Dallas, USA

Solution by Polyahedra, Polk State College, USA

Letting $(m,n) = (1,1)$ and $(m,n) = (1,-1)$, we get $142 + 7k = a^2$ and $-138 + 7k = b^2$ for some nonnegative integers a and b. Considering the quadratic residues modulo 7, we see that $a \equiv \pm 3$ and $b \equiv \pm 3 \pmod 7$, thus $\{a+b, a-b\} \subseteq \{0, \pm 1\} \pmod 7$. Since $(a+b)(a-b) = 280 = 2^3 \cdot 5 \cdot 7$, $a+b$ and $a-b$ must be both even, and it is then easy to deduce that $a+b = 20$ and $a-b = 14$. Therefore, $a = 17$ and $b = 3$, which yields $k = 21$. Finally, if $k = 21$, then the given expression becomes $(5m^2 + 5n^2 + 7mn)^2$.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romanial; Daniel Văcaru, Pitești, România; Moubinool Omarjee, Lycée Henri IV, Paris, France; Sundaresh Harige, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Adam John Frederickson, Utah Valley University, UT, USA; Ray Kwak, Gwinnett School of Mathematics Science and Technology, GA, USA; Ivan Hadinata, Jember, Indonesia.

J608. Let a, b, c, d be integers such that ad is odd and bc is even. Prove that if all solutions to the equation

$$ax^3 + bx^2 + cx + d = 0 \qquad (*)$$

are real numbers then at least one of them is irrational.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Daniel Pascuas, Barcelona, Spain

We only have to prove that if a, b, c, d are integers such that ad is odd and the three solutions q_1, q_2, q_3 to the equation $(*)$ are rational numbers, then bc is odd.

Let $q_j = \frac{m_j}{n_j}$ be the irreducible fraction expression of q_j, that is, m_j and n_j are coprime integers and $n_j > 0$. Since a and d are integers and ad is odd, both a and d are also odd, and in particular $a \neq 0$ and $d \neq 0$. Then the rational root theorem shows that m_j and n_j divide d and a, respectively, and so both m_j and n_j are odd. It follows that $q_j = \frac{m'_j}{n}$, where $n = n_1 n_2 n_3$, $m'_1 = m_1 n_2 n_3$, $m'_2 = n_1 m_2 n_3$, and $m'_3 = n_1 n_2 m_3$ are odd integers. Now, since

$$ax^3 + bx^2 + cx + d = a(x - q_1)(x - q_2)(x - q_3) = a\left(x - \frac{m'_1}{n}\right)\left(x - \frac{m'_2}{n}\right)\left(x - \frac{m'_3}{n}\right),$$

we have that $b = -a\,\dfrac{m'_1 + m'_2 + m'_3}{n}$ and $c = a\,\dfrac{m'_1 m'_2 + m'_2 m'_3 + m'_3 m'_1}{n^2}$, so

$$n^3 bc = -a^2(m'_1 + m'_2 + m'_3)(m'_1 m'_2 + m'_2 m'_3 + m'_3 m'_1). \qquad (\star)$$

But n^3 and $-a^2(m'_1 + m'_2 + m'_3)(m'_1 m'_2 + m'_2 m'_3 + m'_3 m'_1)$ are odd integers (because so are n, a, m'_1, m'_2, and m'_3). Therefore $(\star)$ shows that bc must be also odd.

Also solved by Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Polyahedra, Polk State College, USA; Adam John Frederickson, Utah Valley University, UT, USA; Ivan Hadinata, Jember, Indonesia; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, România; Moubinool Omarjee, Lycée Henri IV, Paris, France; Soham Dutta, DPS Ruby Park, West Bengal, India; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J609. Find all triples (x, y, z) of real numbers for which

$$x^2 - 9y - 2\sqrt{z-4} + 22 = y^2 - 9z - 2\sqrt{x-4} + 22 = z^2 - 9z - 2\sqrt{y-4} + 22 = 0.$$

Proposed by Mihaly Bencze, Braşov, România and Neculai Stanciu, Buzău, România

Solution by Daniel Văcaru, Pitești, România

Summing these equations, we obtain

$$x^2 - 9y - 2\sqrt{z-4} + 22 = y^2 - 9z - 2\sqrt{x-4} + 22 = x^2 + y^2 + z^2 - 9x - 9y - 9z - 2\sqrt{x-4} - 2\sqrt{y-4} - 2\sqrt{z-4} + 66 = 0 \, (1)$$

We can write (1) as

$$x^2 - 10x + 25 + y^2 - 10y + 25 + z^2 - 10z + 25 + x - 2\sqrt{x-4} + 1 + y - 2\sqrt{y-4} + 1 + z - 2\sqrt{z-4} + 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow (x-5)^2 + (y-5)^2 + (z-5)^2 + \left(\sqrt{x-4} - 1\right)^2 + \left(\sqrt{y-4} - 1\right)^2 + \left(\sqrt{z-4} - 1\right)^2 = 0.$$

It follows that $x = y = z = 5$, because sum of squares is 0 only when all terms are 0.

Also solved by Ivan Hadinata, Jember, Indonesia; Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Polyahedra, Polk State College, USA; Ray Kwak, Gwinnett School of Mathematics Science and Technology, GA, USA; G. C. Greubel, Newport News, VA, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Theo Koupelis, Cape Coral, FL, USA; Tigran Gevorgyan, Quantum College, Yerevan, Armenia; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J610. Let a, b, c be pairwise distinct real numbers in the interval $[0, 1]$. Find the minimum value of

$$\frac{1}{\sqrt{|a-b|}} + \frac{1}{\sqrt{|b-c|}} + \frac{1}{\sqrt{|c-a|}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Theo Koupelis, Cape Coral, FL, USA

Without loss of generality let the value of b be between the values of a and c. The minimum value of the given expression is given when the denominators of the three fractions achieve their maximum values. This is clearly the case when a and c take the values 0 and 1. Therefore, we need to examine the cases $(a, b) = (0, 1)$ and $(1, 0)$. In both cases we get

$$\begin{aligned} S_{\min}(a,b,c) = S(0,b,1) = S(1,b,0) = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{1-b}} + 1 \geq \frac{2}{[b(1-b)]^{1/4}} + 1 \\ = \frac{2}{\left[\frac{1}{4} - \left(b - \frac{1}{2}\right)^2\right]^{1/4}} + 1 \geq 2 \cdot 4^{1/4} + 1 = 2\sqrt{2} + 1. \end{aligned}$$

Equality occurs when $(a, b, c) = (0, \frac{1}{2}, 1), (1, \frac{1}{2}, 0)$ and their cyclic permutations.

Also solved by Ivan Hadinata, Jember, Indonesia; Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Polyahedra, Polk State College, USA; Ray Kwak, Gwinnett School of Mathematics Science and Technology, GA, USA; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J611. Let a, b, c be positive real numbers. Prove that

$$3(ab+bc+ca) \le a\sqrt{b^2+8c^2} + b\sqrt{c^2+8a^2} + c\sqrt{a^2+8b^2} \le (a+b+c)^2$$

Proposed by Tran Tien Manh, Vinh City, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

RHS we use the following inequality taken from the book *"Straight from the Book", Titu Andreescu and Gabriel Dospinescu, XYZ Press, page 222*

$$\sqrt{c^2+8b^2} \le 3b+c-\frac{3bc}{2b+c} \iff \frac{4b^2(b-c)^2}{(2b+c)^2} \ge 0$$

Our inequality is impled by

$$\sum_{\text{cyc}} a\left(3b+c-\frac{3bc}{2b+c}\right) \le (a+b+c)^2$$

Moreover by Cauchy–Schwarz

$$\sum_{\text{cyc}} \frac{1}{2b+c} \ge \frac{9}{3(a+b+c)} = \frac{3}{a+b+c}$$

$$\sum_{\text{cyc}} 4ab - \frac{3abc}{a+b+c} \le (a+b+c)^2 \iff \sum_{\text{cyc}} a^3 + 3abc \ge \sum_{\text{cyc}} a^3b + ab^3$$

which is Schür's inequality

L.h.s. By the concavity of the function $\sqrt{x}$

$$\sqrt{b^2+c^2+\ldots+c^2} \ge (a+8c)/3$$

thus

$$3(ab+bc+ca) \le \sum_{\text{cyc}} \frac{a^2+8ac}{3} \iff a^2+b^2+c^2 \ge ab+bc+ca$$

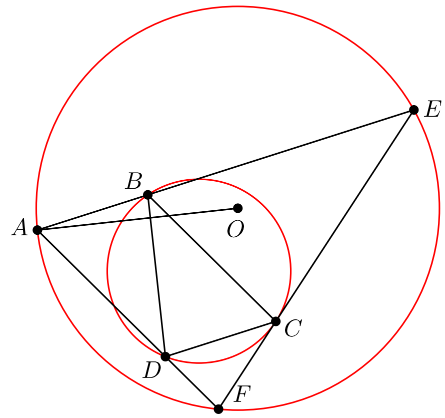
which evidently holds true.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Cape Coral, FL, USA.

J612. Let $ABCD$ be a parallelogram. The tangent at C to the circumcircle of the triangle BCD intersects line AB at E and line AD at F. Let O be the center of the circumcircle of the triangle AEF. Prove that AO is perpendicular to BD.

Proposed by Mihai Miculiţa, Oradea, România

Solution by Polyahedra, Polk State College, USA



Since $\angle DAB = \angle CBE$ and $\angle ABD = \angle CDB = \angle ECB$, we have $\angle BDA = \angle BEC$, so BD is an antiparallel to EF in $\triangle AEF$. Since O and the orthocenter of $\triangle AEF$ are isogonal conjugates with respect to $\triangle AEF$, the orthocenter of $\triangle ABD$ lies on AO.

Also solved by Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Ray Kwak, Gwinnett School of Mathematics Science and Technology, GA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Theo Koupelis, Cape Coral, FL, USA; Marios Zarogoulidis, Thessaloniki, Greece.

Senior problems

S607. Let k be a positive integer, $a = 1 + k^2(2k^2 + 1)(2k^2 + 2k + 1)$, and let $b = 2k^2$. Prove that $ab^n + 1$ is composite for any positive integer n.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by Theo Koupelis, Cape Coral, FL, USA

Let

$$S_n := ab^n + 1 = b^n + 1 + b^n \cdot k^2(b + 1)(2k^2 + 2k + 1).$$

(i) If $n = 2m + 1$, where m is a non-negative integer, then $(b + 1) \mid (b^{2m+1} + 1)$; therefore $(b + 1) \mid S_{2m+1}$, and S_{2m+1} is composite.

(ii) If $n = 4m + 2$, working modulo $(2k^2 + 2k + 1)$ we get

$$S_{4m+2} \equiv (b^{4m+2} + 1) \equiv (4k^4)^{2m+1} + 1 \equiv (4k^4 + 1) \equiv 0 \mod (2k^2 + 2k + 1),$$

because $4k^4 + 1 = (2k^2 + 2k + 1)(2k^2 - 2k + 1)$. Thus, S_{4m+2} is composite.

(iii) If $n = 4m$, then

$$\begin{aligned} S_{4m} &= b^{4m} + 1 + b^{4m} \cdot k^2(2k^2 - 2k + 1 + 2k)(2k^2 - 2k + 1 + 4k) \\ &= b^{4m} + 1 + b^{4m} \cdot k^2[(2k^2 - 2k + 1)^2 + 6k(2k^2 - 2k + 1) + 8k^2] \\ &= b^{4m} + 1 + b^{4m} \cdot [k^2(2k^2 - 2k + 1)^2 + 6k^3(2k^2 - 2k + 1) + 2(4k^4 + 1) - 2] \\ &= 1 - b^{4m} + b^{4m} \cdot (2k^2 - 2k + 1) \cdot (2k^4 + 4k^3 + 5k^2 + 4k + 2). \end{aligned}$$

Working modulo $(2k^2 - 2k + 1)$ we get

$$\begin{aligned} S_{4m} &\equiv 1 - b^{4m} \equiv -[b^{2m} - 1][b^{2m} + 1] \equiv -[(4k^4)^m - 1][(4k^4)^m + 1] \\ &\equiv -[(-1)^m - 1][(-1)^m + 1] \equiv 0 \mod (2k^2 - 2k + 1), \end{aligned}$$

and thus S_{4m} is composite.

Therefore, S_n is composite for any positive integer n.

Also solved by Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Sundaresh Harige, India.

S608. Prove that:

$$\frac{3}{a^3+b^3+c^3} \le \frac{1}{a(a^2+2bc)} + \frac{1}{b(b^2+2ac)} + \frac{1}{c(c^2+2ab)} \le \frac{1}{abc}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Marian Ursărescu, Roman-Vodă National College, Roman, Romania

For the left side, we use Titu's inequality:

$$\frac{1^2}{a(a^2+2bc)} + \frac{1^2}{b(b^2+2ac)} + \frac{1^2}{c(c^2+2ab)} \ge \frac{9}{a^3+b^3+c^3+6abc}$$

We must show: $\frac{9}{a^3+b^3+c^3+6abc} \ge \frac{3}{a^3+b^3+c^3} \Leftrightarrow$
$3(a^3+b^3+c^3) \ge a^3+b^3+c^3+6abc \Leftrightarrow a^3+b^3+c^3 \ge 3abc$, true because its AM-GM inequality.

For the right side, we must show:

$$\frac{bc}{a(a^2+2bc)} + \frac{ca}{b(b^2+2ac)} + \frac{ab}{c(c^2+2ab)} \le 1 \Leftrightarrow$$

$$\frac{2bc}{a(a^2+2bc)} + \frac{2ca}{b(b^2+2ac)} + \frac{2ab}{c(c^2+2ab)} \le 2 \Leftrightarrow$$

$$\sum_{cyc} \frac{a^2+2bc-a^2}{a^2+2bc} \le 2 \Leftrightarrow \sum_{cyc}\left(1-\frac{a^2}{a^2+2bc}\right) \le 2 \Leftrightarrow$$

$$3-\sum_{cyc}\frac{a^2}{a^2+2bc} \le 2 \Leftrightarrow \sum_{cyc}\frac{a^2}{a^2+2bc} \ge 1; \qquad (1)$$

Again, by Titu's inequality:

$$\sum_{cyc}\frac{a^2}{a^2+2bc} \ge \frac{(a+b+c)^2}{a^2+b^2+c^2+2ab+2bc+2ca} = 1 \Rightarrow (1)$$

its true.

Also solved by Ivan Hadinata, Jember, Indonesia; Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Daniel Pascuas, Barcelona, Spain; Daniel Văcaru, Pitești, România; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S609. Let ABC be a triangle with medians m_a, m_b, m_c and area Δ. Prove that

$$\sqrt{2(m_am_b + m_bm_c + m_cm_a - 3\sqrt{3}\Delta)} \geq \max\{m_a, m_b, m_c\} - \min\{m_a, m_b, m_c\}.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Applying this inequality to the triangle with sides $\left(\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c\right)$ and medians $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$, the proposed inequality becomes

$$\sqrt{2} \cdot \sqrt{ab + bc + ca - 4S\sqrt{3}} \geq \max\{a, b, c\} - \min\{a, b, c\}.$$

or assuming that $a \geq b \geq c$

$$ab + bc + ca - 4S\sqrt{3} \geq \frac{(a-c)^2}{2},$$

that is

$$2ab + 2bc + 4ac - a^2 - c^2 \geq 8S\sqrt{3}.$$

We prove that

$$2ab + 2ac - a^2 \geq 4S\sqrt{3}, \qquad 2bc + 2ac - c^2 \geq 4S\sqrt{3}.$$

For the first inequality (second similar) with Ravi's substitutions, i.e. $a = y + z$, $b = z + x$, $c = x + y$, this becomes

$$(y + z)(4x + y + z) \geq 4\sqrt{3xyz(x + y + z)},$$

which follows from the AM-GM Inequality,

$$y + z \geq 2\sqrt{yz},$$

$$4x + y + z = 3x + x + y + z \geq 2\sqrt{3x(x + y + z)}.$$

Also solved by Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S610. Prove that in any triangle ABC,

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{1}{\sqrt{3}}\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author

The desired inequality is equivalent to

$$\frac{1}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} + \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \ge \sqrt{3}.$$

Setting

$$\tan\frac{A}{2} = x, \tan\frac{B}{2} = y, \tan\frac{C}{2} = z$$

yields

$$xy + yz + zx = 1$$

and the inequality becomes

$$\sqrt{(1+x^2)(1+y^2)(1+z^2)} + xyz \ge \sqrt{3}.$$

Note that

$$1 + x^2 = xy + yz + zx + x^2 = (x+y)(x+z).$$

Similarly

$$1 + y^2 = (y+z)(y+x),$$
$$(1+z^2) = (z+x)(z+y).$$

Hence our inequality reduces to

$$(x+y)(y+z)(z+x) + xyz \ge \sqrt{3}$$

or

$$(x+y+z)(xy+yz+zx) \ge \sqrt{3}.$$

Now we have

$$x + y + z \ge \sqrt{3(xy+yz+zx)} = \sqrt{3}$$

and the proof is completed.

Also solved by Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Ivan Hadinata, Jember, Indonesia; Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Daniel Văcaru, Pitești, România; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Mihaly Bencze, Braşov, România; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Marian Ursărescu, Roman-Vodă National College, Roman, Romania; Titu Zvonaru, Comănești, România.

S611. Let p, q be two prime numbers such that $p \equiv -23 \pmod{60}$, $q \equiv -47 \pmod{120}$ and $q = 2p - 1$, we shall prove that $n = pq$ satisfies the following relations.

$$n | 2^n - 2, n | 3^n - 3, \; n \nmid 5^n - 5.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Not that $p \neq q$. Furthermore, $q \equiv 1 (\mod 8)$ and $q \equiv 1 (\mod 12)$. Hence, q divides $2^{\frac{q-1}{2}} - 1, 3^{\frac{q-1}{2}} - 1$. Thus, q divides $2^{p-1} - 1, 3^{p-1} - 1$. It follows that pq divides $2^{p-1} - 1, \; 3^{p-1} - 1$. Note that

$$pq - 1 = (p - 1)(2p + 1).$$

Hence,

$$p.q | 2^{pq-1} - 1 | 2^{pq} - 2.$$

By the same argument

$$p.q | 3^{pq-1} - 1 | 3^{pq} - 3.$$

Now, we shall prove that pq doesn't divide $5^{pq} - 5$. Indeed, $q \equiv -7 \pmod{20}$ and hence $5^{\frac{q-1}{2}} = 5^{p-1} \equiv -1 (\mod q)$. Hence,

$$5^{pq-1} = 5^{(p-1)(2p+1)} \equiv (5^{p-1})^{2p+1} \equiv (-1)^{2p+1} \equiv -1 (\mod q).$$

We are done.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romanial; Theo Koupelis, Cape Coral, FL, USA.

S612. Given a triangle ABC and a point O on the plane. Define $D = AO \cap BC, E = BO \cap AC, F = CO \cap AB, Q = DF \cap AC, R = DE \cap AB$ and $AD \cap RQ = S$. Let the line passing through B and parallel to DE intersects AC at X and the line passing through C and parallel to DF intersects AB at Y. Let $XY \cap BC = T$. If SB, SC, SE, SF intersect AT at K, L, M, N; prove that $|KN| = |NA| = |AM| = |ML|$.

Proposed by Bariş Koyunku, INKA Schools, Istanbul, Turkey

Solution by the author

Let $P = EF \cap BC$. Since AD, BE and CF are concurrent, from Desargues's theorem, it follows that P, Q, R are collinear. Also, we have $(P, D; B, C) = -1$. Hence, $(SP \cap AT, A; K, L) \overset{S}{=} (P, D; B, C) = -1$. Then, in order to prove that $|AK| = |AL|$, we need to show that $SP // AT \Leftrightarrow RQ // AT$. By Menelaus's theorem, we get

$$\frac{|TC|}{|TB|} \cdot \frac{|BY|}{|YA|} \cdot \frac{|AX|}{|XC|} = 1 \Rightarrow \frac{|TC|}{|TB|} = \frac{|YA|}{|BY|} \cdot \frac{|XC|}{|AX|}$$

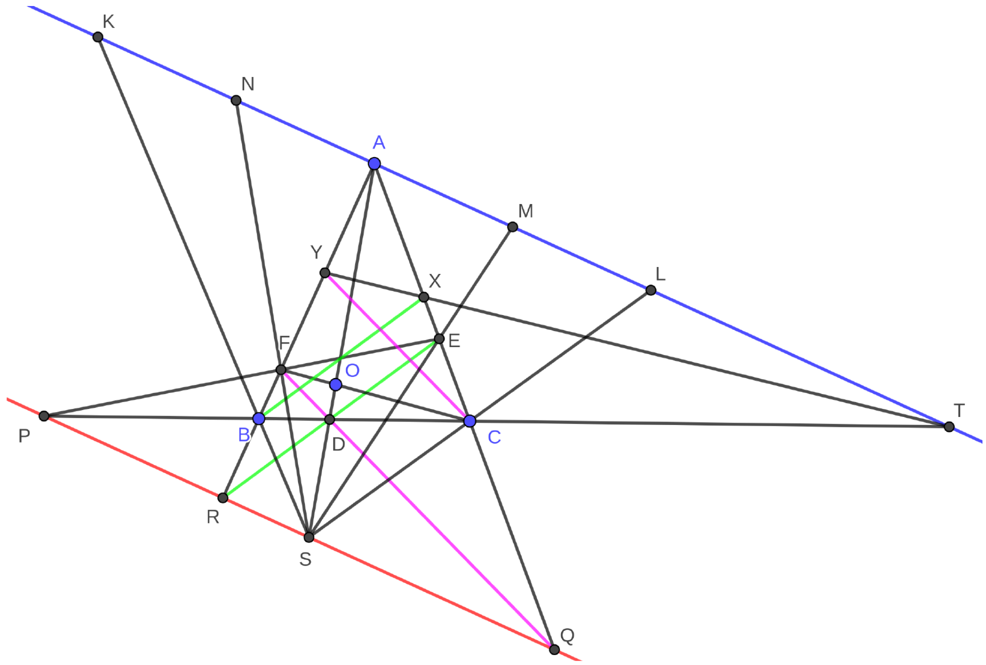
Moreover, since $DF // CY$, we find that $\frac{|AY|}{|AF|} = \frac{|AC|}{|AQ|}$ and $\frac{|BF|}{|BY|} = \frac{|BD|}{|BC|}$. Multiply these to get $\frac{|AY|}{|BY|} = \frac{|AC| \cdot |BD| \cdot |AF|}{|AQ| \cdot |BC| \cdot |BF|}$.

Simiarly, since $DE // BX$, we find that $\frac{|XC|}{|EC|} = \frac{|BC|}{|DC|}$ and $\frac{|AE|}{|AX|} = \frac{|AR|}{|AB|}$. Multiply these to get $\frac{|XC|}{|AX|} = \frac{|BC| \cdot |AR| \cdot |EC|}{|DC| \cdot |AB| \cdot |AE|}$.

Also, by Ceva's theorem, we get $\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1$. Hence,

$$\frac{|TC|}{|TB|} = \frac{|YA|}{|BY|} \cdot \frac{|XC|}{|AX|} = \frac{|AC| \cdot |BD| \cdot |AF|}{|AQ| \cdot |BC| \cdot |BF|} \cdot \frac{|BC| \cdot |AR| \cdot |EC|}{|DC| \cdot |AB| \cdot |AE|} =$$

$$\frac{|BC|}{|BC|} \cdot \left(\frac{|AF|}{|BF|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|EC|}{|AE|} \right) \cdot \frac{|AC|}{|AB|} \cdot \frac{|AR|}{|AQ|} = \frac{|AC|}{|AB|} \cdot \frac{|AR|}{|AQ|} \Longrightarrow \frac{|TC|}{|TB|} = \frac{|AC|}{|AB|} \cdot \frac{|AR|}{|AQ|}.$$



Assume that the line passing through B and parallel to RQ intersects AC at B'. We get $\dfrac{|TC|}{|TB|} = \dfrac{|AC|}{|AB|} \cdot \dfrac{|AR|}{|AQ|} = \dfrac{|AC|}{|AB|} \cdot \dfrac{|AB|}{|AB'|} = \dfrac{|AC|}{|AB'|} \Rightarrow AT//BB'$. Hence, $AT//RQ$. Yields, $|AK| = |AL|$.

Also, since $RQ//AT$, we get $(\infty, N; K, A) \overset{S}{=} (R, F; B, A) = -1$. Therefore, $|NA| = |NK|$. Similarly, $|MA| = |ML|$. Now, combining $|AK| = |AL|, |NA| = |NK|$ and $|MA| = |ML|$ gives us the desired result.

Note: Notice that this is a purely affine problem. So we may assume that $\triangle ABC$ is equilateral or O is the orthocenter of $\triangle ABC$.

U607. Let p be a prime and let m, n be integers with $m \geq n \geq 0$. Prove that

$$\binom{2p+m}{2p+n} + \binom{m}{n} + \binom{m}{2p+n} + 2\binom{m}{p+n} \equiv \left[2\binom{p+m}{p+n} + 2\binom{p+m}{2p+n}\right] \pmod{p^2}.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by Theo Koupelis, Cape Coral, FL, USA

Using the binomial expansion and Cauchy's residue theorem we have

$$\binom{2p+m}{2p+n} = \frac{1}{2\pi i}\int_{|z|=r} \frac{(1+z)^{2p+m}}{z^{2p+n+1}}\, \mathrm{d}z,$$

and similar expressions for the other terms of the given equation. Setting

$$S := \binom{2p+m}{2p+n} + \binom{m}{n} + \binom{m}{2p+n} + 2\binom{m}{p+n} - 2\binom{p+m}{p+n} - 2\binom{p+m}{2p+n},$$

we have

$$\begin{aligned}
2\pi i \cdot S &= \int_{|z|=r} \frac{(1+z)^m}{z^{2p+n+1}} \cdot \left[(1+z)^{2p} + z^{2p} + 1 + 2z^p - 2z^p(1+z)^p - 2(1+z)^p\right] \mathrm{d}z \\
&= \int_{|z|=r} \frac{(1+z)^m}{z^{2p+n+1}} \cdot \left[(1+z)^p - 1 - z^p\right]^2 \mathrm{d}z \\
&= \int_{|z|=r} \frac{1}{z^{2p+n+1}} \sum_{t=0}^{m} \binom{m}{t} z^t \cdot \left[\sum_{k=1}^{p-1} \binom{p}{k} z^k\right]^2 \mathrm{d}z \\
&= \int_{|z|=r} \frac{1}{z^{2p+n+1}} \sum_{t=0}^{m} \sum_{k}^{p-1} \sum_{\ell=1}^{p-1} \binom{m}{t}\binom{p}{k}\binom{p}{\ell} z^{t+k+\ell}\, \mathrm{d}z \\
&= 2\pi i \cdot \sum_{k=1}^{p-1} \sum_{\ell=1}^{p-1} \binom{m}{2p+n-k-\ell}\binom{p}{k}\binom{p}{\ell}.
\end{aligned}$$

and thus

$$S = \sum_{k=1}^{p-1} \sum_{\ell=1}^{p-1} \binom{m}{2p+n-k-\ell}\binom{p}{k}\binom{p}{\ell}.$$

But $p \mid \binom{p}{k}$, and $p \mid \binom{p}{\ell}$ for $k, \ell \in \{1, 2, \ldots, p-1\}$, and $\binom{m}{2p+n-k-\ell}$ is either equal to zero when $n \leq m < 2p+n-k-l$ or a positive integer for $m \geq 2p+n+k+\ell$. Therefore, $p^2 \mid S$.

U608. Consider the formal power series

$$f(x) = x + x^2 + x^4 + x^8 + x^{16} + \cdots.$$

Find the coefficient of x^{10} in $f(f(x))$.

Proposed by Treanungkul Mal, Ideal Public School, West Bengal, India

Solution by G. C. Greubel, Newport News, VA, USA
The question amounts to determining the coefficients of

$$f(f(x)) = f(x) + f^2(x) + f^4(x) + f^8(x) + f^{10}(x) + \cdots.$$

This solution uses the method of determining the powers by expanding the series by multiplying the appropriate terms and collecting the necessary values. This yields

$$\begin{aligned} f(x) &= x + x^2 + x^4 + x^8 + x^{16} + \cdots \\ f^2(x) &= x^2 + 2\,x^3 + x^4 + 2\,x^5 + 2\,x^6 + x^8 + 2\,x^9 + 2\,x^{10} + 2\,x^{12} + \cdots \\ f^4(x) &= x^4 + 4\,x^5 + 6\,x^6 + 8\,x^7 + 13\,x^9 + 10\,x^{10} + 16\,x^{11} + 18\,x^{12} + \cdots \\ f^8(x) &= x^8 + 8\,x^9 + 28\,x^{10} + 64\,x^{11} + 126\,x^{12} + \cdots \end{aligned}$$

and

$$\begin{aligned} f(f(x)) = x + 2\,x^2 + 2\,x^3 + 3\,x^4 + 6\,x^5 + 8\,x^6 + 8\,x^7 + 16\,x^8 \\ + 22\,x^9 + 40\,x^{10} + 80\,x^{11} + 146\,x^{12} + \cdots. \end{aligned}$$

The desired value is then $[x^{10}]\,f(f(x)) = 40$.

Also solved by Suel Heybatova, Baku, Azerbaijan; Ivan Hadinata, Jember, Indonesia; Adam John Frederickson, Utah Valley University, UT, USA; Daniel Pascuas, Barcelona, Spain; Corneliu Mănescu-Avram, Ploieşti, Romania; Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Theo Koupelis, Cape Coral, FL, USA; Yunyong Zhang, Chinaunicom; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

U609. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(x)\geq 2x$ for all $x\in[0,1]$. We denote

$$I_f=\int_0^1 xf(x)^3\,dx \quad\text{and}\quad I_g=\int_0^1 x^3f(x)\,dx.$$

Find the minimum of I_f-3I_g.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Adam John Frederickson, Utah Valley University, UT, USA

$$\begin{aligned}
I_f-3I_g &= \int_0^1 \left(xf(x)^3-3x^3f(x)\right)\,dx\\
&= \int_0^1 xf(x)\left(f(x)^2-3x^2\right)\,dx\\
&\geq \int_0^1 x(2x)\left((2x)^2-3x^2\right)\,dx\\
&= \int_0^1 2x^4\,dx\\
&= \frac{2}{5}.
\end{aligned}$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ivan Hadinata, Jember, Indonesia; Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Daniel Pascuas, Barcelona, Spain; Corneliu Mănescu-Avram, Ploieşti, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U610. If n is a positive integer find the value of

$$\int_0^\infty x^{n-1}e^{-x}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k\cos\left(x+\frac{k\pi}{2}\right)\right\}dx.$$

Proposed by Seán M. Stewart, King Abdullah Univ. of Science and Technology, Thuwal, Saudi Arabia

Solution by Matthew Too, Brockport, NY, USA

Using Euler's formula and the binomial theorem, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\cos\left(x+\frac{k\pi}{2}\right)&=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\,\Re\left\{e^{i\left(x+\frac{k\pi}{2}\right)}\right\}=\Re\left\{e^{ix}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{ik\pi/2}\right\}\\&=\Re\left\{e^{ix}\sum_{k=0}^{n}\binom{n}{k}(-i)^k\right\}=\Re\left\{e^{ix}(1-i)^n\right\}\end{aligned}$$

and so

$$\begin{aligned}\int_0^\infty x^{n-1}e^{-x}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k\cos\left(x+\frac{k\pi}{2}\right)\right\}dx&=\int_0^\infty x^{n-1}e^{-x}\,\Re\left\{e^{ix}(1-i)^n\right\}dx\\&=\Re\left\{\int_0^\infty[(1-i)x]^{n-1}e^{-(1-i)x}(1-i)\,dx\right\}.\end{aligned}$$

To evaluate the integral above, consider the contour integral $\oint_C z^{n-1}e^{-z}\,dz$ where C is the negatively orientated triangle with vertices 0, R, and $R-Ri$ for large R. Since the integrand is holomorphic for all $z\in\mathbb{C}$ and the contour C is closed, then according to Cauchy's integral theorem,

$$\begin{aligned}\oint_C z^{n-1}e^{-z}\,dz&=\int_{[0,R]}z^{n-1}e^{-z}\,dz+\int_{[R,R-Ri]}z^{n-1}e^{-z}\,dz+\int_{[R-Ri,0]}z^{n-1}e^{-z}\,dz\\&=\int_0^R t^{n-1}e^{-t}\,dt-iR^n\int_0^1(1-it)^{n-1}e^{-R(1-it)}\,dt-\int_0^R[(1-i)t]^{n-1}e^{-(1-i)t}(1-i)\,dt\\&=0\end{aligned}$$

which implies that as $R\to\infty$,

$$\int_0^\infty[(1-i)t]^{n-1}e^{-(1-i)t}(1-i)\,dt=\Gamma(n)-\lim_{R\to\infty}iR^n\int_0^1(1-it)^{n-1}e^{-R(1-it)}\,dt.$$

The limit of the integral on the right is 0 because

$$\begin{aligned}\left|iR^n\int_0^1(1-it)^{n-1}e^{-R(1-it)}\,dt\right|&\le R^n\int_0^1\left|(1-it)^{n-1}e^{-R(1-it)}\right|dt=R^ne^{-R}\int_0^1(1+t^2)^{\frac{n-1}{2}}\,dt\\&\le R^ne^{-R}\int_0^1 2^{\frac{n-1}{2}}\,dt=2^{\frac{n-1}{2}}R^ne^{-R}\xrightarrow{R\to\infty}0.\end{aligned}$$

Therefore,

$$\int_0^\infty x^{n-1}e^{-x}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k\cos\left(x+\frac{k\pi}{2}\right)\right\}dx=\Re\{\Gamma(n)\}=\Gamma(n)=(n-1)!$$

and we are done.

Also solved by Daniel Pascuas, Barcelona, Spain; Theo Koupelis, Cape Coral, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Yunyong Zhang, Chinaunicom; G. C. Greubel, Newport News, VA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U611. We call a set A of real numbers *nice* if there are non-zero distinct numbers $x_1, \dots, x_n \in A$ and integers $y_1, \cdots, y_n$ (not all z zero) such that

$$x_1^{y_1} \dots x_n^{y_n} = 1.$$

Let $P(x)$ be a second degree polynomial with rational coefficients. Prove that the set

$$\{P(n) : n \in \mathbb{Z}, n > 2021\},$$

is *nice*.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let $P(x) = ax^2 + bx + c$ where a, b, c are rational numbers and $a \neq 0$ it is easy to verify that for each non-zero z;

$$P(x + \frac{z}{a}P(x)) = \frac{z^2}{a}P(x)P(x + \frac{1}{z}).$$

Far a moment, assume that $a, b, c \in \mathbb{Z}$ and $a, c > 0$. Taking $z = 1$ then $P(x + \frac{1}{a}P(x)) = \frac{1}{a}P(x)P(x+1)$ thus;

$$\frac{P(x_1 + a^{-1}P(x_1))}{P(x_2 + a^{-1}P(x_2))} = \frac{P(x_1)P(x_1+1)}{P(x_2)P(x_2+1)}.$$

Plugging $z = \frac{a}{c}$ then $P(x + \frac{1}{c}P(x)) = \frac{a}{c^2}P(x)P(x + \frac{c}{a})$. There are x_3, x_4 such that $x_1 + a^{-1}P(x_1) = x_3 + \frac{c}{a}, x_2 + a^{-1}P(x_2) = x_4 + \frac{c}{a}$. It follows that

$$\frac{P(x_1)P(x_1+1)}{P(x_2)P(x_2+1)} = \frac{P(x_1 + a^{-1}P(x_1))}{P(x_2 + a^{-1}P(x_2))} = \frac{P(x_3 + \frac{c}{a})}{P(x_4 + \frac{c}{a})} = \frac{P(x_4)P(x_3 + c^{-1}P(x_3))}{P(x_3)P(x_4 + c^{-1}P(x_4))}.$$

Letting $x_1 = acy_1, x_2 = acy_2$ then $x_3 = cy_1(a^2cy_1 + a + b), x_4 = cy_2(a^2cy_2 + a + b)$. Thus, c divides $P(x_3), P(x_4)$. Hence. for large positive integers y_1, y_2 the equality

$$\frac{P(x_1)P(x_1+1)}{P(x_2)P(x_2+1)} = \frac{P(x_4)P(x_3 + c^{-1}P(x_3))}{P(x_3)P(x_4 + C^{-1}P(x_4))},$$

implies that the set $\{P(n) : n \in \mathbb{Z}, n > 2021\}$ is *nice*.

Now, if a, b, c are rational there are positive integers A, B such that the polynomial

$$P_1(x) = (-1)^{\text{sgn}a} A \cdot P(x - B) = a_1x^2 + b_1x + c_1.$$

Has integer coefficients and $a_1, c_1 > 0$. Since in the relation

$$P(x_1)P(x_1+1)P(x_3)P(x_4 + c^{-1}P(x_4)) \cdot (P(x_2)P(x_2+1)P(x_4)P(x_3 + c^{-1}P(x_3)))^{-1} = 1.$$

The sum of the exponents, i.e,. $y_1, \cdots, y_8$ is zero then applying it to $P_1(x)$ we obtain the respective relation for $P(x)$.

U612. If n is a nonnegative integer and $m \in \{0, 1, 2, ..., n\}$, evaluate

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n+m}.$$

Proposed by Seán M. Stewart, King Abdullah Univ. of Science and Technology, Thuwal, Saudi Arabia

Solution by the author

Let $[z^n]$ denote the coefficient operator extracting the coefficient of z^n in a formal power series $A(z)$. Using the Binomial theorem, we obtain

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n+m} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} [z^{n+m}](1+z)^{2n-2k} \\ &= [z^{n+m}](1+z)^{2n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{1}{(1+z)^2}\right)^k \\ &= [z^{n+m}](1+z)^{2n} \left(1 - \frac{1}{(1+z)^2}\right)^n \\ &= [z^{n+m}] \left((1+z)^2 - 1\right)^n \\ &= [z^{n+m}] z^n (2+z)^n \\ &= [z^m](2+z)^n = \binom{n}{m} 2^{n-m},\end{aligned}$$

as required to prove. Note that in the penultimate line the following coefficient operator extraction rule of $[z^{p-q}]A(z) = [z^p]z^q A(z)$ has been used.

Also solved by Daniel Pascuas, Barcelona, Spain; Theo Koupelis, Cape Coral, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Yunyong Zhang, Chinaunicom; G. C. Greubel, Newport News, VA, USA.

Olympiad problems

O607. Let $a_1, \ldots, a_k$ and b be positive integers, and let $f(t) = (t + a_1) \cdots (t + a_k)$. Prove that there are infinitely many positive integers n such that $P(f(n)) \geq P(f(n+b))$, where $P(m)$ denotes the greatest prime divisor of the integer $m \geq 2$. (For instance, $P(81) = P(3) = 3$, or $P(56) = P(14) = 7$.)

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by the authors

We may assume, without loss of generality, that $a_1 \leq \cdots \leq a_k$. Since $a_k < b + a_k$, some $1 \leq i \leq k$ must exist such that we have

$$b + a_1 \leq \cdots \leq b + a_{i-1} \leq a_k < b + a_i \leq \cdots \leq b + a_k.$$

(For $i = 1$ no inequalities at the left of a_k will be written, as all $b + a_j$, $1 \leq j \leq k$ are greater than a_k.)

We first show that we can choose infinitely many primes $p > b$ such that $p + b - a_k + a_j$ is not a prime for any $i \leq j \leq k$. In order to do that, we start by picking distinct primes $q_i, \ldots, q_k$ such that $a_k - b - a_j$ and q_j are relatively prime for each $i \leq j \leq k$ (which we can do since every such $a_k - b - a_j$ is nonzero, actually strictly negative). Then, according to the Chinese remainder theorem, the system of congruences

$$x \equiv a_k - b - a_j \bmod q_j, \quad i \leq j \leq k$$

has integer solutions x, and, moreover, all these solutions are the terms of the arithmetic sequence $(x_0 + lq_i \cdots q_k)_{l \in \mathbb{Z}}$ with common difference $q_i \cdots q_k$ (and first term x_0, one of the solutions). Since any solution (including x_0) is relatively prime to any of $q_i, \ldots, q_k$ (by the conditions imposed above), we also have that the first term x_0 and the common difference $q_i \cdots q_k$ of this progression (formed with the solutions of the system of congruences) are relatively prime. By Dirichlet's theorem, the progression contains infinitely many primes, hence infinitely many such primes can be chosen such that, moreover, they are greater than $\max\{q_i, \ldots, q_k, b\}$. Thus, for such a prime p, we have

$$p + b - a_k + a_j \equiv 0 \bmod q_j,$$

hence $p + b - a_k + a_j$ is not a prime (it is divisible by q_j, and greater than q_j) for any $i \leq j \leq k$, and, also, we have $p > b$.

Now consider $n = p - a_k$ for such a prime p. We clearly have

$$P(f(n)) = P((n + a_1) \cdots (n + a_k)) = p$$

since $n + a_k = p$, and any of $n + a_1, \ldots, n + a_{k-1}$ is at most equal to $n + a_k = p$, and hence it has only prime factors at most equal to p.

On the other hand, any of $n + b + a_1, \ldots, n + b + a_{i-1}$ has prime factors at most equal to p, since any of these numbers is also at most equal to $n + a_k = p$. Also, none of $n + b + a_j$, $i \leq j \leq k$ can have prime factors greater than p since these numbers are not prime and are less than $2p$:

$$n + b + a_j = p + b - a_k + a_j \leq p + b < 2p, \quad i \leq j \leq k.$$

Thus, for any such n (and there are infinitely many), we have

$$P(f(n+b)) = P((n + b + a_1) \cdots (n + b + a_k)) \leq p = P(f(n)),$$

which is the desired result.

Remarks:

1) The result remains true when the integers $a_1, \ldots, a_k$ are not necessarily positive; the proof is the same, if one takes care to choose $n > -a_1$, in order that all the factors of $f(n)$ and $f(n+b)$ be positive. However, this is not much of a generalization, so we prefer this form.

2) Note that any $n + b + a_j = p + b - a_k + a_j$ $(i \le j \le k)$ is not only less than $2p$, but also greater than p, hence actually $P((n+b+a_i)\cdots(n+b+a_k))$ is less than p. (Anyway, no such $n+b+a_j$ is a prime, so it cannot equal p.) If $i = 1$, that is, if all $b + a_j$ $(1 \le j \le k)$ are greater than a_k, this means that $P(f(n+b))$ is less than p, so the above proof shows that there are infinitely many n for which $P(f(n)) > P(f(n+b))$. Probably this is true in all cases, but this proof doesn't show it.

Also solved by Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

O608. Find the minimum possible value of the positive integer k such that the equation

$$x^2 + kxy + y^2 = 2022$$

is solvable in positive integers x and y.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution by the author

The equality is symmetric about x, y, so without loss of generality we may assume that $x \le y$. Let $y = x + n, n \ge 0, n \in \mathbb{Z}$.

$$F(k) = x^2 + kxy + y^2 = x^2 + kx(x+n) + (x+n)^2 = (k+2)x^2 + (k+2)nx + n^2$$

$$F(1) = 3x^2 + 3nx + n^2 = 2022 = 3.674$$

Hence $3|n$. Let $n = 3n_1$; $n_1 \ge 0$.

$$x^2 + 3n_1x + 3{n_1}^2 = 674 = 672 + 2 = 3.224 + 2.$$

$674 \equiv 2 \pmod 3$ but $x^2 + 3n_1x + 3{n_1}^2 \equiv x^2 \equiv 0, 1 \pmod 3$, hence no solutions for $k = 1$.

$$F(2) = x^2 + 2xy + y^2 = (x+y)^2$$

$F(2)$ is a perfect square, but 2022 is not, so no solutions for $k = 2$.

$$F(3) = 5x^2 + 5nx + n^2 = 2022$$

$2022 \equiv 2 \pmod 5$ but $5x^2 + 5nx + n^2 \equiv n^2 \equiv 0, 1, 4 \pmod 5$, hence no solutions for $k = 3$.

$$F(4) = 6x^2 + 6nx + n^2 = 2022 = 6.337$$

Hence $6|n$. Let $n = 6n_1$; $n_1 \ge 0$.

$$x^2 + 6n_1x + 6{n_1}^2 = 337 = 336 + 1 = 6.56 + 1.$$

Hence $x^2 \equiv 1 \pmod 6$ or $x \equiv 1, -1 \pmod 6$.

Case 1: $x = 6x_1 + 1 \equiv 1 \pmod 6$; $x_1 \ge 0$

$$6{x_1}^2 + (6n_1 + 2)x_1 + {n_1}^2 + n_1 - 56 = 0$$

Solving about x_1, we receive

$$x_1 = \frac{1}{6}\left(-1 - 3n_1 \pm \sqrt{337 + 3{n_1}^2}\right)$$

but $\frac{1}{6}\left(-1 - 3n_1 - \sqrt{337 + 3{n_1}^2}\right) < 0$, hence $x_1 = \frac{1}{6}\left(-1 - 3n_1 + \sqrt{337 + 3{n_1}^2}\right) \ge 0$.

$$-1 - 3n_1 + \sqrt{337 + 3{n_1}^2} = 0 \Leftrightarrow (3n_1 + 1)^2 = 337 + 3{n_1}^2$$
$$\Leftrightarrow 6{n_1}^2 + 6n_1 - 336 = 0 \Leftrightarrow {n_1}^2 + n_1 - 56 = 0 \Leftrightarrow n_1 = 7$$

The function $-1 - 3n_1 + \sqrt{337 + 3{n_1}^2}$ is decreasing, so for $n_1 > 7$ no positive solutions.

It is easy to check that for $0 \le n_1 \le 6$, the sum $337 + 3{n_1}^2$ is not a perfect square, and for $n_1 = 7, \sqrt{337 + 3{n_1}^2} = 22$ and $x_1 = 0$.

Hence $x = 6x_1 + 1 = 1, n = 6n_1 = 42, y = x + n = 43$ is a solution.

Case 2: $x = 6x_1 + 5 \equiv -1 \pmod 6$; $x_1 \geq 0$

$$6{x_1}^2 + (6n_1 + 10)x_1 + {n_1}^2 + 5n_1 - 52 = 0$$

Solving about x_1, we receive

$$x_1 = \frac{1}{6}\left(-5 - 3n_1 \pm \sqrt{337 + 3{n_1}^2}\right)$$

but $\frac{1}{6}\left(-5 - 3n_1 - \sqrt{337 + 3{n_1}^2}\right) < 0$, hence $x_1 = \frac{1}{6}\left(-5 - 3n_1 + \sqrt{337 + 3{n_1}^2}\right) \geq 0$. For $n_1 = 7, \sqrt{337 + 3{n_1}^2} = 22$ but $x_1 < 0$, so for $n_1 \geq 7$ no positive solutions. For $0 \leq n_1 \leq 6$, the sum $337 + 3{n_1}^2$ is not a perfect square, so that no solutions in this case.

Finally, the minimum value of k is $k = 4$, and the solutions are $x = 1, y = 43$ and $x = 43, y = 1$.

Note: The maximum value of k is $k = 2020$, and the solution is $x = 1, y = 1$.

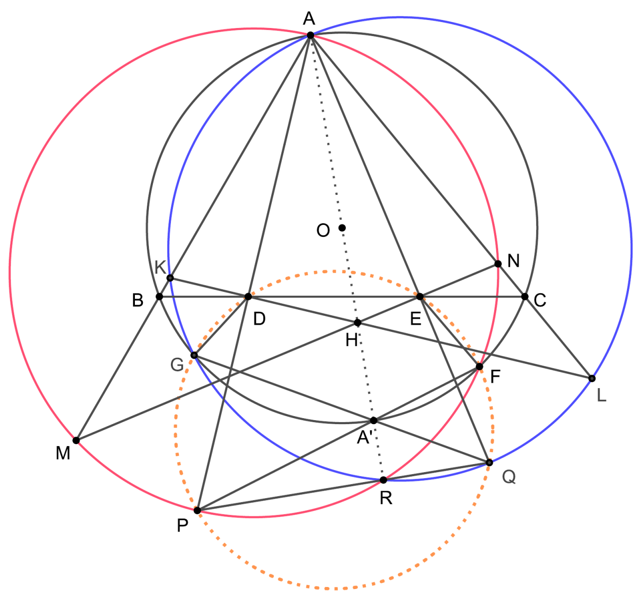
Also solved by Farakh Novruzova, ADA University, Azerbaijan; Ivan Hadinata, Jember, Indonesia; Aaron Kim, Bronx High School of Science, Bronx, NY, USA; Adam John Frederickson, Utah Valley University, UT, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Prajnanaswaroopa S, Bengaluru, Karnataka; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Cape Coral, FL, USA; Sundaresh Harige, India.

O609. Let ABC be a triangle with circumcircle $\Gamma(O)$ and the points D, E on the side BC such that $\angle BAD = \angle CAE$. The perpendicular lines through D to AD and E to AE intersects AB, AC at K, L and M, N, respectively. The circumcircles of the triangles AMN and AKL intersect again Γ at F and G, respectively. Prove that the points D, E, F, G are concyclic.

Proposed by Marius Stănean, Zalău, România

Solution by the author

Let A' be the antipode of A on Γ and let $\alpha = \angle BAD = \angle CAE$. Denote by $\omega_1(O_1)$ and $\omega_2(O_2)$ the circumcircles of the triangles AMN and AKL, respectively. Let $P = AD \cap \omega_1$ and $Q = AE \cap \omega_2$.



Since

$$\angle AMP = 180^\circ - \alpha - \angle APM = 180^\circ - \alpha - \angle ANM = \angle AEN = 90^\circ$$

it follows that AP is the diameter of ω_1 so the points P, A', F are collinears since $PA' \parallel O_1O$ and $O_1O \perp AF$(radical axis of ω_1 and Γ) and $\angle A'FA = 90^\circ$. Similarly results that AQ is the diameter of ω_2 and the points Q, A', G are collinears. Let $R = \omega_1 \cap \omega_2$, $(R \neq A)$ then $\angle ARP = \angle ARQ = 90^\circ$ so the points P, R, Q are collinear.

We have

$$\angle ENF = \angle MNF \overset{\omega_1}{=} \angle MAF = \angle BAF \overset{\Gamma}{=} \angle BCF = \angle ECF$$

so $ENCF$ is cyclic. Also, we have

$$\begin{aligned}\angle DPF =& \angle APF = \angle AA'F - \angle PAA' = 90^\circ - \angle A'AF - (\angle BAA' - \alpha)\\ =& 90^\circ + \alpha - \angle BAF,\end{aligned}$$

$$\begin{aligned}\angle DEF =& \angle DEM + \angle MEF = \angle NEC + \angle NCF = \angle ANE - \angle C + \angle NCF\\ =& 90^\circ - \alpha + \angle ECF = 90^\circ - \alpha + \angle BCF = 90^\circ - \alpha + \angle BAF\end{aligned}$$

so $DEFP$ is cyclic. Similarly results that $DEQG$ is cyclic.

From $\triangle ADK \sim \triangle AMP (\angle ADK = \angle AMP = 90^\circ)$ it follows that

$$AD \cdot AP = AK \cdot AM$$

and from $\triangle AEN \sim \triangle ALQ (\angle AEN = \angle ALQ = 90^\circ)$ it follows that

$$AE \cdot AQ = AN \cdot AL.$$

Since $\triangle AMN \sim \triangle ALK$ $(\angle ANM = \angle AKL = 90^\circ - \alpha)$ we deduce that

$$\frac{AN}{AK} = \frac{AM}{AL} \Longleftrightarrow AN \cdot AL = AK \cdot AM.$$

Therefore (using power of point)

$$AD \cdot AP = AE \cdot AQ \Longrightarrow DEQP \text{ is cyclic.}$$

Finaly, we conclude that the points D, E, F, Q, P, G are concyclic.

Also solved by Theo Koupelis, Cape Coral, FL, USA.

O610. Let a, b, c be positive real numbers. Prove that

$$a^2\sqrt{\frac{a+b}{a+c}} + b^2\sqrt{\frac{b+c}{b+a}} + c^2\sqrt{\frac{c+a}{c+b}} \geq ab + bc + ca.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Theo Koupelis, Cape Coral, FL, USA

We have

$$\sqrt{\frac{a+b}{a+c}} = \frac{a+b}{\sqrt{(a+b)(a+c)}} \geq \frac{2(a+b)}{2a+b+c},$$

with similar expressions for the other two terms. Thus it is sufficient to show that

$$S := \frac{a^2(a+b)}{2a+b+c} + \frac{b^2(b+c)}{a+2b+c} + \frac{c^2(c+a)}{a+b+2c} \geq \frac{ab+bc+ca}{2}.$$

However, using Cauchy-Schwarz we get

$$\left[\sum_c (a+b)(2a+b+c)\right] \cdot S \geq \left[a(a+b) + b(b+c) + c(c+a)\right]^2,$$

or

$$S \geq \frac{(a^2+b^2+c^2+ab+bc+ca)^2}{3(a^2+b^2+c^2) + 5(ab+bc+ca)}.$$

Therefore, setting $x = a^2 + b^2 + c^2$ and $y = ab + bc + ca$, it is sufficient to show that

$$\frac{(x+y)^2}{3x+5y} \geq \frac{y}{2} \iff 2x^2 + xy \geq 3y^2 \iff (x-y)(2x+3y) \geq 0,$$

which is obvious because $x \geq y > 0$. Equality occurs when $a = b = c$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jiang Lianjun, Quanzhou County, 2nd Middle School, Guilin, China; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O611. Let $m \geq 2$ and $a_1, \ldots, a_m$ be positive integers such that $a_1, \ldots a_m$ are not all equal. Prove that $(n + a_1) \cdots (n + a_m)$ is the m^{th} power of a positive integer for at most finitely many positive integers n.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by Daniel Pascuas, Barcelona, Spain

Without loss of generality, we may assume that $a_1 \leq \cdots \leq a_m$. Then

$$(n + a_1)^m \leq (n + a_1) \cdots (n + a_m) \leq (n + a_m)^m, \quad \text{for every positive integer } n,$$

and so if $(n + a_1) \cdots (n + a_m) = N^m$, for some positive integers n and N, then $n + a_1 \leq N \leq n + a_m$, that is, $N = n + k$, for some integer k such that $a_1 \leq k \leq a_m$. It follows that we only have to show that, for any integer k such that $a_1 \leq k \leq a_m$, the equation

$$(x + a_1) \cdots (x + a_m) = (x + k)^m \tag{$*$}$$

has at most finitely many integer solutions. Indeed, we prove that $(*)$ has at most finitely many real solutions. Just note that, since the numbers $a_1, \ldots, a_m$ are not all equal to k, the polynomial

$$P_k(x) = (x + a_1) \cdots (x + a_m) - (x + k)^m$$

is not identically zero, and therefore it has at most finitely many real zeros, which means that $(*)$ has at most finitely many real solutions.

Also solved by Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

O612. Let a, b, c, d, e be positive real numbers such that

$$a^3 + b^3 + c^3 + d^3 + e^3 + abcde = 6$$

Prove that $a + bcde \le 2$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

Let us start with the hypothesis $a + bcde > 2$. It can be obtained by setting conditions $a < \sqrt[3]{6} < 2$, therefore $bcde > 2 - a > 0$.

$$6 = a^3 + b^3 + c^3 + d^3 + e^3 + abcde \ge a^3 + 4(bcde)^{\frac{3}{4}} + a(bcde) \ge a^3 + 4(2-a)^{\frac{3}{4}} + a(2-a),$$

namely, $a^3 + 4(2-a)^{\frac{3}{4}} + a(2-a) - 6 < 0$, which is not true, therefore, contradiction and $a + bcde \le 2$. Now, let us prove the correctness of $a^3 + 4(2-a)^{\frac{3}{4}} + a(2-a) - 6 \ge 0$.

First, order $x = \sqrt[4]{2-a} < \sqrt[4]{2}$. Then, the equation is equivalent to

$$(2-x^4)^3 - (2-x^4)^2 + 2(2-x^4) - 6 + 4x^3 \ge 0$$

$$\Leftrightarrow 2 + 4x^3 - 10x^4 + 5x^8 - x^{12} \ge 0$$

$$\Leftrightarrow x^6(x-1)^2 \cdot g(x) \ge 0$$

Now, prove that $g(x) > 0$ $(0 < x < \sqrt[4]{2})$ among

$$g(x) = \frac{2}{x^6} + \frac{4}{x^5} + \frac{6}{x^4} + \frac{12}{x^3} + \frac{8}{x^2} + \frac{4}{x} - (4x + 3x^2 + 2x^3 + x^4).$$

Obviously, function $g(x)$ is decreasing on $(0, \sqrt[4]{2})$ and when $x = \sqrt[4]{2}, x^4 = 2$

$$x^6 g(x) = 2 + 4x + 6x^2 + 12x^3 + 16 + 8x - (8x^3 + 12 + 8x + 4x^2) = 6 + 4x + 2x^2 + 4x^3 > 0$$

So, when $0 < x < \sqrt[4]{2}$, $g(x) > 0$ and the conclusion follows.

Also solved by Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Theo Koupelis, Cape Coral, FL, USA.