Junior problems

J613. Find all integers $n \geq 2$ for which both $n-1$ and n^2+1 divide n^4+2039.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Since both $n-1$ and n^2+1 divide n^4-1, they must both divide $2040 = 2^3 \cdot 3 \cdot 5 \cdot 17$. If n is even, then $n^2+1 \equiv 1 \pmod 4$, so $n^2+1 \in \{5, 17, 5 \cdot 17\}$, which yields $n = 2, 4$. If n is odd, then $n^2+1 \equiv 2 \pmod 8$, so $n^2+1 \in \{2 \cdot 5, 2 \cdot 17, 2 \cdot 5 \cdot 17\}$, which yields $n = 3, 13$. Finally, it is easy to check that $n = 2, 3, 4, 13$ do satisfy the required conditions.

Also solved by Ivan Hadinata, Jember, Indonesia; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Daniel Văcaru, Pitești, Romania; G. C. Greubel, Newport News, VA, USA; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Matthew Too, Brockport, NY, USA; Sundaresh H R, Shivamogga, India; Vishwesh Ravi Shrimali, Jaipur, India; Aaron Kim, Bronx Science, NY, USA; Daniel Pascuas, Barcelona, Spain; Brian Bradie, Newport News, VA, USA; Anderson Torres, Brazil; Adam John Frederickson, Utah Valley University, UT, USA; Theo Koupelis, Cape Coral, FL, USA; Jiang Lianjun, Quanzhou Middle School, GuiLin, China; Soham Bhadra, India; Titu Zvonaru, Comănești, România.

J614. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{\sqrt{1+b^2c+bc^2}} + \frac{b}{\sqrt{1+c^2a+ca^2}} + \frac{c}{\sqrt{1+a^2b+ab^2}} \geq \frac{a+b+c}{\sqrt{3}}.$$

Proposed by Mircea Becheanu, Canada

Solution by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy
Using $abc = 1$ the inequality is

$$\sum_{\text{cyc}} \frac{a}{\sqrt{1+\frac{b}{a}+\frac{c}{a}}} = \frac{a\sqrt{a}+b\sqrt{b}+c\sqrt{c}}{\sqrt{a+b+c}} \geq \frac{a+b+c}{\sqrt{3}}$$

that is

$$\sqrt{3}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}) \geq (a+b+c)^{3/2}$$

This is power–means–inequality

$$\frac{(a^{3/2}+b^{3/2}+c^{3/2})^{2/3}}{3^{2/3}} \geq \frac{a+b+c}{3} \iff 3^{1/3}(a^{3/2}+b^{3/2}+c^{3/2})^{2/3} \geq a+b+c$$

and then by elevating to the fractional power $3/2$ we get the desired inequality

$$3^{1/2}(a^{3/2}+b^{3/2}+c^{3/2}) \geq (a+b+c)^{3/2}$$

Also solved by Ivan Hadinata, Jember, Indonesia; Polyahedra, Polk State College, USA; Aaron Kim, Bronx Science, NY, USA; Daniel Pascuas, Barcelona, Spain; Brian Bradie, Newport News, VA, USA; Anderson Torres, Brazil; Corneliu Mănescu-Avram, Ploieşti, Romania; Batakogias Panagiotis, Velestino, Greece; Daniel Văcaru, Pitești, Romania; Etisha Sharma, Agra College, Agra, India; Henry Ricardo, Westchester Area Math Circle, NY, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Matthew Too, Brockport, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sicheng Du, Shenzhen, Guangdong, China; Toyesh Prakash Sharma, Agra College, Agra, India; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Cape Coral, FL, USA; Jiang Lianjun, Quanzhou Middle School, GuiLin, China; Titu Zvonaru, Comănești, România.

J615. Prove that in any triangle ABC,

$$\frac{m_b m_c}{(m_b+m_c)^2} \le \frac{2a^2+bc}{8a^2+(b+c)^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Theo Koupelis, Cape Coral, Florida, USA

Let a, b, c be the side-lengths BC, AC, AB, respectively. Let BD, CE be the medians of the triangle ABC from vertices B, C, respectively, with corresponding lengths m_b, m_c. Let $x = m_b m_c$. Using the well-known expressions $4m_b^2 = 2(a^2+c^2) - b^2$ and $4m_c^2 = 2(a^2+b^2) - c^2$ we get

$$\frac{m_b m_c}{(m_b+m_c)^2} = \frac{x}{m_b^2+m_c^2+2x} = \frac{4x}{4a^2+b^2+c^2+8x}.$$

The desired inequality becomes

$$\frac{4x}{4a^2+b^2+c^2+8x} \le \frac{2a^2+bc}{8a^2+(b+c)^2} \Longleftrightarrow x \le \frac{a^2}{2} + \frac{bc}{4},$$

which is obvious by applying Ptolemy's inequality for the quadrilateral $BCDE$.

Also solved by Polyahedra, Polk State College, USA; Aaron Kim, Bronx Science, NY, USA; Sicheng Du, Shenzhen, Guangdong, China; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Elizabeth Standera, SUNY Brockport, USA; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Soham Bhadra, India; Titu Zvonaru, Comănești, România.

J616. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a(b+c)^2}{a+3} + \frac{b(c+a)^2}{b+3} + \frac{c(a+b)^2}{c+3} \le 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by the author
The AM-GM inequality yields

$$2a(b+c) \le \frac{(2a+b+c)^2}{4}.$$

It follows that

$$\frac{a(b+c)^2}{2a+b+c} \le \frac{(2a+b+c)(b+c)}{8}.$$

Establishing similar two inequalities and adding them up we get

$$\begin{aligned}\sum_{\text{cyc}} \frac{a(b+c)^2}{2a+b+c} &\le \sum_{\text{cyc}} \frac{2a(b+c)+(b+c)^2}{8} \\ &= \frac{(a+b+c)^2 + ab + bc + ca}{4} \\ &\le \frac{(a+b+c)^2}{3} \\ &= 3\end{aligned}$$

as desired.

Also solved by Ivan Hadinata, Jember, Indonesia; Polyahedra, Polk State College, USA; Aaron Kim, Bronx Science, NY, USA; Daniel Pascuas, Barcelona, Spain; Brian Bradie, Newport News, VA, USA; Adam John Frederickson, Utah Valley University, UT, USA; Sicheng Du, Shenzhen, Guangdong, China; Toyesh Prakash Sharma, Agra College, Agra, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Soham Dutta, India; Sundaresh H R, Shivamogga, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Cape Coral, FL, USA; Jiang Lianjun, Quanzhou Middle School, GuiLin, China; Titu Zvonaru, Comănești, România.

J617. Prove that triangle ABC is equilateral if and only if

$$2(a^2\cos A + b^2\cos B + c^2\cos C) \geq \sqrt{3(a^4+b^4+c^4)}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Polyahedra, Polk State College, USA

We may assume that $a \geq b \geq c$. Then $\cos A \leq \cos B \leq \cos C$. Therefore, by Chebyshev's inequality,

$$a^2\cos A + b^2\cos B + c^2\cos C \leq \frac{1}{3}(a^2+b^2+c^2)(\cos A + \cos B + \cos C).$$

By Euler's inequality, $\cos A + \cos B + \cos C = 1 + \frac{r}{R} \leq \frac{3}{2}$. By the Cauchy-Schwarz inequality,

$$2(a^2\cos A + b^2\cos B + c^2\cos C) \leq a^2+b^2+c^2 \leq \sqrt{3(a^4+b^4+c^4)},$$

with equality if and only if $a = b = c$.

Also solved by Sicheng Du, Shenzhen, Guangdong, China; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Cape Coral, FL, USA; Titu Zvonaru, Comănești, România.

J618. Let triangle ABC have side lengths a, b, c and interior angle bisector lengths w_a, w_b, w_c. Prove that

$$w_a(bc - a^2) + w_b(ca - b^2) + w_c(ab - c^2) \geq 0.$$

Proposed by Marius Stănean, Zalău, România

Solution by the author

Suppose that $a \leq b \leq c$. We have

$$bc(b+c) \geq ca(c+a) \geq ab(a+b),$$

and

$$-a^2(b+c) \geq -b^2(c+a) \geq -c^2(a+b)$$

so

$$bc(b+c) - a^2(b+c) \geq ca(c+a) - b^2(c+a) \geq ab(a+b) - c^2(a+b).$$

We show that

$$\frac{w_a}{b+c} \geq \frac{w_b}{c+a} \geq \frac{w_c}{a+b}.$$

We prove the first inequality and the other similarly

$$w_a^2(c+a)^2 \geq w_b^2(b+c)^2 \iff b(b+c-a)(a+c)^4 \geq a(c+a-b)(b+c)^4$$

$$\iff (b-a)\left[c^5 + (a+b)c^4 + 2abc^3 + 2ab(a+b)c^2 + abc(3a^2 - ab + 3b^2) + ab(a^3 + b^3)\right] \geq 0,$$

clearly true.

According to Chebyshev's Inequality, we have

$$3\sum_{cyc}\left[bc(b+c) - a^2(b+c)\right]\frac{w_a}{b+c} \geq \sum_{cyc}\left[bc(b+c) - a^2(b+c)\right]\sum_{cyc}\frac{w_a}{b+c} = 0$$

as we wished to prove.

Also solved by Polyahedra, Polk State College, USA; Sicheng Du, Shenzhen, Guangdong, China; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Cape Coral, FL, USA; Titu Zvonaru, Comănești, România.

Senior problems

S613. Solve the equation $2(\sin x+\cos x)+\sec x+\csc x=4\sqrt{2}$

Proposed by Adrian Andreescu, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy
Evidently $x\neq k\pi$, $x\neq \pi/2+k\pi$, $k\in\mathbb{Z}$. The equation is

$$2\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]+\frac{\sqrt{2}}{\sin x\cos x}\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]=4\sqrt{2}$$
$$\iff \quad 2\sin(x+\frac{\pi}{4})+\frac{\sin(x+\frac{\pi}{4})}{\sin x\cos x}=4$$
$$\iff \quad \sin(x+\frac{\pi}{4})(1+\sin(2x))=2\sin(2x) \underbrace{\iff}_{y=x+\pi/4} \sin y=\frac{-2\cos(2y)}{2\sin^2 y}$$
$$\iff \quad \sin^3 y-2\sin^2 y+1=(\sin y-1)(\sin^2 y-\sin y-1)=0$$

$$\sin y=1 \quad\iff\quad y=\frac{\pi}{2}+2k\pi \implies x=\frac{\pi}{4}+2k\pi$$

$$\sin^2 y-\sin y-1=0 \quad\iff\quad \sin y=\frac{1\pm\sqrt{5}}{2} \implies \sin y=\frac{1-\sqrt{5}}{2}$$

$$y=\arcsin\frac{1-\sqrt{5}}{2},\quad y=\pi-\arcsin\frac{1-\sqrt{5}}{2}$$

hence

$$x=-\arcsin\frac{\sqrt{5}-1}{2}-\frac{\pi}{4},\quad x=\frac{3\pi}{4}+\arcsin\frac{\sqrt{5}-1}{2}$$

Thus we have

$$x_1=\frac{\pi}{4}+2k,\quad x_2=\frac{3\pi}{4}+\arcsin\frac{\sqrt{5}-1}{2}+2k\pi,\quad x_3=\frac{\pi}{2}-x_2$$

Also solved by Aaron Kim, Bronx Science, NY, USA; Adam John Frederickson, Utah Valley University, UT, USA; Theo Koupelis, Cape Coral, FL, USA; Daniel Văcaru, Pitești, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Sundaresh H R, Shivamogga, India; Matthew Too, Brockport, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Vicente Vicario Garcia, Sevilla, Spain; Arkady Alt, San Jose, CA, USA; Soham Bhadra, India; Marian Ursărescu, Roman-Vodă National College, Roman, Romania.

S614. Let a, b, c be positive real numbers such that $(a+b)(b+c)(c+a) = 9abc$. Prove that

$$\sqrt[3]{2abc} \le \max(a, b, c) \le \sqrt[3]{4abc}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Suppose that $a = \max(a, b, c)$. The given condition can also be written like this

$$a^2(b+c) + bc(b+c) + a(b+c)^2 = 9abc \Longrightarrow bc = \frac{a^2(b+c) + a(b+c)^2}{9a - b - c}.$$

We need to show that

$$\frac{1}{4} \le \frac{bc}{a^2} \le \frac{1}{2} \Longleftrightarrow \frac{1}{4} \le \frac{\frac{b+c}{a} + \frac{(b+c)^2}{a^2}}{9 - \frac{b+c}{a}} \le \frac{1}{2} \Longleftrightarrow 1 \le \frac{b+c}{a} \le \frac{3}{2}.$$

We have $(a-b)(a-c) \ge 0 \Longrightarrow bc \ge a(b+c) - a^2$ and from AM-GM Inequality $bc \le \dfrac{(b+c)^2}{4}$. Therefore

$$a(b+c) - a^2 \le \frac{a^2(b+c) + a(b+c)^2}{9a - b - c} \le \frac{(b+c)^2}{4}$$

or if we denote $x = \dfrac{b+c}{a} > 0$,

$$x - 1 \le \frac{x + x^2}{9 - x} \le \frac{x^2}{4}.$$

Solving this, we deduce $x \in \left[1, \dfrac{3}{2}\right] \cup [3, 4]$. Since $x = \dfrac{b+c}{a} \le \dfrac{a+a}{a} = 2$, we deduce $x \in \left[1, \dfrac{3}{2}\right]$.

The equality holds in the left hand side when $a = b = 2c$ and in right hand side when $a = 2b = 2c$.

Also solved by Adam John Frederickson, Utah Valley University, UT, USA; Sicheng Du, Shenzhen, Guangdong, China; Theo Koupelis, Cape Coral, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

S615. Let ABC be a triangle. Prove that

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ca}\geq\frac{4\left(a^2+b^2+c^2\right)}{ab+bc+ca}-\frac{2r}{R}.$$

Proposed by Titu Zvonaru, Comăneşti, România

Solution by Arkady Alt, San Jose, CA, USA

Let $x:=s-a, y:=s-b, z:=s-c, p:=xy+yz+zx, q:=xyz$. Then, assuming $s=1$ (due homogeneity), we obtain $x,y,z>0, x+y+z=1, a=1-x, b=1-y, c=1-z,\quad abc=p-q, \sum ab=1+p, \sum a^2=2(1-p), \sum a^3=\left(\sum a\right)^3+3abc-3\sum a\cdot\sum ab=8+3(p-q)-3\cdot 2\cdot(1+p)=2+3(p-q)-6p$,

$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}=\sqrt{xyz}=\sqrt{q},$$

$R=\dfrac{abc}{4rs}=\dfrac{p-q}{4\sqrt{q}}$ and

$$\delta=\sum\frac{a^2}{bc}-\frac{4\sum a^2}{\sum ab}+\frac{2r}{R}=\frac{\sum a^3}{abc}-\frac{4\sum a^2}{\sum ab}+\frac{2r}{R}=$$

$$\frac{2+3(p-q)-6p}{p-q}-\frac{8(1-p)}{1+p}+\frac{8q}{p-q}=\frac{2(1-3p)+8q}{p-q}-\frac{8(1-p)}{1+p}+3.$$

Since $3p=3\sum ab\leq\left(\sum a\right)^2=1$ and $9q\geq 4p-1$
(Schure's Inequality $\sum a(a-b)(a-c)\geq 0$ in p,q notation and normalized by $\sum x=1$) then

$$\delta=\frac{2(1-3p)+8q}{p-q}+3--\frac{8(1-p)}{1+p}\geq\frac{2(1-3p)+8\cdot\dfrac{4p-1}{9}}{p-\dfrac{4p-1}{9}}+3-\frac{8(1-p)}{1+p}=$$

$$\frac{(5-11p)(1-3p)}{(1+p)(1+5p)}\geq 0.$$

Also solved by Sicheng Du, Shenzhen, Guangdong, China; Theo Koupelis, Cape Coral, FL, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Jiang Lianjun, Quanzhou Middle School, GuiLin, China.

S616. Let ABC be a triangle with centroid G. The ray AG intersects the side BC and the circumcircle at A_1, A_2, respectively. Pairs of points B_1, B_2 and C_1, C_2 are defined similarly. Prove that

$$\text{(i)} \quad A_1A_2 + B_1B_2 + C_1C_2 \geq \frac{\sqrt{3}}{2} \cdot \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}.$$

$$\text{(ii)} \quad \frac{A_1A_2}{AA_2 + 2A_1A_2} + \frac{B_1B_2}{BB_2 + 2B_1B_2} + \frac{C_1C_2}{CC_2 + 2C_1C_2} = \frac{1}{2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Văcaru, Pitești, Romania

(i) We have

$$AA_1 \cdot A_1A_2 = BA_1 \cdot CA_1 \Leftrightarrow A_1A_2 = \frac{a^2}{4m_a} \; (1)$$

We obtain

$$A_1A_2 + B_1B_2 + C_1C_2 = \frac{1}{4}\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right) = \frac{1}{4}\left(\frac{a^3}{am_a} + \frac{b^3}{bm_b} + \frac{c^3}{cm_c}\right) (1)$$

But we have a

$$a^2 + b^2 + c^2 \geq 2\sqrt{3}am_a \Leftrightarrow \left(a^2 + b^2 + c^2\right)^2 \geq 12a^2 \frac{2\left(b^2 + c^2\right) - a^2}{4} \Leftrightarrow \left(a^2 + b^2 + c^2\right)^2 \geq 6a^2b^2 + 6a^2c^2 - 3a^4 \Leftrightarrow$$

$$\Leftrightarrow a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \geq 6a^2b^2 + 6a^2c^2 - 3a^4 \Leftrightarrow 4a^4 + b^4 + c^4 + 2b^2c^2 \geq 4a^2b^2 + 4a^2c^2 \Leftrightarrow$$

$$\Leftrightarrow 4a^4 + \left(b^2 + c^2\right)^2 \geq 4a^2\left(b^2 + c^2\right),$$

which is AM ≥ GM.We obtain

$$A_1A_2 + B_1B_2 + C_1C_2 \overset{(1)}{=} \frac{1}{4}\left(\frac{a^3}{am_a} + \frac{b^3}{bm_b} + \frac{c^3}{cm_c}\right) \geq \frac{1}{4}\left(\frac{a^3}{\frac{a^2+b^2+c^2}{2\sqrt{3}}} + \frac{b^3}{\frac{a^2+b^2+c^2}{2\sqrt{3}}} + \frac{c^3}{\frac{a^2+b^2+c^2}{2\sqrt{3}}}\right) = \frac{\sqrt{3}}{2} \cdot \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}.$$

(ii) We have

$$\frac{A_1A_2}{AA_1 + 3A_1A_2} = \frac{1}{\frac{AA_1}{A_1A_2} + 3} = \frac{1}{\frac{AA_1^2}{AA_1 \cdot A_1A_2} + 3} = \frac{1}{\frac{AA_1^2}{\frac{a^2}{4}} + 3} = \frac{1}{\frac{4AA_1^2}{a^2} + 3} = \frac{1}{\frac{2(b^2+c^2)-a^2}{a^2} + 3} = \frac{a^2}{2\left(a^2 + b^2 + c^2\right)} \; (1)$$

Following the same path, we have

$$\frac{B_1B_2}{BB_2 + 2B_1B_2} = \frac{b^2}{2\left(a^2 + b^2 + c^2\right)} \; (2)$$

and

$$\frac{C_1C_2}{CC_2 + 2C_1C_2} = \frac{c^2}{2\left(a^2 + b^2 + c^2\right)} \; (3)$$

Adding relationships (1), (2), (3), we obtain

$$\frac{A_1A_2}{AA_2 + 2A_1A_2} + \frac{B_1B_2}{BB_2 + 2B_1B_2} + \frac{C_1C_2}{CC_2 + 2C_1C_2} = \frac{1}{2}.$$

Also solved by Sicheng Du, Shenzhen, Guangdong, China; Theo Koupelis, Cape Coral, FL, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Jiang Lianjun, Quanzhou Middle School, GuiLin, China.

S617. Consider the polynomial $f(X) = 1 + X + X^2 + \cdots + X^{2023}$. Find the coefficient of X^{2023}in the polynomial $f(X^5) f(X^9)$.

Proposed by Mircea Becheanu, Canada

Solution by Anderson Torres, São Paulo, Brazil

$f(X^5)$ is a sum of powers, all multiple of 5; $f(X^9)$ is a sum of powers, all multiple of 9. Multiplying them generate a polynomial whose terms are basically sums of powers in the form X^{5a+9b}. Then the problem is asking us to calculate how many non-negative solutions are for $5a + 9b = 2023$. Notice that we can write $a = 9z + 5$ and $b = 222 - 5z$ for any integer z. The obvious limitations are $0 \le a, b \le 2023$, which imply $0 \le z \le 224$ and $0 \le z \le 44$.

Therefore, the answer is 45.

Also solved by Aaron Kim, Bronx Science, NY, USA; Daniel Pascuas, Barcelona, Spain; Brian Bradie, Newport News, VA, USA; Adam John Frederickson, Utah Valley University, UT, USA; Theo Koupelis, Cape Coral, FL, USA; Daniel Văcaru, Pitești, Romania; G. C. Greubel, Newport News, VA, USA; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Jishnu Ghosh, South Point High School, Kolkata, West Bengal,India; Matthew Too, Brockport, NY, USA; Sundaresh H R, Shivamogga, India; Vishwesh Ravi Shrimali, Jaipur, India.

S618. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{1}{abc}+\frac{4\sqrt{2}}{a^2+b^2+c^2}\geq\frac{9+4\sqrt{2}}{ab+bc+ca}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Suppose that $a=\max(a,b,c)$. In homogeneous form, the inequality can be written as

$$\frac{(a+b+c)(ab+bc+ca)}{abc}-9-\frac{4\sqrt{2}(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2}\geq 0,$$

or

$$\frac{2a(b-c)^2+(b+c)(a-b)(a-c)}{abc}-\frac{4\sqrt{2}\left[(b-c)^2+(a-b)(a-c)\right]}{a^2+b^2+c^2}\geq 0.$$

It remains to show that

$$a^2+b^2+c^2\geq 2\sqrt{2}bc,$$

which is true, because

$$a^2+b^2+c^2\geq 3\left(\frac{b^2+c^2}{2}\right)\geq 3bc\geq 2\sqrt{2}bc$$

and

$$(a^2+b^2+c^2)(b+c)\geq 4\sqrt{2}abc,$$

is true because

$$(a^2+b^2+c^2)(b+c)\geq(a^2+2bc)2\sqrt{bc}\geq 2a\sqrt{2bc}\cdot 2\sqrt{bc}=4\sqrt{2}abc.$$

The equality holds when $a=b=c$ and $a=\sqrt{2}b=\sqrt{2}c$.

Also solved by Sicheng Du, Shenzhen, Guangdong, China; Theo Koupelis, Cape Coral, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U613. Prove that there are infinitely many polynomials $P(x)$ with real coefficients such that

$$P(x)^2 + P(y)^2 + P(z)^2 + 2P(x)P(y)P(z) = 1,$$

for all real numbers x, y, z which satisfy the condition $x^2 + y^2 + z^2 = xyz + 4$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Li Zhou, Polk State College, USA
First, real x, y, z satisfying $x^2 + y^2 + z^2 = xyz + 4$ can be parametrized by

$$x = u(e^r + e^{-r}),\ y = v(e^s + e^{-s}),\ z = uv(e^{r+s} + e^{-r-s}),$$

where $u, v, \in \{\pm 1\}$ and r, s are either both real or both purely imaginary.

Now for any $n \in \mathbb{N}$, $-u(e^{nr} + e^{-nr})/2$ can be expressed unqiuely as a polynomial in $u(e^r + e^{-r})$, with real coefficients (similar to the Chebyshev polynomial). Denote this polynomial by P_n. That is,

$$P_n\left(u(e^r + e^{-r})\right) = -\frac{u(e^{nr} + e^{-nr})}{2}$$

for all real or purely imaginary r. Then we can readily check that

$$\begin{aligned} &P_n\left(u(e^r + e^{-r})\right)^2 + P_n\left(v(e^s + e^{-s})\right)^2 + P_n\left(uv(e^{r+s} + e^{-r-s})\right)^2 \\ &\qquad + 2P_n\left(u(e^r + e^{-r})\right) P_n\left(v(e^s + e^{-s})\right) P_n\left(uv(e^{r+s} + e^{-r-s})\right) \\ &= \left(\frac{e^{nr} + e^{-nr}}{2}\right)^2 + \left(\frac{e^{ns} + e^{-ns}}{2}\right)^2 + \left(\frac{e^{nr+ns} + e^{-nr-ns}}{2}\right)^2 \\ &\qquad - \frac{1}{4}(e^{nr} + e^{-nr})(e^{ns} + e^{-ns})(e^{nr+ns} + e^{-nr-ns}) = 1. \end{aligned}$$

Also solved by Arkady Alt, San Jose, CA, USA; Adam John Frederickson, Utah Valley University, UT, USA.

U614. Let $a_k > 0$, $k = 1, 2, \dots$ and $r, s > 0$. Prove that for $s \geq r$ if S_1 converges, also S_2 converges

$$S_1 = \sum_{k=3}^{\infty} \frac{1}{a_k (\ln(\ln a_k))^r}, \qquad S_2 = \sum_{k=3}^{\infty} \frac{1}{a_k (\ln(\ln k))^s}$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

Solution by the author

It suffices to prove the $s = r$ case.

Let $c_k > 0$ and $d_k > 0$ be two sequences such that $\sum c_k < +\infty$ and $\sum d_k = +\infty$. It follows $\limsup d_k / c_k = \infty$ because otherwise $d_k / c_k \leq A$ definitively for a certain positive number A and then $\sum d_k$ also would converge.

Let's apply this result to our case supposing by contradiction that

$$\sum_{k=3}^{\infty} \frac{1}{a_k (\ln(\ln k))^r} = \infty$$

It follows

$$\limsup \frac{(\ln(\ln a_k))^r}{(\ln(\ln k))^r} = \infty$$

Let $K \doteq \{k_i\}_{i=1}^{\infty}$ the largest subsequence of the naturals such

$$k \in K \Longrightarrow (\ln(\ln a_k))^r \geq A (\ln(\ln k))^r, \qquad A > 1$$

$$\ln(\ln a_k) \geq A^{1/r} \ln(\ln k) \quad \Longleftrightarrow \quad a_k \geq e^{(\ln k)^{A^{1/r}}}$$

$$\sum_{k=3, k \in K}^{\infty} \frac{1}{a_k (\ln(\ln k))^r} \leq \sum_{k=3, k \in K}^{\infty} \frac{1}{e^{(\ln k)^{A^{1/r}}} A (\ln(\ln k))^r} < \infty$$

because

$$\lim_{k \to \infty} \frac{k^{A^{1/r}}}{e^{(\ln k)^{A^{1/r}}}} = e^{A^{1/r} \ln k - (\ln k)^{A^{1/r}}} = e^{-\infty} = 0$$

Hence

$$\sum_{k=3, k \notin K}^{\infty} \frac{1}{a_k (\ln(\ln k))^r} = \infty$$

but this would mean

$$\limsup_{k \to \infty, k \notin K} \frac{(\ln(\ln a_k))^r}{(\ln(\ln k))^r} = \infty$$

contradicting that K is the largest set.

If $s < r$ the result is untrue. Let's take $a_k = k \ln k (\ln(\ln k))^{1-r+\delta}$, $\delta > 0$

$$\ln \ln a_k = \ln \ln (k \ln k (\ln(\ln k))^{1-r+\delta}) \geq \ln \ln k$$

hence

$$\sum_{k=3}^{\infty} \frac{1}{a_k (\ln(\ln a_k))^r} \leq \sum_{k=3}^{\infty} \frac{1}{k \ln k (\ln(\ln k))^{1-r+\delta} (\ln(\ln k))^r} =$$
$$= \sum_{k=3}^{\infty} \frac{1}{k \ln k (\ln(\ln k))^{1+\delta}} < \infty$$

This may be seen by the Cauchy–condensation–test

$$\sum_{k=3}^{\infty} \frac{1}{k \ln k (\ln(\ln k))^{1+\delta}} < \infty \iff \sum_{k=3}^{\infty} \frac{2^k}{2^k \ln(2^k)(\ln(\ln 2^k))^{1+\delta}} < \infty$$

$$\sum_{k=3}^{\infty} \frac{2^k}{2^k \ln(2^k)(\ln(\ln 2^k))^{1+\delta}} = \sum_{k=3}^{\infty} \frac{1}{k \ln 2 (\ln k + \ln\ln 2))^{1+\delta}}$$

By applying Cauchy's test again

$$\sum_{k=3}^{\infty} \frac{1}{k \ln 2 (\ln(k \ln 2))^{1+\delta}} < \infty \iff \sum_{k=3}^{\infty} \frac{2^k}{2^k \ln 2 (\ln 2^k + \ln\ln 2))^{1+\delta}} < \infty$$

and this is true because the general term of the last series goes to zero as $k^{-1-\delta}$

On the other hand $\ln\ln a_k \leq C \ln\ln k$ for any $k \geq 3$ if C is large enough.

$$\sum_{k=3}^{\infty} \frac{1}{a_k (\ln(\ln k))^s} \geq \sum_{k=3}^{\infty} \frac{1}{k \ln k (\ln(\ln k))^{1-r+s+\delta}} = \infty$$

if $\delta < r - s$

U615. Evaluate

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos(\sin(\cos x)) - 1}{(\frac{\pi}{2} - x)^2}.$$

Proposed by Mircea Becheanu, Canada

Solution by Marian Ursărescu, Roman-Vodă, National College, Roman, Romania

Let $\dfrac{\pi}{2} - x = t, x \to \dfrac{\pi}{2} \Rightarrow t \to 0$. We calculate

$$L = \lim_{t\to 0} \frac{\cos(\sin(\cos(\frac{\pi}{2} - t))) - 1}{t^2}$$

But

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos\alpha}{2} \text{ and}$$

$$\cos(\frac{\pi}{2} - t) = \sin t(3)$$

From three equation above we get:

$$L = \lim_{t\to 0} \frac{\cos(\sin(\sin t)) - 1}{t^2} = \lim_{t\to 0} \frac{-2\sin^2(\frac{\sin(\sin t)}{2})}{t^2} =$$

$$= -2\lim_{t\to 0} \left(\frac{\sin(\frac{\sin(\sin t)}{2})}{\frac{\sin(\sin t)}{2}}\right)^2 \cdot \frac{1}{4} \left(\frac{\sin(\sin t)}{\sin t}\right)^2 \cdot \left(\frac{\sin t}{t}\right)^2 = -\frac{1}{2}$$

Also solved by Aaron Kim, Bronx Science, NY, USA; Daniel Pascuas, Barcelona, Spain; Brian Bradie, Newport News, VA, USA; Adam John Frederickson, Utah Valley University, UT, USA; Theo Koupelis, Cape Coral, FL, USA; Vicente Vicario Garcia, Sevilla, Spain; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, NY, USA; Ibrahim Huseynov, BSCS; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Matthew Too, Brockport, NY, USA; Magdalene Hantho, SUNY Brockport, NY, USA; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Prajnanaswaroopa S, Bengaluru, Karnataka, India; Soham Dutta, India; Sundaresh H R, Shivamogga, India; Vishwesh Ravi Shrimali, Jaipur, India; Yunyong Zhang, China; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U616. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(f(f(x))) + f(f(x)) - f(x) - x = 0,$$

for all $x \in \mathbb{R}$. Prove that $f(f(x)) = x$ for all $x \in \mathbb{R}$.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

Solution by the authors

The function is injective because $f(x) = f(y)$ implies $f^{[n]}(x) = f^{[n]}(y)$ (for any n), hence $f(x) = f(y)$ leads, by the given functional equation, to $x = y$. Also, f is surjective. Indeed, being continuous and injective, f is strictly monotone. Let $m = \inf_{x\in\mathbb{R}} f(x)$ (which exists in $\overline{\mathbb{R}}$, by the monotony of f) and let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim_{n\to\infty} f(x_n) = m$. Assume that m is a real number. By the continuity of f, we have $f(m) = \lim_{n\to\infty} f(f(x_n))$ and $f(f(m)) = \lim_{n\to\infty} f(f(f(x_n)))$, therefore, because

$$x_n = f(f(f(x_n))) + f(f(x_n)) - f(x_n), \quad \forall n \in \mathbb{N}^*,$$

we infer that the sequence $(x_n)_{n\geq 1}$ is convergent. Now, if $l = \lim_{n\to\infty} x_n$ we get (again by the continuity of f) $m = \lim_{n\to\infty} f(x_n) = f(l)$. This, obviously, is not possible for a function $f : \mathbb{R} \to \mathbb{R}$ which is strictly monotone. (For instance, if f was strictly increasing, we would get the contradiction $f(t) < f(l) = m = \inf_{x\in\mathbb{R}} f(x)$ when $t < l$.) Thus the assumption $m \in \mathbb{R}$ is wrong, hence $\inf_{x\in\mathbb{R}} f(x)$ must be either ∞ or $-\infty$. Similarly, we see that $\sup_{x\in\mathbb{R}} f(x)$ is either ∞ or $-\infty$. As f is continuous and strictly monotone, the fact that $f(\mathbb{R}) = \mathbb{R}$ follows, that is, f is surjective.

Thus any solution f of the given functional equation is bijective and we can define (for positive integer n)

$$f^{[-n]} = f^{-1} \circ \cdots \circ f^{-1},$$

where f^{-1} (the inverse of f) appears n times. We still denote $f^{[0]}$ the identity of $\mathbb{R}$, that is, $f^{[0]}(x) = x$ for all $x \in \mathbb{R}$. Then the equation

$$f^{[n+3]}(x) + f^{[n+2]}(x) - f^{[n+1]}(x) - f^{[n]}(x) = 0,$$

is easy to check for any $x \in \mathbb{R}$ and any $n \in \mathbb{Z}$ (just replace x with $f^{[n]}(x)$ in the given functional equation). Then, by using the theory of linear recurrences (or just by inducting on n, in both directions) we get

$$f^{[n]}(x) = \frac{x + 2f(x) + f^{[2]}(x)}{4} + (-1)^n \left(\frac{3x - 2f(x) - f^{[2]}(x)}{4} + \frac{f^{[2]}(x) - x}{2} n \right)$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$. In particular, for $n = 2m$ even we get

$$f^{[2m]}(x) = x + m(f^{[2]}(x) - x)$$

for all $x \in \mathbb{R}$ and all $m \in \mathbb{Z}$.

As f is continuous and injective, it must be strictly monotone, which implies that any function $f^{[2m]}$ is strictly increasing. Thus, for any real numbers x and y, with $x < y$, and any $m \in \mathbb{Z}$, we have

$$f^{[2m]}(x) < f^{[2m]}(y) \Leftrightarrow x + m(f^{[2]}(x) - x) < y + m(f^{[2]}(y) - y),$$

which leads to

$$\frac{x}{m} + f^{[2]}(x) - x < \frac{y}{m} + f^{[2]}(y) - y$$

for positive m, and to

$$\frac{x}{m} + f^{[2]}(x) - x > \frac{y}{m} + f^{[2]}(y) - y$$

for negative m. Passing to the limit for $m \to \infty$ in the first inequality, and for $m \to -\infty$ in the second, we get

$$f^{[2]}(x) - x \leq f^{[2]}(y) - y$$

and

$$f^{[2]}(x) - x \geq f^{[2]}(y) - y$$

respectively. Thus, for any $x < y$ we actually have

$$f^{[2]}(x) - x = f^{[2]}(y) - y,$$

that is, the function $x \mapsto f^{[2]}(x) - x$ is constant. So, there is $c \in \mathbb{R}$ such that

$$f^{[2]}(x) - x = c$$

for every $x \in \mathbb{R}$; replacing here x with $f(x)$ yields

$$f^{[3]}(x) - f(x) = c$$

for every real number x. Now we see that the above two relations together with the initial equation

$$f^{[3]}(x) + f^{[2]}(x) - f(x) - x = 0,$$

imply $c = 0$, and, consequently

$$f^{[2]}(x) - x = 0$$

follows for every $x \in \mathbb{R}$, as desired.

Also solved by Toyesh Prakash Sharma, Agra College, Agra, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U617. The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$; $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Evaluate

$$\lim_{n\to\infty}\left[\left(1+\frac{F_1}{F_{2n}}\right)\left(1+\frac{F_2}{F_{2n}}\right)\cdots\left(1+\frac{F_n}{F_{2n}}\right)\right]^{L_n}.$$

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

Solution by the author

Let L be the proposed limit. Then, by taking logarithms we get

$$\begin{aligned}
\ln L &= \lim_{n\to\infty} L_n \sum_{k=1}^{n} \ln\left(1+\frac{F_k}{F_{2n}}\right) \\
&= \lim_{n\to\infty} L_n \sum_{k=1}^{n} \left(\frac{F_k}{F_{2n}} - \frac{F_k^2}{2F_{2n}^2}\right) \\
&= \lim_{n\to\infty} L_n \left(\frac{F_{n+2}-1}{F_nL_n} - \frac{F_nF_{n+1}}{2F_n^2L_n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{\alpha^{n+2}}{\alpha^n} - \frac{\alpha^{n+1}}{2\alpha^{3n}}\right) \\
&= \alpha+1
\end{aligned}$$

where we have used that $\ln(1+x) = x - \frac{x^2}{2} + O(x^3)$, and $\sum_{k=1}^{n} F_k = F_{n+2} - 1$, $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$, and $F_{2n} = F_nL_n$, and also the Binet's formulas for Fibonacci and Lucas numbers, that is $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$, $L_n = \alpha^n + \beta^n$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

Hence, $L = e^{\alpha+1}$.

Also solved by Brian Bradie, Newport News, VA, USA; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; G. C. Greubel, Newport News, VA, USA; Jishnu Ghosh, South Point High School, Kolkata, West Bengal,India; Matthew Too, Brockport, NY, USA; Yunyong Zhang, China; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U618. Let d be a positive integer and $n = \binom{d}{2}$. Let $z_1, \dots, z_d$ be complex numbers on a unit circle. For every integers i, j such that $1 \le i < j \le d$ we consider the positive real number $x = |z_i - z_j|^2$. These numbers are arranged in some order to obtain a sequence $x_1, x_2, \dots, x_n$. Prove that

$$\sum_{1 \le i < j \le n} x_i x_j \le \frac{d^4 - 3d^2}{2}$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Matthew Too, Brockport, NY, USA

We know due to symmetry that

$$2 \sum_{1 \le i < j \le n} x_i x_j = \left(\sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2 = \left(\sum_{1 \le i < j \le d} |z_i - z_j|^2 \right)^2 - \sum_{1 \le i < j \le d} |z_i - z_j|^4.$$

Since $|z_i - z_j|^2 = (z_i - z_j)(\overline{z_i} - \overline{z_j}) = 2 - (z_i \overline{z_j} + z_j \overline{z_i})$, then

$$\sum_{1 \le i < j \le d} |z_i - z_j|^2 = \sum_{1 \le i < j \le d} \left[2 - (z_i \overline{z_j} + z_j \overline{z_i}) \right] = 2\binom{d}{2} - \sum_{1 \le i \ne j \le d} z_i \overline{z_j} = d(d-1) - \left(\left| \sum_{i=1}^{d} z_i \right|^2 - d \right) = d^2 - \left| \sum_{i=1}^{d} z_i \right|^2$$

and

$$\begin{aligned}
\sum_{1 \le i < j \le d} |z_i - z_j|^4 &= \sum_{1 \le i < j \le d} \left[2 - (z_i \overline{z_j} + z_j \overline{z_i}) \right]^2 = \sum_{1 \le i < j \le d} \left[4 - 4(z_i \overline{z_j} + z_j \overline{z_i}) + (z_i \overline{z_j} + z_j \overline{z_i})^2 \right] \\
&= 4\binom{d}{2} - 4 \sum_{1 \le i \ne j \le d} z_i \overline{z_j} + \sum_{1 \le i < j \le d} \left(z_i^2 \overline{z_j}^2 + z_j^2 \overline{z_i}^2 + 2 \right) \\
&= 6\binom{d}{2} - 4 \left(\left| \sum_{i=1}^{d} z_i \right|^2 - d \right) + \sum_{1 \le i \ne j \le d} z_i^2 \overline{z_j}^2 \\
&= 3d(d-1) + 4d - 4 \left| \sum_{i=1}^{d} z_i \right|^2 + \left(\left| \sum_{i=1}^{d} z_i^2 \right|^2 - d \right) \\
&= 3d^2 - 4 \left| \sum_{i=1}^{d} z_i \right|^2 + \left| \sum_{i=1}^{d} z_i^2 \right|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{1 \le i < j \le n} x_i x_j &= \frac{1}{2} \left[\left(\sum_{1 \le i < j \le d} |z_i - z_j|^2 \right)^2 - \sum_{1 \le i < j \le d} |z_i - z_j|^4 \right] \\
&= \frac{1}{2} \left[\left(d^2 - \left| \sum_{i=1}^{d} z_i \right|^2 \right)^2 - \left(3d^2 - 4 \left| \sum_{i=1}^{d} z_i \right|^2 + \left| \sum_{i=1}^{d} z_i^2 \right|^2 \right) \right] \\
&= \frac{1}{2} \left[d^4 - 2d^2 \left| \sum_{i=1}^{d} z_i \right|^2 + \left| \sum_{i=1}^{d} z_i \right|^4 - 3d^2 + 4 \left| \sum_{i=1}^{d} z_i \right|^2 - \left| \sum_{i=1}^{d} z_i^2 \right|^2 \right] \\
&= \frac{1}{2} \left[d^4 - 3d^2 + \left| \sum_{i=1}^{d} z_i \right|^4 - \left| \sum_{i=1}^{d} z_i^2 \right|^2 - (2d^2 - 4) \left| \sum_{i=1}^{d} z_i \right|^2 \right].
\end{aligned}$$

Note that according to the triangle inequality, $\left|\sum_{i=1}^d z_i\right| \le d$, so

$$\left|\sum_{i=1}^d z_i\right|^4 - \left|\sum_{i=1}^d z_i^2\right|^2 - (2d^2-4)\left|\sum_{i=1}^d z_i\right|^2 \le d^2\left|\sum_{i=1}^d z_i\right|^2 - \left|\sum_{i=1}^d z_i^2\right|^2 - (2d^2-4)\left|\sum_{i=1}^d z_i\right|^2$$
$$= -\left|\sum_{i=1}^d z_i^2\right|^2 - (d^2-4)\left|\sum_{i=1}^d z_i\right|^2 \le 0$$

for all $d > 1$. The conclusion follows.

Olympiad problems

O613. Solve in integers the equation
$$x^3 - 7xy + y^3 = 2023.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Theo Koupelis, Cape Coral, FL, USA

We have $2023 = 7 \cdot 17^2$ and thus $xy \neq 0$. Clearly, x, y cannot both be negative; also, if (x, y) is a solution, so is (y, x).

(i) Let $x, y > 0$. Then the given equation is equivalent to
$$(x+y)^3 - 2023 = xy\,[7 + 3(x+y)] \le \frac{(x+y)^2}{4} \cdot [7 + 3(x+y)]\,.$$

Simplifying we get $(x+y)^3 - 7(x+y)^2 \le 8092$ and thus $x + y \le 22$. But $(x+y)^3 > 2023$ and thus $x + y \ge 13$. Therefore $x + y \in \{13, 14, \ldots, 22\}$. Substituting into the given equation we get that only $x + y = 15$ leads to an integer value for xy, namely $xy = 26$. Therefore, x, y are solutions of the equation $t^2 - 15t + 26 = 0$, and thus $(x, y) = (2, 13)$ or $(13, 2)$ are acceptable solutions.

(ii) Let $y > 0$ and $x = -z < 0$. Then the given equation becomes $(y - z)^3 + 3yz(y - z) + 7yz = 2023$.

Case 1: If $y = z$, then $y = z = 17$ and thus $(x, y) = (-17, 17)$ is an acceptable solution. By symmetry so is $(x, y) = (17, -17)$.

Case 2: If $y > z$ we have
$$yz = \frac{2023 - (y - z)^3}{7 + 3(y - z)}.$$

But $yz \ge 1$ and thus $y - z \le 12$. Therefore $y - z \in \{1, 2, \ldots, 12\}$. Substituting into the given equation we find that only the values $y - z = 2$ and $y - z = 7$ lead to an integer value for yz, namely $yz = 155$ and $yz = 60$, respectively. The equation $t^2 - 2t + 155 = 0$ has no integer solutions, and the equation $t^2 - 7t + 60 = 0$ has the solution $(y, z) = (12, 5)$. Thus, an acceptable solution to the given equation is $(x, y) = (-5, 12)$ and by symmetry so is $(x, y) = (12, -5)$.

Case 3: If $y < z$, we rewrite the given equation as
$$(z - y)^3 + yz\,[3(z - y) - 7] = -2023.$$

For $z - y \ge 3$ the left-hand-side is positive and the right-hand-side is negative. Thus, we must have $z - y = 1$ or $z - y = 2$, which leads to $zy = 506$ or $zy = 2031$, respectively. The equation $y(1 + y) = 506$ has the solutions $y = 22$ and $y = -23$, which lead to the solutions $(x, y) = (22, -23)$ and $(x, y) = (-23, 22)$. The equation $y(2 + y) = 2031$ does not have integer solutions.

In summary, the solutions to the given equation are $(x, y) = (2, 13)$, $(13, 2)$, $(17, -17)$, $(-17, 17)$, $(-5, 12)$, $(12, -5)$, $(22, -23)$, and $(-23, 22)$.

Also solved by Aaron Kim, Bronx Science, NY, USA; Daniel Pascuas, Barcelona, Spain; Anderson Torres, Brazil; Adam John Frederickson, Utah Valley University, UT, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Garib Guluzade, ADA University, Baku, Azerbaijan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Soham Bhadra, India.

O614. Let a, b, c, d be positive real numbers such that

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}=1.$$

Prove that

$$abc+bcd+cda+dab+36 \geq 12(a+b+c+d).$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

With the substitutions $\frac{1}{1+a} \to a, \frac{1}{1+b} \to b, \frac{1}{1+c} \to c, \frac{1}{1+d} \to d$, it follows that $a+b+c+d=1$ and the inequality is equivalent with

$$\sum_{cyc} \frac{(1-a)(1-b)(1-c)}{abc} - 12\sum_{cyc}\frac{1-a}{a} + 36 \geq 0,$$

or

$$\sum_{cyc} a - 2\sum_{cyc} ab - 9\sum_{cyc} abc + 80abcd \geq 0.$$

We homogenize the inequality and we get the following inequality

$$\left(\sum_{cyc} a\right)^4 - 2\left(\sum_{cyc} a\right)^2 \sum_{cyc} ab - 9\sum_{cyc} a \sum_{cyc} abc + 80abcd \geq 0.$$

Without loss of generality assume that $a+b+c+d=4$. The inequality becomes

$$64 - 8\sum_{cyc} ab - 9\sum_{cyc} abc + 20abcd \geq 0.$$

Let $x=a-1$, $y=b-1$, $z=c-1$, $t=d-1$, $x+y+z+t=0$, $x,y,z,t \in [-1,3]$. We need to prove that

$$-6\sum_{cyc} xy + 11\sum_{cyc} xyz + 20xyzt \geq 0,$$

or

$$3\sum_{cyc} x^2 + 11\sum_{cyc} xyz + 20xyzt \geq 0,$$

or

$$9(x^2+y^2+z^2+t^2) + 11(x^3+y^3+z^3+t^3) + 60xyzt \geq 0.$$

Assuming that $x \leq y \leq z \leq t$ it is clearly that $x \leq 0 \leq t$. We have the following cases

1. $0 \leq y$, then $y+z+t=-x \leq 1 \Longrightarrow yzt < 1$. Since

$$3(y^2+z^2+t^2) \geq (y+z+t)^2 = x^2,$$

$$9(y^3+z^3+t^3) \geq (y+z+t)^3 = -x^3, \text{ (Hölder's Inequality)}$$

$$27yzt \leq (y+z+t)^3 = -x^3, \text{ (AM-GM Inequality)}$$

we need to prove that

$$9x^2 + 3x^2 + 11x^3 - \frac{11x^3}{9} - \frac{20x^4}{9} \geq 0$$

that is

$$\frac{4}{9}x^2(x+1)(27-5x) \geq 0,$$

clearly true.

2. $y \leq 0 \leq z$, which is true because

$$9(x^2+y^2) + 9(x^3+y^3) = 9x^2(x+1) + 9y^2(y+1) \geq 0,$$

$$8(z^3+t^3) \geq 2(z+t)^3 = -2(x+y)^3 \geq -2(x^3+y^3),$$

hence

$$9(x^2+y^2)+8(z^3+t^3)+11(x^3+y^3) \geq 0$$

and obviously $9(z^2+t^2)+3(z^3+t^3)+60xyzt \geq 0$.

3. $z \leq 0$, then $-x-y-z=t\leq 3$. The inequality can be rewritten as

$$9\left[x^2+y^2+z^2+(x+y+z)^2\right]+11\left[x^3+y^3+z^3-(x+y+z)^3\right]-60xyz(x+y+z)\geq 0,$$

or

$$6(x^2+y^2+z^2+xy+yz+zx)-11\sum_{cyc} z(x^2+y^2)-22xyz-20xyz(x+y+z)\geq 0,$$

or

$$6(x^2+y^2+z^2+xy+yz+zx)-11\sum_{cyc} z(x-y)^2-4xyz\left[22+5(x+y+z)\right]\geq 0,$$

clearly true. The equality holds when $x=y=z=t=0$ or $x=-1$, $y=z=t=\frac{1}{3}$.

Also solved by Sicheng Du, Shenzhen, Guangdong, China; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Cape Coral, FL, USA.

O615. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$abc(\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}) \le 3$$

Proposed by Tran Tien Manh, Vinh City, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy

$$3\frac{1}{3}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}) \le 3\sqrt{\frac{a^2+b^2+c^2}{3}}$$

thus we prove

$$a+b+c=3 \implies (abc)^2(a^2+b^2+c^2) \le 3$$

Moreover by $abc \le (a+b+c)^3/27 \le 1$ we come to prove

$$a+b+c=3 \implies abc(a^2+b^2+c^2) \le 3$$

Let's change variables $a+b+c=3$, $ab+bc+ca=3v^2$, $abc=w^3$. The inequality reads as

$$u=1 \implies w^3(9u^2-6v^2) \le 3$$

that is

$$f(w^3) \doteq w^3(9-6v^2) \le 3 \tag{1}$$

The function $f(w^3)$ is linear increasing thus it holds if and only if it holds true for the minimum value of w. The minimum value of w is attained when $c=0$ (or cyclic) or $b=c$ (or cyclic).

$c=0$ is forbidden by the hypotheses but if we let a, b, c, assume also that value we can observe that $w=0$ and the inequality clearly holds true.

If $c=b$ whence $a=(3-b)/2$ we have that $abc(a^2+b^2+c^2) \le 3$ is equivalent to

$$\frac{3}{8}(a^3-6a^2+11a-8)(a-1)^2 \le 3 \iff h(a) \doteq a^3-6a^2+11a-8 \le 0 \quad 0 \le a \le 3$$

$$h'(a) = (a-2-\frac{1}{\sqrt{3}})(a-2+\frac{1}{\sqrt{3}}) \ge 0 \iff 0 \le a \le 2-\frac{1}{\sqrt{3}}, \quad 2+\frac{1}{\sqrt{3}} \le a \le 3$$

$$h(a) = \frac{2\sqrt{3}}{9}-2, \quad h(3) = -2$$

hence $h(a) < 0$ for any $0 \le a \le 3$ and this concludes the proof.

Also solved by Sicheng Du, Shenzhen, Guangdong, China; Batakogias Panagiotis, Velestino, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Theo Koupelis, Cape Coral, FL, USA; Jiang Lianjun, Quanzhou Middle School, GuiLin, China; Titu Zvonaru, Comănești, România.

O616. Let a, b, c be positive numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq 2(a^2+b^2+c^2)-(ab+bc+ca).$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Theo Koupelis, Cape Coral, FL, USA

Let $p=a+b+c=3$, $q=ab+bc+ca>0$, and $r=abc>0$. The positive numbers a, b, c are roots of the cubic $f(t)=t^3-3t^2+qt-r$, where $f'(t)=3t^2-6t+q$ and $f''(t)=6(t-1)$. From AM-GM we get $q \leq p^2/3=3$ and $r \leq (p/3)^3=1$. Setting $q=3(1-\omega^2)$, where $0 \leq \omega < 1$, we get that the extrema of $f(x)$ occur when $t=1 \pm \omega$, with $f(1-\omega) \geq 0$ and $f(1+\omega) \leq 0$. That is, $r \leq r_1=(1-\omega)^2(1+2\omega)$ and $r \geq r_2=(1+\omega)^2(1-2\omega)$. Clearly $r_2 \leq r_1 \Leftrightarrow -4\omega^3 \leq 0$, and $r_1 \leq 1 \Leftrightarrow \omega^2(2\omega-3) \leq 0$, both of which are obvious.

The desired inequality is

$$\frac{q^2-2pr}{r^2}-2\left(p^2-2q\right)+q \geq 0.$$

After substituting $p=3$ and $q=3(1-\omega^2)$, it is sufficient to show that

$$3(1+5\omega^2)r^2+6r-9(1-\omega^2)^2 \leq 0,$$

which is satisfied when

$$0 < r \leq r_0 = \frac{\sqrt{1+3(1+5\omega^2)(1-\omega^2)^2}-1}{1+5\omega^2}.$$

Thus, it is sufficient to show that

$$\begin{aligned} r_1 \leq r_0 &\iff (1+5\omega^2)(1-\omega)^2(1+2\omega)+1 \leq \sqrt{1+3(1+5\omega^2)(1-\omega^2)^2} \\ &\iff \omega^2(1-\omega)^2(1+5\omega^2)\left[20\omega^3(1-\omega)+2\omega^2+(3\omega-1)^2\right] \geq 0, \end{aligned}$$

which is obvious. Equality occurs when $\omega=0$, or when $a=b=c=1$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Soham Bhadra, India; Jiang Lianjun, Quanzhou Middle School, GuiLin, China.

O617. Let $a < b < c < d$ be positive integers. Prove that

$$\gcd(a! + 1, b! + 1, c! + 1, d! + 1) < d^{\frac{d-a}{3}}.$$

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Theo Koupelis, Cape Coral, FL, USA

Let $n = \gcd(a! + 1, b! + 1, c! + 1, d! + 1)$. Then $n \mid [(b! + 1) - (a! + 1)]$ or $n \mid (b! - a!)$ or $n \mid a![(a + 1)\cdots b - 1]$. But $(a! + 1, a!) = 1$ and thus $n \mid [(a + 1)\cdots b - 1]$. Therefore, $n \le (a + 1)\cdots b < b^{b-a} < d^{b-a}$. Similarly we get $n < d^{c-b}$, and $n < d^{d-c}$. Multiplying we get $n^3 < d^{d-a}$ or $n < d^{\frac{d-a}{3}}$.

Also solved by Soham Bhadra, India.

O618. Let ABC be an acute triangle. Prove that

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-\frac{3}{2}\ge k\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-\frac{1}{8}\right),$$

where $k=4\left(1+\sqrt{2}-\sqrt{2+\sqrt{2}}\right)^2$. When does equality hold?

Proposed by Marius Stănean, Zalău, România

Solution by the author

With the substitutions $A\to\pi-2A$, $B\to\pi-2B$, $C\to\pi-2C$ the our inequality becomes

$$\cos A+\cos B+\cos C-\frac{3}{2}\ge k\left(\cos A\cos B\cos C-\frac{1}{8}\right),$$

where $A,B,C\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$. Now, let $x=2\cos A$, $y=2\cos B$, $z=2\cos C$, $x,y,z\in\left[0,\sqrt{2}\right]$ and $x^2+y^2+z^2+xyz=4$. We need to prove that

$$4(x+y+z)-12\ge k(xyz-1).$$

Without loss of generality, we may assume that $x\ge y\ge z$, then $x\ge 1$. Choose a number $t>0$ for which $x^2+2t^2+xt^2=4$. Clearly, $t^2=2-x\le 1$ and

$$x^2+2t^2+xt^2=x^2+y^2+z^2+xyz$$

so

$$y^2+z^2-2t^2=x(t^2-yz).$$

If we assume that $t^2<yz$, then it follows that $y^2+z^2<2t^2$. On the other hand

$$t^2<yz\le\frac{y^2+z^2}{2}\Longrightarrow 2t^2<y^2+z^2$$

which is a contradiction. It follows that $t^2\ge yz$ and $y^2+z^2\ge 2t^2$. Denote

$$f(x,y,z)=4(x+y+z)-12\ge k(xyz-1).$$

We notice that

$$(y+z)^2+yz(x-2)=4-x^2\Longleftrightarrow(y+z)^2=(2-x)(2+x+yz)=t^2(2+x+yz)$$

so

$$4t^2-(y+z)^2=t^2(2-x-yz)\Longrightarrow 2t-y-z=\frac{(2-x)(t^2-yz)}{2t+y+z}.$$

$$\begin{aligned}f(x,y,z)-f(x,t,t)&=4(y+z-2t)+kx(t^2-yz)\\&=(t^2-yz)\left(kx-\frac{4(2-x)}{2t+y+z}\right)\\&=\frac{(t^2-yz)[2ktx+kx(y+z)+4x-8]}{2t+y+z}.\end{aligned}$$

We evaluate

$$\begin{aligned}y+z=&2\cos B+2\cos C=4\cos\frac{B+C}{2}\cos\frac{B-C}{2}\\=&4\sin\frac{A}{2}\cos\frac{C-B}{2}\ge 4\sin\frac{\pi}{8}\cos\frac{\pi}{8}=\sqrt{2},\end{aligned}$$

since $\frac{A}{2} \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right]$ and $\frac{C-B}{2} \in \left[0, \frac{\pi}{8}\right]$. We have

$$\begin{aligned} 2ktx + kx(y+z) + 4x - 8 \geq & 2kx\sqrt{2-x} + k\sqrt{2}x + 4x - 8 \\ \geq & 2.4x\sqrt{2-x} + 1.2\sqrt{2}x + 4x - 8 \geq 0 \end{aligned}$$

which is true for $x \in [1, \sqrt{2}]$, $k \approx 1.2834$(also using the computer), therefore $f(x,y,z) \geq f(x,t,t)$. It remains to show that

$$f(x,t,t) \geq 0 \Longleftrightarrow$$
$$4(x+2t) - 12 - kxt^2 + k \geq 0 \Longleftrightarrow$$
$$(t-1)^2[k(t+1)^2 - 4] \geq 0 \Longleftrightarrow$$
$$\sqrt{k}(t-1)^2\left[\sqrt{k}(t+1) + 2\right]\left(t - \sqrt{2-\sqrt{2}}\right) \geq 0$$

which is true because $t^2 = 2 - x \geq 2 - \sqrt{2} \Longrightarrow t - \sqrt{2-\sqrt{2}} \geq 0$. The equality holds when $t = 1$ which means $x = y = z = 1$, or $t = \sqrt{2-\sqrt{2}}$ which means $a = \sqrt{2}, b = c = \sqrt{2-\sqrt{2}}$ and its cyclic permutations.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Theo Koupelis, Cape Coral, FL, USA.