On a Combinatorial Identity

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Abstract

In this note we present a combinatorial proof for an identity proposed at Putnam Mathematical Competition [1]. Our approach allows for various generalizations. We propose two of them in the second part of the paper.

1 Introduction

The following problem was proposed at the 81st Putnam Mathematical Competition (2021): A2. Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}. \tag{1}$$

The answer is 2^{2k} so we propose a combinatorial proof for

$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j} = 2^{2k} \tag{2}$$

Unlike the two solutions presented in [1], our proof is purely combinatorial. The approach consists in counting NE lattice paths of specific lengths in two different ways. This type of approach is intuitive and allows for various generalizations. In the second half of the paper, we propose two such generalizations: a non-symmetric 2-dimensional variation of (2), which includes (2) as a particular case, and a symmetric 3-dimensional extension of (2). The symmetric form can naturally be generalized to any dimension $d \geq 2$, by following the same line of thought.

2 NE lattice paths

We remind here the definition and two basic results relative to NE lattice paths [2].

Definition 1. *A* ***NE (North-East) lattice path*** *L is a sequence*

$$(v_0, v_1, ..., v_k)$$

of points (vectors) in $\mathbf{Z}^2$ such that $v_{i+1} - v_i$ is either $(1,0)$ or $(0,1)$. We call ***step*** *each pair (v_i, v_{i+1}) of consecutive points in L. The step is of* ***type N (North)*** *if $v_{i+1} - v_i = (0,1)$ or of* ***type E (East)*** *otherwise. We say that L is a (NE lattice) path* ***from*** *v_0 (the* ***beginning*** *of L)* ***to*** *v_k (the* ***end*** *of L). The* ***length*** *of L is defined as being k (the number of steps).*

Remark 1. *A NE lattice path L can alternatively be defined as a pair $(v_0; (s_1, ..., s_k))$, where*

1. *$v_0 \in \mathbf{Z}^2$ is the beginning point of L*
2. *$(s_1, s_2, ...s_k)$ is the sequence of steps of L. .*

In this alternative definition, we will denote by the letter N (respectively E) a step of type N (resp. of type E). For example, $L = ((1,4), (1,5), (2,5))$ is the same as $L = ((1,4); (N,E))$.

Proposition 1. *Let v be a point in $\mathbf{Z}^2$. There are 2^k NE lattice paths of length k beginning in v.*

Proof. Such a path can be written as $(v; (s_1, ..., s_k))$. v is constant while each step s_i can be either "N" or "E". Therefore, there are 2^k such paths. $\square$

Proposition 2. *Let $v = (a, b)$ and $w = (c, d)$ be points in $\mathbf{Z}^2$ such that $a \leq c$ and $b \leq d$. Then, there are*

$$\binom{(c-a)+(d-b)}{c-a}$$

NE lattice paths from v to w.

Proof. In order to get from v to w, one needs to perform $(c-a)+(d-b)$ steps, $c-a$ of them being of type E and $d-b$ of type N. This can be achieved in

$$\binom{(c-a)+(d-b)}{(c-a)}$$

ways. $\square$

3 A combinatorial proof

The proof of (2) that we are proposing in this section consists in counting in two different ways a certain set of NE lattice paths.

Let $\mathcal{L}_{2k}$ be the set of NE lattice paths beginning in $(0, 0)$ and having length $2k$. From Proposition 1

$$\mid \mathcal{L}_{2k} \mid = 2^{2k} \tag{3}$$

Let S_k be the square $\{(x, y) \in \mathbf{R}^2 \mid 0 \leq x, y \leq k\}$. Any path in $\mathcal{L}_{2k}$ belongs to exactly one of the following sets:

1. $\mathcal{C}_k = \{L \in \mathcal{L}_{2k} \mid L \text{ ends in } (k, k)\}$
2. $\mathcal{E}_k(j) = \{L \in \mathcal{L}_{2k} \mid L \text{ quits } S_k \text{ at } (k, j)\}$, with $0 \leq j \leq k-1$
3. $\mathcal{N}_k(j) = \{L \in \mathcal{L}_{2k} \mid L \text{ quits } S_k \text{ at } (j, k)\}$, with $0 \leq j \leq k-1$.

Therefore,

$$\mid \mathcal{L}_{2k} \mid = \mid \mathcal{C}_k \mid + \sum_{j=0}^{k-1} \mid \mathcal{E}_k(j) \mid + \sum_{j=0}^{k-1} \mid \mathcal{N}_k(j) \mid \tag{4}$$

From Proposition 2

$$\mid \mathcal{C}_k \mid = \binom{2k}{k} \tag{5}$$

For any $0 \leq j \leq k-1$, a lattice path $L \in \mathcal{E}_k(j)$ is the concatenation of

1. a subpath S_1 from $(0, 0)$ to (k, j)
2. a step of type E
3. a subpath S_2 of length $k-j-1$ beginning at $(k+1, j)$.

From Proposition 2, S_1 can be chosen in $\binom{k+j}{j}$ ways, while, from Proposition 1, S_2 can be chosen in 2^{k-j-1} ways. Therefore

$$\mid \mathcal{E}_k(j) \mid = 2^{k-j-1}\binom{k+j}{j} \tag{6}$$

By symmetry, $\mid \mathcal{N}_k(j) \mid = \mid \mathcal{E}_k(j) \mid$ so

$$\mid \mathcal{N}_k(j) \mid = 2^{k-j-1}\binom{k+j}{j} \tag{7}$$

By replacing (5), (6) and (7) in (4), we obtain

$$\mid \mathcal{L}_{2k} \mid = \binom{2k}{k} + 2\sum_{j=0}^{k-1} 2^{k-j-1}\binom{k+j}{j} = \sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j} \tag{8}$$

From (3) and (8) we obtain (2), the relation to be proven.

4 Two generalizations

In this section we propose two generalization of the identity (2). The first one is obtained by considering NE lattice paths of length $k+l$, with k, l independent, nonnegative integers. For $k = l$ we get (2). The second one is a variation of (2) where we consider the set of 3-dimensional lattice paths of length $3k$, with k a nonnegative integer.

4.1 A non-symmetric generalization

Proposition 3. *For k, l nonnegative integers, the following identity holds:*

$$\binom{k+l}{k} + \sum_{j=0}^{l-1} 2^{l-j-1}\binom{k+j}{j} + \sum_{j=0}^{k-1} 2^{k-j-1}\binom{l+j}{j} = 2^{k+l} \tag{9}$$

Proof. Let $\mathcal{L}_{k+l}$ be the set of NE lattice paths beginning in $(0,0)$ and having length $k+l$. From Prop. 1

$$\mid \mathcal{L}_{k+l} \mid = 2^{k+l} \tag{10}$$

Let $R_{k,l}$ be the rectangle $\{(x,y) \in \mathbf{R}^2 \mid 0 \le x \le k, 0 \le y \le l\}$. Any path in $\mathcal{L}_{k+l}$ belongs to exactly one of the following sets:

1. $\mathcal{C}_{k,l} = \{L \in \mathcal{L}_{k+l} \mid L \text{ ends in } (k,l)\}$
2. $\mathcal{E}_k(j) = \{L \in \mathcal{L}_{k+l} \mid L \text{ quits } R_{k,l} \text{ at } (k,j)\}$, with $0 \le j \le l-1$
3. $\mathcal{N}_l(j) = \{L \in \mathcal{L}_{k+l} \mid L \text{ quits } R_{k,l} \text{ at } (j,l)\}$, with $0 \le j \le k-1$.

Therefore,

$$\mid \mathcal{L}_{k+l} \mid = \mid \mathcal{C}_{k,l} \mid + \sum_{j=0}^{l-1} \mid \mathcal{E}_k(j) \mid + \sum_{j=0}^{k-1} \mid \mathcal{N}_l(j) \mid \tag{11}$$

From Proposition 2

$$\mid \mathcal{C}_{k,l} \mid = \binom{k+l}{k} \tag{12}$$

For any $0 \le j \le l-1$, a lattice path $L \in \mathcal{E}_k(j)$ is the concatenation of

1. a subpath S_1 from $(0,0)$ to (k,j)
2. a step of type E
3. a subpath S_2 of length $l-j-1$ and beginning at $(k+1,j)$.

From Proposition 2, S_1 can be chosen in $\binom{k+j}{j}$ ways, while, from Proposition 1, S_2 can be chosen in 2^{l-j-1} ways. Therefore

$$\mid \mathcal{E}_k(j) \mid = 2^{l-j-1}\binom{k+j}{j} \tag{13}$$

Similarly, for any $0 \leq j \leq k-1$, a lattice path $L \in \mathcal{N}_l(j)$ is the concatenation of

1. a subpath S_1 from $(0,0)$ to (j,l)
2. a step of type N
3. a subpath S_2 of length $k-j-1$ and beginning at $(j,l+1)$.

From Proposition 2, S_1 can be chosen in $\binom{l+j}{j}$ ways, while, from Proposition 1, S_2 can be chosen in 2^{k-j-1} ways. Therefore

$$\mid \mathcal{N}_l(j) \mid = 2^{k-j-1}\binom{l+j}{j} \tag{14}$$

By replacing (12), (13) and (14) in (11), we obtain

$$\mid \mathcal{L}_{k+l} \mid = \binom{k+l}{k} + \sum_{j=0}^{l-1} 2^{l-j-1}\binom{k+j}{j} + \sum_{j=0}^{k-1} 2^{k-j-1}\binom{l+j}{j} \tag{15}$$

From (10) and (15) we obtain (9), the relation to be proven. □

4.2 A 3-dimensional form

The notion of NE lattice paths can naturally be extended to more dimensions. In what follows, we will consider the particular case of 3-dimensional paths and prove a 3-dimensional version of (2). We will start by introducing some definitions and basic results.

Definition 2. *A* ***UNE (Up-North-East) lattice path*** *L is a sequence $(v_0, v_1, ..., v_k)$ of points (vectors) in $\mathbf{Z}^3$ such that $v_{i+1} - v_i$ belongs to*

$$\{(1,0,0),(0,1,0),(0,0,1)\}$$

for every $0 \leq i \leq k-1$. We call ***step*** *each pair (v_i, v_{i+1}) of consecutive points in L. The step is of* ***type U (Up)*** *if $v_{i+1} - v_i = (0,0,1)$, of* ***type N (North)*** *if $v_{i+1} - v_i = (0,1,0)$ or of* ***type E (East)*** *if $v_{i+1} - v_i = (1,0,0)$. We say that L is a (UNE lattice) path* ***from*** *v_0 (the* ***beginning*** *of L)* ***to*** *v_k (the* ***end*** *of L). The* ***length*** *of L is defined as being k (the number of steps).*

Remark 2. *A UNE lattice path L can alternatively be defined as a pair $(v_0; (s_1, ..., s_k))$, where*

1. *$v_0 \in \mathbf{Z}^3$ is the beginning point of L*
2. *$(s_1, s_2, ...s_k)$ is the sequence of steps of L. In this alternative definition, we will denote by the letter U,N or E, a step of type U, N or E respectively.*

Proposition 4. *Let v be a point in $\mathbf{Z}^3$. There are 3^k UNE lattice paths of length k beginning in v.*

Proof. Such a path can be written as $(v; (s_1, ..., s_k))$. v is constant while each step s_i can be either U, N or E. Therefore, there are 3^k such paths. $\square$

Proposition 5. *Let $v = (a, b, c)$ and $w = (d, e, f)$ be points in $\mathbf{Z}^3$ such that $a \leq d$ and $b \leq e$ and $c \leq f$. Then, there are*

$$\binom{(d-a)+(e-b)+(f-c)}{(d-a),(e-b),(f-c)}$$

UNE lattice paths from v to w.

Proof. In order to get from v to w, one needs to perform $(d-a)+(e-b)+(f-c)$ steps, $d-a$ of them being of type E, $e-b$ of type N and $f-c$ of type U. This can be achieved in

$$\binom{(d-a)+(e-b)+(f-c)}{(d-a),(e-b),(f-c)} = \frac{((d-a)+(e-b)+(f-c))!}{(d-a)!(e-b)!(f-c)!)}$$

ways. $\square$

We now state and proof the 3-dimensional equivalent of (2).

Proposition 6. *For k a nonnegative integer, the following identity holds:*

$$\sum_{i,j=0}^{k} 3^{2k-i-j} \binom{k+i+j}{k,i,j} = 3^{3k} \tag{16}$$

Proof. Let $\mathcal{L}_{3k}$ be the set of UNE lattice paths beginning in $(0, 0, 0)$ and having length $3k$. From Proposition 4

$$\mid \mathcal{L}_{3k} \mid = 3^{3k} \tag{17}$$

Let C_k be the cube $\{(x, y, z) \in \mathbf{R}^3 \mid 0 \leq x \leq k, 0 \leq y \leq k, 0 \leq z \leq k\}$. A path L in $\mathcal{L}_{3k}$ can fall in exactly one of the following cases:

1. L ends in (k, k, k). Let us denote by $\mathcal{C}_k$ the set of those lattice paths.

2. L quits C_k with a step of type U. If (v, v') is that step, than v must be of the form (i, j, k), where $0 \leq i, j \leq k$ and $(i, j) \neq (k, k)$. For each such pair i, j, let us denote by $\mathcal{U}_k(i, j)$ the set of those L that quit the cube in (i, j, k), with a step of type U.

3. L quits C_k with a step of type N. If (v, v') is that step, than v must be of the form (i, k, j), where $0 \leq i, j \leq k$ and $(i, j) \neq (k, k)$. For each such pair (i, j), let us denote by $\mathcal{N}_k(i, j)$ the set of those L that quit the cube in (i, k, j), with a step of type N.

4. L quits C_k with a step of type E. If (v, v') is that step, than v must be of the form (k, i, j), where $0 \leq i, j \leq k$ and $(i, j) \neq (k, k)$. For each such pair (i, j), let us denote by $\mathcal{E}_k(i, j)$ the set of those L that quit the cube in (k, i, j), with a step of type E.

As all these sets are (pairwise) disjoint, we have

$$\mid \mathcal{L}_{3k} \mid = \mid \mathcal{C}_k \mid + \sum_{i,j=0,(i,j)\neq(k,k)}^{k} \mid \mathcal{U}_k(i,j) \mid + \sum_{i,j=0,(i,j)\neq(k,k)}^{k} \mid \mathcal{N}_k(i,j) \mid + \sum_{i,j=0,(i,j)\neq(k,k)}^{k} \mid \mathcal{E}_k(i,j) \mid \tag{18}$$

By symmetry, $\mathcal{U}_k(i, j) \mid = \mathcal{E}_k(i, j) \mid = \mathcal{E}_k(i, j) \mid$ for every pair (i, j). Therefore,

$$\mid \mathcal{L}_{3k} \mid = \mid \mathcal{C}_k \mid + 3 \cdot \sum_{i,j=0,(i,j)\neq(k,k)}^{k} \mid \mathcal{U}_k(i,j) \mid \tag{19}$$

Each path L in $\mathcal{U}_k(i, j)$ is a concatenation of

1. a subpath S_1 from $(0,0,0)$ to (i,j,k)
2. a step of type U
3. a subpath of length $3k-i-j-k-1=2k-i-j-1$, starting at $(i,j,k+1)$.

From Proposition 5, S_1 can be chosen in

$$\binom{k+i+j}{k,i,j}$$

ways. From Proposition 4, there are $3^{2k-i-j-1}$ choices for S_2. Therefore,

$$\mathcal{U}_k(i,j)\mid= 3^{2k-i-j-1}\binom{k+i+j}{k,i,j} \tag{20}$$

Finally, from Proposition 5,

$$\mid \mathcal{C}_k \mid = \binom{3k}{k,k,k} \tag{21}$$

By using (20) and (21) in (19), we get

$$\begin{aligned}\mid \mathcal{L}_{3k}\mid = \binom{3k}{k,k,k} + \sum_{i\neq j=0}^{k} 3^{2k-i-j}\binom{k+i+j}{k,i,j} = \\ \sum_{i,j=0}^{k} 3^{2k-i-j}\binom{k+i+j}{k,i,j}\end{aligned} \tag{22}$$

From (17) and (21) we get (16). □

5 Conclusion

We have used the same combinatorial technique to prove three identities that all follow a similar pattern. The first one is the symmetric 2-dimensional version of that pattern, the second is the non-symmetric 2-dimensional version, while the third is the symmetric 3-dimensional of it. It should be obvious from the identities and proves above how to state and prove the symmetric n-dimensional version of the identity, for any $n \geq 2$. We let this as an exercise for the reader.

References

[1] https://kskedlaya.org/putnam-archive

[2] Stanley, Richard (2012) *Enumerative Combinatorics, Volume 1 (2 ed.)*, Cambridge University Press, p. 21.